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The Tame Kernel of Multi-Cyclic Number Fields

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Abstract. There are many results about the structures of the tame kernels of the number fields. In this paper, we study the structure of those fields F, which are the composition of some cyclic number fields, whose degrees over $\mathbb{Q}$ are the same prime number q. Then, for any odd prime $p \neq q$, we prove that the p-primary part of $K_2\mathscr{O}_F$ is the direct sum of the p-primary part of the tame kernels of all the cyclic intermediate fields of $F/\mathbb{Q}$. Moreover, by the approach we developed, we can extend the results to any abelian totally real base field K with trivial p-primary tame kernel.

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1. Introduction

Let L be an algebraic number field and $\mathscr{O}_L$ the ring of integers of L. It is well-known that $K_2\mathscr{O}_L$ is the tame kernel of L. The structure of $K_2\mathscr{O}_L$ has been extensively investigated by many authors (see e.g., [2, 5, 11], and the references therein). In particular, in [8–10], some results about the tame kernels of quadratic number fields have been obtained. In [2, 4, 13], structures on the tame kernels of cubic cyclic number fields have been studied. Recently, in [14], Zhou studied the structure of multi-quadratic fields and found that, for any odd prime p, $(K_2\mathscr{O}_N)_p = \bigoplus(K_2\mathscr{O}_H)_p$, where N is a multi-quadratic field and H runs over all quadratic subfields of N.

In this paper, we develop a new approach and extend Zhou's results to multi-cyclic number fields when all the fields involved are totally real. Specifically, let F be a multi-cyclic number field, for any odd prime $p \neq q$, we prove that $(K_2\mathscr{O}_F)_p = \bigoplus(K_2\mathscr{O}_E)_p$, where E runs over all the cyclic subfields of F and q is the degree of E over $\mathbb{Q}$. Moreover, by the approach we developed, we can extend the results to any abelian totally real base field K with trivial p-primary tame kernel. Based on Browkin's and Zhou's numerical results in [2, 13], we can determine the structure of the odd part of the tame kernels of multi-cyclic number fields with two ramified primes p_1, p_2, where $7 \leq p_1, p_2 \leq 100$.

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2. The odd part of the tame kernel $K_2\mathcal{O}_F$

Theorem 2.1. *Let $p_1, \ldots, p_t$ be odd primes, and g_i be a primitive root modulo p_i, $i = 1,2,\ldots,t$. If there is an odd prime q satisfying $q|(p_i-1)$, let $\alpha_i = \sum_{j\in<g_i^q>} \zeta_{p_i}^j$, $i = 1,2,\ldots,t$. Let $F = \mathbb{Q}(\alpha_1,\ldots,\alpha_t)$. Then for any odd prime $p \neq q$, $(K_2\mathcal{O}_F)_p = \bigoplus(K_2\mathcal{O}_E)_p$, where E runs over all the cyclic subfields of F.*

Prior to proving Theorem 2.1, we first introduce some preliminary results. Let Γ be a finite group and S be a finite Γ-set, with Γ acting on the right. Then S can be written as a disjoint union of Γ-orbits S_i. Furthermore, if $s_i \in S_i$ has stabilizer Λ_i, S_i is isomorphic as a Γ-set to the right coset space $\Lambda_i\backslash\Gamma$. Thus, we have an isomorphism of Γ-sets $S \cong \dot{\bigcup_i}\Lambda_i\backslash\Gamma$ (disjoint union).

Suppose that two Γ-sets, S and T, have the property that $\mathbb{C}(S)$ and $\mathbb{C}(T)$ are isomorphic $\mathbb{C}\Gamma$-modules, where $\mathbb{C}(S)$ and $\mathbb{C}(T)$ are the $\mathbb{C}$-vector spaces on S and T. Suppose that S and T have orbit decompositions:

$$S \cong \dot{\bigcup_a}\Lambda_a\backslash\Gamma, \quad T \cong \dot{\bigcup_b}\Upsilon_b\backslash\Gamma.$$

In the following part, we set Γ=Gal$(F/\mathbb{Q})$, where F is defined in Theorem 2.1. Then, in [3], the following result is obtained.

Lemma 2.1. [3, Theorem 73] *Let N_a (resp. N_b) denote the subfield of F fixed by Λ_a (resp. Υ_b). Then*

$$\prod_a \zeta_{N_a}(x) = \prod_b \zeta_{N_b}(x).$$

By the definition of Γ, we know that Γ is an elementary group of order q^t, and can be written as the direct product of t cyclic groups of order q. Let S and T denote Γ-sets with orbit decompositions:

$$S = (\Gamma\backslash\Gamma)^{(k)}\dot{\bigcup}\Gamma, \quad \text{where} \quad k = q^{t-1}+q^{t-2}+\cdots+q,$$

$$T = \dot{\bigcup_\Delta}\Delta\backslash\Gamma,$$

where $\Gamma\backslash\Gamma$ is a singleton with a trivial Γ-action, and $(\Gamma\backslash\Gamma)^{(k)}$ denotes the disjoint union of k copies of this Γ-set, and the union in the definition of T extends over the $q^{t-1}+q^{t-2}+\cdots+1$ distinct subgroups of Δ of order q^{t-1} in Γ.

The trace of $\gamma\in\Gamma$ on $\mathbb{C}(S)$ is given by the number of γ-fixed elements in S, denoted $|S^{<\gamma>}|$. Thus to show $\mathbb{C}(S)\cong\mathbb{C}(T)$, it suffices to prove the following lemma.

Lemma 2.2. *For each $\gamma\in\Gamma$, $|S^{<\gamma>}| = |T^{<\gamma>}|$.*

Proof. The result is clear when $\gamma = 1$, since $|S^{<\gamma>}| = q^t+q^{t-1}+\cdots+q = |T^{<\gamma>}|$. So let $\gamma\in\Gamma$, $\gamma\neq 1$. We note that

$$(\Gamma\backslash\Gamma)^{<\gamma>} = \Gamma\backslash\Gamma \quad \text{and} \quad \Gamma^{<\gamma>} = \emptyset,$$

$$(\Delta\backslash\Gamma)^{<\gamma>} = \begin{cases} \Delta\backslash\Gamma, & \text{if } \gamma\in\Delta; \\ \emptyset, & \text{otherwise.}\end{cases}$$

For each $\gamma \neq 1$, there are $q^{t-2}+q^{t-3}+\cdots+1$ distinct Δ containing γ. Therefore $|S^{<\gamma>}| = (q^{t-2}+q^{t-3}+\cdots+1)\cdot q = |T^{<\gamma>}|$. ∎

Thus we get $\mathbb{C}(S) \cong \mathbb{C}(T)$. This result in conjunction with Lemma 2.1 yields the following relation

$$\left(-\frac{1}{12}\right)^k \zeta_F(-1) = \zeta_{\mathbb{Q}}(-1)^k \zeta_F(-1) = \prod \zeta_E(-1), \tag{2.1}$$

where F and E are defined in Theorem 2.1, $k = q^{t-1}+q^{t-2}+\cdots+q$.

The argument we use to derive (2.1) is similar with that in [3], which is used to prove (7.21), except that in [3] only the case $t=2$ is considered. In fact, (2.1) can also be obtained by Satz 3 in [1]. In [14], Zhou gives the following result.

Lemma 2.3. [14, Theorem 5] *Let N/K be an abelian extension with Galois group G of order n and $p \nmid n$. Then $(K_2\mathscr{O}_N)_p = \sum (K_2\mathscr{O}_H)_p$, where H runs over all intermediate fields cyclic over K.*

Now we introduce the Birch-Tate conjecture, which can be used to compute the order of $K_2\mathscr{O}_F$. The conjecture states that whenever M is a totally real number field,

$$\#K_2\mathscr{O}_M = \omega_2(M)|\zeta_M(-1)|, \tag{2.2}$$

where ζ_M is the Dedekind zeta function of the field M, and

$$\omega_2(M) = 2\prod_{l\ prime} l^{n_l},$$

where n_l is the largest integer n such that M contains $\mathbb{Q}(\zeta_{l^n}+\zeta_{l^n}^{-1})$, the maximal real subfield of $\mathbb{Q}(\zeta_{l^n})$. The conjecture is known to be true when M is abelian over $\mathbb{Q}$ and is known to be true in general up to a power of 2. (See [6, 7, 12].)

We now give a proof of Theorem 2.1.

Proof. By the Birch-Tate conjecture, we have

$$\#K_2\mathscr{O}_F = \omega_2(F)|\zeta_F(-1)|, \tag{2.3}$$

$$\#K_2\mathscr{O}_E = \omega_2(E)|\zeta_E(-1)|. \tag{2.4}$$

In almost all cases, $\omega_2(E) = 24$, however, there are some special cases, in which $\omega_2(E) = p\cdot 24$ for some odd primes p. In those cases, the corresponding $\omega_2(F)$ also equals to $p\cdot 24$. For any $n \in \mathbb{N}$ and prime p, we denote $n_p := p^{v_p(n)}$, where $v_p(n)$ is the p-adic valuation of n. Then, combining (2.1), (2.2), (2.3) and (2.4), we get the following result, for any odd prime p,

$$(\#K_2\mathscr{O}_F)_p = \prod (\#K_2\mathscr{O}_E)_p, \tag{2.5}$$

where E runs over all the cyclic subfields of F.

For example, when $p=3$,

$$\begin{aligned}(\#K_2\mathscr{O}_F)_3 &= \omega_2(F)_3 \cdot |\zeta_F(-1)|_3 = 3\cdot(12^k)_3 \cdot \prod |\zeta_E(-1)|_3\\ &= 3^{q^{t-1}+q^{t-2}+\ldots+q+1} \cdot \prod |\zeta_E(-1)|_3,\end{aligned}$$

$$\begin{aligned}\prod (\#K_2\mathscr{O}_E)_3 &= \prod \omega_2(E)_3 \cdot \prod |\zeta_E(-1)|_3 = 3^{\frac{q^t-1}{q-1}} \cdot \prod |\zeta_E(-1)|_3\\ &= 3^{q^{t-1}+q^{t-2}+\ldots+q+1} \cdot \prod |\zeta_E(-1)|_3.\end{aligned}$$

Thus we get

$$(\#K_2\mathscr{O}_F)_3 = \prod(\#K_2\mathscr{O}_E)_3.$$

By Lemma 2.3, it is easy to see that

$$(K_2\mathscr{O}_F)_p = \sum(K_2\mathscr{O}_E)_p, \tag{2.6}$$

where $p \neq q$ and E runs over all the cyclic subfields of F. Then by (2.5) and (2.6), for any odd prime $p \neq q$, we can get

$$(K_2\mathscr{O}_F)_p = \bigoplus(K_2\mathscr{O}_E)_p,$$

where E runs over all the cyclic subfields of F. This completes the proof. ∎

In fact, our approach in Theorem 2.1 also gives a more general result for any abelian totally real base field K with trivial p-primary tame kernel.

With the notation of Theorem 2.1, we can get the following Corollary.

Corollary 2.1. *Let $F = \mathbb{Q}(\alpha_1, ..., \alpha_t)$. For any odd prime $p \neq q$, we can get*

$$p^i\text{-rank } K_2\mathscr{O}_F = \sum p^i\text{-rank } K_2\mathscr{O}_E,$$

where $i > 0$, E runs over all the cyclic subfields of F.

In particular, when $q = 3$, F is the composition of some cubic cyclic fields. Then we can get the following theorem.

Theorem 2.2. *If p is a prime number and $p \equiv 5 \ (mod\, 6)$, then the p^i*-rank of $\mathrm{K}_2\mathscr{O}_\mathrm{F}$ *is even, where $i > 0$. Moreover, if k_i is the largest integer n such that $2^n | p^i$*-rank $K_2\mathscr{O}_E$, *where E runs over all the cyclic subfields of F, then $2^{\sum k_i} | p^i$*-rank $K_2\mathscr{O}_F$.

Proof. By [13, Theorem 3.13] we know that p^i-rank of $K_2\mathscr{O}_E$ is even, for $i > 0$. Thus the result follows from Corollary 2.1. ∎

3. Applications

Let F be a bicubic field with exactly two ramified primes $p_1,\ p_2$. For $7 \leq p_1, p_2 < 100$, by the results of Browkin and Haiyan Zhou (see [2, 13]), as an application of Theorem 2.1, it is straightforward to get the structure of the p-primary part of $K_2\mathscr{O}_F$, $p > 3$. For example, when $p_1 = 7$ and $p_2 = 31$, for $p > 3$, the p-primary part of $K_2\mathscr{O}_F$ is $\mathbb{Z}/7 \times \mathbb{Z}/13 \times \mathbb{Z}/37 \times \mathbb{Z}/37 \times \mathbb{Z}/61$.

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