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Weighted Endpoint Estimates for Multilinear Commutators of Marcinkiewicz Integrals

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Abstract. Let $\mu_{\Omega,\vec{b}}$ be the multilinear commutator generalized by μ_Ω, the n-dimensional Marcinkiewicz integral, with $\mathrm{Osc}_{\exp L^\tau}(\mathbb{R}^n)$ functions for $\tau \geq 1$, where $\mathrm{Osc}_{\exp L^\tau}(\mathbb{R}^n)$ is a space of Orlicz type satisfying that $\mathrm{Osc}_{\exp L^\tau}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ if $\tau = 1$ and $\mathrm{Osc}_{\exp L^\tau}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$ if $\tau > 1$. The authors establish the weighted weak $L\log L$-type estimates for $\mu_{\Omega,\vec{b}}$ when Ω satisfies a kind of Dini conditions.

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1. Introduction and main result

Denote by S^{n-1} the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $\mathrm{d}x' = \mathrm{d}\sigma(x')$. Let $\Omega(x) \in L^1(S^{n-1})$ be homogeneous function of degree zero in $\mathbb{R}^n$ satisfying

$$\int_{S^{n-1}} \Omega(x')\mathrm{d}x' = 0, \tag{1.1}$$

where $x' = x/|x|$ $(x \neq 0)$. The n-dimensional Marcinkiewicz integral introduced by Stein [11] is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.$$

A weight will always means a positive locally integrable function. As usual, we denote by A_p $(1 \leq p \leq \infty)$ the Muckenhoupt weights classes (see [4, 12] for details). For a weight ω on $\mathbb{R}^n$, we write $\|f\|_{L^p_\omega(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\mathrm{d}x)^{1/p}$ and $\omega(E) = \int_E \omega(x)\mathrm{d}x$.

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In 2004, Ding, Lu and Zhang [1] studied the weighted weak $L\log L$-type estimates for the commutators of the Marcinkiewicz integral, which is defined by

$$\mu_{\Omega,b}^m(f)(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{(b(x)-b(y))^m\Omega(x-y)}{|x-y|^{n-1}}f(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}},\ m\in\mathbb{Z}^+,\ b\in\mathrm{BMO}(\mathbb{R}^n),$$

when the kernel Ω satisfies the $\mathrm{Lip}_\alpha(0<\alpha\le 1)$ condition, that is, there exists a constant $C>0$ such that

$$(1.2)\qquad |\Omega(x')-\Omega(y')|\le C|x'-y'|^\alpha,\quad \forall\, x',y'\in S^{n-1}.$$

In 2008, Zhang [13] established the weighted weak $L(\log L)^{1/r}$-type estimates for the multilinear commutators of the Marcinkiewicz integral when $\omega\in A_1$, and Ω satisfies (1.1) and (1.2). Let $\Omega\in L^r(S^{n-1})$ $(r\ge 1)$, the integral modulus of continuity of order r of Ω is defined by

$$\omega_r(\delta)=\sup_{|\rho|<\delta}\left(\int_{S^{n-1}}|\Omega(\rho x')-\Omega(x')|^r\mathrm{d}x'\right)^{1/r},$$

where ρ is a rotation in $\mathbb{R}^n$ with $|\rho|=\sup_{x'\in S^{n-1}}|\rho x'-x'|$. We say $\Omega\in L^r(S^{n-1})$ $(r\ge 1)$ satisfies the L^r-Dini condition if $\int_0^1\omega_r(\delta)\delta^{-1}\mathrm{d}\delta<\infty$. Recently, Zhang [14] also considered the following result.

Theorem 1.1. [14] *Let* $b\in\mathrm{BMO}(\mathbb{R}^n),\Omega\in L^r(S^{n-1})$ *for some* $r>1$, *and* $\omega^{r'}\in A_1$. *If* Ω *satisfies* (1.1) *and*

$$(1.3)\qquad \int_0^1\frac{\omega_r(\delta)}{\delta}\left(\log\frac{1}{\delta}\right)^m\mathrm{d}\delta<\infty,$$

then for all $\lambda>0$, *there has*

$$\omega(\{x\in\mathbb{R}^n:\mu_{\Omega,b}^m(f)(x)>\lambda\})\le C\int_{\mathbb{R}^n}\frac{|f(y)|}{\lambda}\left(1+\log^+\frac{|f(y)|}{\lambda}\right)^m\omega(y)\mathrm{d}y,$$

where C *is a positive constant independent of* f *and* λ.

In this paper, by applying the calderón-Zygmund decomposition theory, we will study the weighted weak $L\log L$-type estimates for the multilinear commutators generated by μ_Ω and $\mathrm{Osc}_{\exp L^r}(\mathbb{R}^n)$ functions, in analogy with the results established by Pérez and Trujillo-González in [7] for the multilinear commutators of Calderón-Zygmund operators. Before stating our results, we first recall some notation.

Let m be a positive integer and $\vec{b}=(b_1,b_2,\dots,b_m)$, we define the multilinear commutators $\mu_{\Omega,\vec{b}}$ by

$$\mu_{\Omega,\vec{b}}(f)(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)f(y)}{|x-y|^{n-1}}\prod_{j=1}^m(b_j(x)-b_j(y))\,\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}}.$$

It is easy to see, when $m=1$, $\mu_{\Omega,\vec{b}}$ is the commutator of Marcinkiewicz integral and when $b_1=\cdots=b_m$, $\mu_{\Omega,\vec{b}}$ is the higher commutator of Marcinkiewicz integral.

To state the weak type estimate for the multilinear commutator $\mu_{\Omega,\vec{b}}$, we need to introduce the following notation. As in [7], given any positive integer m, for all $1\le j\le m$, we denote by $\mathscr{C}_j^m$ the family of all finite subsets $\sigma=\{\sigma(1),\sigma(2),\dots,\sigma(j)\}$ of $\{1,2,\dots,m\}$

with j different elements. For any $\sigma \in \mathscr{C}_j^m$, we define the complementary sequence $\sigma' = \{1,2,\ldots,m\} \setminus \sigma$.

In the following, we will always assume that Ω be homogeneous function of degree 0, and let $\vec{b} = (b_1, b_2, \ldots, b_m)$ be a finite family of locally integrable functions. For all $1 \le j \le m$ and $\sigma = \{\sigma(1), \sigma(2), \ldots, \sigma(j)\} \in \mathscr{C}_j^m$, we write for any i-tuple $(\tau_1, \tau_2, \ldots, \tau_m)$ with $\tau_j \ge 1$ for $1 \le j \le m$, $1/\tau_\sigma = 1/\tau_{\sigma(1)} + \cdots + 1/\tau_{\sigma(j)}$ and $1/\tau_{\sigma'} = 1/\tau - 1/\tau_\sigma$, where $1/\tau = 1/\tau_1 + \cdots + 1/\tau_m$, we will denote $\vec{b}_\sigma = (b_{\sigma(1)}, b_{\sigma(2)}, \ldots, b_{\sigma(j)})$ and the product $b_\sigma = b_{\sigma(1)} b_{\sigma(2)} \cdots b_{\sigma(j)}$. With this notation, we write

$$\|\vec{b}_\sigma\|_{\mathrm{Osc}_{\exp L^{\tau_\sigma}}(\mathbb{R}^n)} = \|b_{\sigma(1)}\|_{\mathrm{Osc}_{\exp L^{\tau_{\sigma(1)}}}(\mathbb{R}^n)} \cdots \|b_{\sigma(j)}\|_{\mathrm{Osc}_{\exp L^{\tau_{\sigma(j)}}}(\mathbb{R}^n)}.$$

In particular, we write

$$(b(x) - b(y))_\sigma = \left(b_{\sigma(1)}(x) - b_{\sigma(1)}(y)\right) \cdots \left(b_{\sigma(j)}(x) - b_{\sigma(j)}(y)\right),$$

and

$$(b_B - b(y))_\sigma = \left((b_{\sigma(1)})_B - b_{\sigma(1)}(y)\right) \cdots \left((b_{\sigma(j)})_B - b_{\sigma(j)}(y)\right),$$

where B is any ball in $\mathbb{R}^n$, $x, y \in \mathbb{R}^n$, and $f_B = |B|^{-1} \int_B f(y) \mathrm{d}y$. For any $\sigma \in \mathscr{C}_j^m$, we set

$$\mu_{\Omega,\vec{b}_\sigma}(f)(x) = \left(\int_0^\infty \left| \int_{|x-y| \le t} \frac{\Omega(x-y) f(y)}{|x-y|^{n-1}} \prod_{i=1}^{j} \left(b_{\sigma(i)}(x) - b_{\sigma(i)}(y)\right) \mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{\frac{1}{2}}.$$

If $\sigma = \{1,2,\ldots,m\}$, then σ' is an empty set, we understand $\mu_{\Omega,\vec{b}_\sigma} = \mu_{\Omega,\vec{b}}$ and $\mu_{\Omega,\vec{b}_{\sigma'}} = \mu_\Omega$. Our main result can be stated as follows.

Theorem 1.2. *Let $b_j \in \mathrm{Osc}_{\exp L^{\tau_j}}, \tau_j \ge 1$ $(1 \le j \le m), \Omega \in L^r(S^{n-1})$ for some $r > 1$, and $\omega^{r'} \in A_1$. If Ω satisfies* (1.1) *and* (1.3)*, then for all $\lambda > 0$, there has*

$$\omega\left(\left\{x \in \mathbb{R}^n : \mu_{\Omega,\vec{b}}(f)(x) > \lambda\right\}\right) \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \frac{|f(y)|}{\lambda}\right)^m \omega(y) \mathrm{d}y,$$

where C is a positive constant independent of f and λ.

Remark 1.1. Noting that $\mathrm{Osc}_{\exp L^1} = \mathrm{BMO}$ and $\mathrm{Osc}_{\exp L^\tau} \subset \mathrm{BMO}$ for $\tau > 1$. For more information on Orlicz space see [10].

Obviously, condition (1.3) is slightly stronger than the L^r-Dini condition, but much more weaker than the Lip_α condition. Noting that $\mu_{\Omega,\vec{b}}$ coincides with $\mu_{\Omega,b}^m$ when $b_j = b$ for $j = 1, 2, \ldots, m$. So, Theorem 1.2 improves the main results in [13, 14].

Throughout this paper, C denotes a constant that is independent of the main parameters involved but whose value may differ from line to line. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$. For $A \sim B$, we mean that there is a constant $C > 0$ such that $C^{-1} B \le A \le CB$.

2. Preliminaries and lemmas

In this section, we will formulate some lemmas and preliminaries.

Lemma 2.1. [2] *Suppose that $0<\alpha<n, r>1$ and Ω satisfies the L^r-Dini condition. If there is a constant C_0 with $0<C_0<1/2$ such that $|y|<C_0K$, then*

$$\left(\int_{K<|x|<2K}\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(x)}{|x|^{n-\alpha}}\right|^r \mathrm{d}x\right)^{1/r} \leq CK^{n/r-n+\alpha}\left(\frac{|y|}{K}+\int_{|y|/(2K)<\delta<|y|/K}\frac{\omega_r(\delta)}{\delta}\mathrm{d}\delta\right).$$

Lemma 2.2. [3] *Suppose $\Omega\in L^r(S^{n-1})$ for some $r>1$ and $\omega^{r'}\in A_1$. Then for any $\lambda>0$, there is a constant $C>0$ independent of f and λ, such that*

$$\omega(\{x\in\mathbb{R}^n:\mu_\Omega(f)(x)>\lambda\})\leq C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}.$$

Lemma 2.3. *Let $\omega\in A_1, 1<p<\infty$, $b_j\in \mathrm{Osc}_{\exp L^{\tau_j}}$, $\tau_j\geq 1$ $(1\leq j\leq m), \Omega\in L^r(S^{n-1})$ for some $r>1$ and satisfies* (1.1) *and* (1.3). *Then, there is a constant $C>0$ independent of f, such that*

$$\|\mu_{\Omega,\vec{b}}(f)\|_{L^p_\omega(\mathbb{R}^n)}\leq C\|\vec{b}\|_{\mathrm{Osc}_{\exp L^{\tau}}}\|f\|_{L^p_\omega(\mathbb{R}^n)}.$$

The idea of the proof of Lemma 2.3 comes from the [13, corollary 1.3]. We omit the details. We also need a few facts of Orlicz spaces, see [10] for more information. A function $\varphi:[0,+\infty)\to[0,+\infty)$ is called a Young function if φ is continuous, convex and increasing with $\varphi(0)=0$ and $\varphi(t)\to+\infty$ as $t\to+\infty$. We defined the φ-average of a function f over a ball B by means of the Luxemburg norm

$$\|f\|_{\varphi,B}=\inf\left\{\lambda>0:\frac{1}{|B|}\int_B\varphi\left(\frac{|f(y)|}{\lambda}\right)\mathrm{d}y\leq 1\right\},$$

which satisfies the following inequalities (see [10, p. 69] or [8, formula (7)])

$$\|f\|_{\varphi,B}\leq\inf\left\{\eta+\frac{\eta}{|B|}\int_B\varphi\left(\frac{|f(y)|}{\eta}\right)\mathrm{d}y\leq 1\right\}\leq 2\|f\|_{\varphi,B}. \tag{2.1}$$

The Young function that we are going to be using is $\Phi_\alpha(t)=t(1+\log^+t)^\alpha$ $(\alpha>0)$ with its complementary Young function $\tilde{\Phi}_\alpha(t)\approx\exp(t^{1/\alpha})$. Denote by $\|f\|_{L(\log L)^\alpha,B}=\|f\|_{\Phi_\alpha,B}$ and $\|f\|_{\exp L^{1/\alpha},B}=\|f\|_{\tilde{\Phi}_\alpha,B}$. When $\alpha=1$, we simply denote by $\Phi(t)=t(1+\log^+t)$ and $\tilde{\Phi}(t)\approx e^t$, and by $\|f\|_{L\log L,B}=\|f\|_{\Phi,B}$ and $\|f\|_{\exp L,B}=\|f\|_{\tilde{\Phi},B}$. By the generalized Hölder's inequality (see [6]), we have

$$\frac{1}{|B|}\int_B|f(y)g(y)|\mathrm{d}y\leq 2\|f\|_{L(\log L)^\alpha,B}\|g\|_{\exp L^{1/\alpha},B}. \tag{2.2}$$

As usual, for a locally integrable function f and a ball B, we denote $f_B=|B|^{-1}\int_B f(y)\mathrm{d}y$. Let $b\in\mathrm{BMO}(\mathbb{R}^n)$, for any ball B and integer $k\geq 0$, there has (see [12, p.141])

$$|b_{2^{k+1}B}-b_B|\leq C(k+1)\|b\|_*, \tag{2.3}$$

where ℓB denotes the ℓ-times concentric expansion of B and $\|b\|_*$ denotes the BMO norm of b. By the John-Nirenberg's inequality, it is not difficult to see that (c.f. [9, p.169])

$$\|b-b_B\|_{\exp L,B}\leq C\|b\|_*. \tag{2.4}$$

Let $M_{L(\log L)^\alpha}(f)(x)=\sup_{B\ni x}\|f\|_{L(\log L)^\alpha,B}$. Denote by M the Hardy-Littlewood maximal function and M^k the k-times iterations of M, then $M_{L(\log L)^k}\approx M^{k+1}$ for $k=0,1,2,\ldots$. We also need the following estimates in the proof of Theorem 1.2.

Lemma 2.4. [13] *Let $1 \le p < \infty, \omega^p \in A_1$ and B be a ball. Then for any $y \in B$ and any positive integer m, there has*

$$\left(\frac{1}{|2^kB|}\int_{2^kB}|b(x)-b_B|^{mp}\omega^p(x)\mathrm{d}x\right)^{1/p} \le C\|b\|_*^m(k+1)^m \inf_{y\in B}\omega(y), \quad k=0,1,2,\ldots.$$

Lemma 2.5. *Let $1 \le p < \infty, \omega^p \in A_1$ and B be a ball. Then for any $y \in B$ and any positive integer m, there has*

$$\left(\frac{1}{|2^kB|}\int_{2^kB}\omega^p(x)\prod_{j=1}^m\left|b_j(x)-(b_j)_B\right|^p\mathrm{d}x\right)^{1/p} \le C\|\vec{b}\|_*(k+1)^m\inf_{y\in B}\omega(y), \quad k=0,1,2,\ldots.$$

Proof. By the Hölder's inequality and Lemma 2.4, we obtain

$$\begin{aligned}&\left(\frac{1}{|2^kB|}\int_{2^kB}\omega^p(x)\prod_{j=1}^m\left|b_j(x)-(b_j)_B\right|^p\mathrm{d}x\right)^{1/p}\\ &\le \prod_{j=1}^m\left(\frac{1}{|2^kB|}\int_{2^kB}\omega^p(x)\left|b_j(x)-(b_j)_B\right|^{p\gamma_j}\mathrm{d}x\right)^{\frac{1}{p\gamma_j}} \le C\prod_{j=1}^m\left(\|b_j\|_*^{\gamma_j}(k+1)^{\gamma_j}\inf_{y\in B}\omega(y)\right)^{\frac{1}{\gamma_j}}\\ &\le C\|\vec{b}\|_*(k+1)^m\inf_{y\in B}\omega(y),\end{aligned}$$

where $1=1/\gamma_1+1/\gamma_2+\cdots+1/\gamma_m$. This completes the proof of Lemma 2.5. ∎

We also need the following notations. For $\omega \in A_\infty$ and a ball B, denote by

$$\|f\|_{L(\log L)^m,B,\omega} = \inf\left\{\lambda>0 : \frac{1}{\omega(B)}\int_B\Phi_m\left(\frac{|f(y)|}{\lambda}\right)\omega(y)\mathrm{d}y \le 1\right\}$$

and

$$\|f\|_{\exp L^{1/m},B,\omega} = \inf\left\{\lambda>0 : \frac{1}{\omega(B)}\int_B\tilde{\Phi}_m\left(\frac{|f(y)|}{\lambda}\right)\omega(y)\mathrm{d}y \le 1\right\}.$$

Similar to (2.1), we have (c.f. [10, p.69])

$$\|f\|_{L(\log L)^m,B,\omega} \approx \inf\left\{\eta+\frac{\eta}{\omega(B)}\int_B\Phi_m\left(\frac{|f(y)|}{\eta}\right)\omega(y)\mathrm{d}y\right\}. \tag{2.5}$$

By (2.2), there also holds the following generalized Hölder's inequality

$$\frac{1}{\omega(B)}\int_B|f_1(y)\cdots f_m(y)g(y)|\omega(y)\mathrm{d}y \le C\|g\|_{L(\log L)^m,B,\omega}\prod_{j=1}^m\|f_j\|_{\exp L,B,\omega}. \tag{2.6}$$

Furthermore, for any $b \in \mathrm{BMO}(\mathbb{R}^n)$, any ball B and any $\omega \in A_\infty$, there has

$$\|b-b_B\|_{\exp L,B,\omega} \le C\|b\|_*, \tag{2.7}$$

Indeed, by John-Nirenberg's inequality, there exist positive constants C_1 and C_2, such that

$$|\{x\in B : |b(x)-b_B|>t\}| \le C_1|B|e^{-C_2t/\|b\|_*}.$$

Noting that $\omega \in A_\infty$, from the proof of [5, Theorem 5], there is a $\delta > 0$, such that

$$\omega(\{x\in B : |b(x)-b_B|>t\}) \le C_1\omega(B)e^{-C_2\delta t/\|b\|_*}.$$

Similar to the proof of [4, Corollary 7.1.7, p. 528], we have

$$\frac{1}{\omega(B)}\int_B \exp\left(\frac{|b(x)-b_B|}{C_3\|b\|_*}\right)\omega(x)\mathrm{d}x \le C, \tag{2.8}$$

which implies (2.7).

3. Proof of Theorem 1.2

Without loss of generality, we may assume that for $j=1,\dots,m$, $\|b_j\|_{\mathrm{Osc}_{\exp L^{\tau_j}}(\mathbb{R}^n)}=1$. In fact, let

$$\tilde{b}_j=\frac{b_j}{\|b_j\|_{\mathrm{Osc}_{\exp L^{\tau_j}}(\mathbb{R}^n)}}$$

for $j=1,\dots,m$. The homogeneity tells us that for any $\lambda>0$,

$$\begin{aligned}&\omega\left(\left\{x\in\mathbb{R}^n:\mu_{\Omega,\vec{b}}(f)(x)>\lambda\right\}\right)\\&=\omega\left(\left\{x\in\mathbb{R}^n:\mu_{\Omega,\tilde{b}}(f)(x)>\lambda/\|\vec{b}\|_{\mathrm{Osc}_{\exp L^{\tau}}(\mathbb{R}^n)}\right\}\right)\end{aligned} \tag{3.1}$$

Noting that $\|\tilde{b}_j\|_{\mathrm{Osc}_{\exp L^{\tau_j}}(\mathbb{R}^n)}=1$ for $j=1,\dots,m$, if when $\|b_j\|_{\mathrm{Osc}_{\exp L^{\tau_j}}(\mathbb{R}^n)}=1$ $(j=1,\dots,m)$, the theorem is true. By (3.1) and the inequality

$$\Phi_s(t_1t_2)\le C\Phi_s(t_1)\Phi_s(t_2)$$

for any $s>0, t_1,t_2\ge 0$, we easily obtain that the theorem still holds for any $b_j\in \mathrm{Osc}_{\exp L^{\tau_j}}(\mathbb{R}^n)$ $(j=1,\dots,m)$.

For a fixed λ, we consider the Calderón-Zygmund decomposition of f at height λ and get a sequence of balls $\{B_i\}$, where B_i is a ball centered at x_i with radius r_i, such that $|f(x)|\le C\lambda$ for *a.e.* $x\in\mathbb{R}^n\setminus\cup_i B_i$ and

$$\lambda<\frac{1}{|B_i|}\int_{B_i}|f(y)|\mathrm{d}y\le 2^n\lambda. \tag{3.2}$$

Moreover, there is an integer $N\ge 1$, independent of f and λ, such that for every point in $\mathbb{R}^n$ belongs to at most N balls in $\{B_i\}$. We decompose $f=g+h$, where

$$g(x)=\begin{cases} f(x), & x\in\mathbb{R}^n\setminus\cup_i B_i,\\ f_{B_i}, & x\in B_i.\end{cases}$$

Then $h(x)=f(x)-g(x)=\sum_i h_i(x)$ with $h_i(x)=(f(x)-f_{B_i})\chi_{B_i}(x)$. Obviously, $\operatorname{supp} h_i\subset B_i$, $\int_{B_i}h_i(y)\mathrm{d}y=0$ and

$$|g(x)|\le 2^n\lambda,\quad a.e.\ x\in\mathbb{R}^n. \tag{3.3}$$

Noting that if $\omega^{r'}\in A_1$ then $\omega\in A_1$, and then $M(\omega)(x)\le C\omega(x)$ for *a.e.* $x\in\mathbb{R}^n$. By (3.2) and the fact that $|B_i|^{-1}\omega(B_i)=|B_i|^{-1}\int_{B_i}\omega(x)\mathrm{d}x\le C\inf_{y\in B_i}\omega(y)$, we have

$$\omega(B_i)\le C|B_i|\inf_{y\in B_i}\omega(y)\le C\lambda^{-1}\int_{B_i}|f(y)|\mathrm{d}y\inf_{y\in B_i}\omega(y)\le C\lambda^{-1}\int_{B_i}|f(y)|\omega(y)\mathrm{d}y. \tag{3.4}$$

Denote by $E=\cup_i(4B_i)$, it follows from (3.4) that

$$\omega(E)\le C\sum_i\int_{B_i}\omega(x)\mathrm{d}x=C\sum_i\omega(B_i)\le C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}.$$

Write

$$\begin{aligned}
&\omega\left(\left\{x\in\mathbb{R}^n:\mu_{\Omega,\vec{b}}(f)(x)>\lambda\right\}\right)\\
&\le\omega\left(\left\{x\in\mathbb{R}^n\setminus E:\mu_{\Omega,\vec{b}}(f)(x)>\lambda\right\}\right)+\omega(E)\\
&\le\omega\left(\left\{x\in\mathbb{R}^n\setminus E:\mu_{\Omega,\vec{b}}(g)(x)>\frac{\lambda}{2}\right\}\right)+\omega\left(\left\{x\in\mathbb{R}^n\setminus E:\mu_{\Omega,\vec{b}}(h)(x)>\frac{\lambda}{2}\right\}\right)+\omega(E)\\
&\le I_1+I_2+C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}.
\end{aligned}$$

We consider I_1 first. For $\omega^{r'}\in A_1$ there has $\omega\in A_1$. Noting that $A_1\subset A_s$ $(s\ge 1)$, then for any $p>r'$, we have $\omega\in A_{p/r'}$. It follows from Lemma 2.3, (3.3) and (3.4) that

$$\begin{aligned}
I_1&\le C\lambda^{-p}\int_{\mathbb{R}^n}\left(\mu_{\Omega,\vec{b}}(g)(x)\right)^p\omega(x)\mathrm{d}x\le C\lambda^{-p}\int_{\mathbb{R}^n}|g(x)|^p\omega(x)\mathrm{d}x\\
&\le C\lambda^{-1}\int_{\mathbb{R}^n}|g(x)|\omega(x)\mathrm{d}x\le C\lambda^{-1}\left(\int_{\mathbb{R}^n\setminus\cup_iB_i}|g(x)|\omega(x)\mathrm{d}x+\int_{\cup_iB_i}|g(x)|\omega(x)\mathrm{d}x\right)\\
&\le C\lambda^{-1}\left(\int_{\mathbb{R}^n}|f(x)|\omega(x)\mathrm{d}x+\sum_i\int_{B_i}|f_{B_i}|\omega(x)\mathrm{d}x\right)\\
&\le C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}+C\lambda^{-1}\sum_i\int_{B_i}\left(|B_i|^{-1}\int_{B_i}|f(y)|\mathrm{d}y\right)\omega(x)\mathrm{d}x\\
&\le C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}+C\lambda^{-1}\sum_i\int_{B_i}|f(y)|\mathrm{d}y\left(|B_i|^{-1}\int_{B_i}\omega(x)\mathrm{d}x\right)\\
&\le C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}+C\lambda^{-1}\sum_i\int_{B_i}|f(y)|\mathrm{d}y\inf_{y\in B_i}\omega(y)
\end{aligned}$$

$$\le C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}+C\lambda^{-1}\sum_i\int_{B_i}|f(y)|\omega(y)\mathrm{d}y\le C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}. \tag{3.5}$$

We remark that the proof of (3.5) implies the following fact, which will be used later.

$$\sum_i\int_{B_i}|f_{B_i}|\omega(x)\mathrm{d}x\le C\|f\|_{L^1_\omega(\mathbb{R}^n)}. \tag{3.6}$$

Now, let us estimate I_2. By the definition of μ_Ω and $\mu_{\Omega,\vec{b}}$, with the aid of the formula

$$\prod_{j=1}^m(b_j(x)-b_j(y))=\sum_{j=0}^m\sum_{\sigma\in\mathscr{C}_j^m}(b(x)-b_{B_i})_\sigma(b_{B_i}-b(y))_{\sigma'},$$

we have

$$\begin{aligned}
&\mu_{\Omega,\vec{b}}(h)(x)\\
&=\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)h(y)}{|x-y|^{n-1}}\sum_{j=0}^m\sum_{\sigma\in\mathscr{C}_j^m}(b(x)-b_{B_i})_\sigma(b_{B_i}-b(y))_{\sigma'}\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac12}\\
&\le\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)h(y)}{|x-y|^{n-1}}\prod_{j=1}^m(b_j(x)-(b_j)_{B_i})\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac12}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)h(y)}{|x-y|^{n-1}}\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}(b(x)-b_{B_i})_\sigma\,(b_{B_i}-b(y))_{\sigma'}\,\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}}\\
&+\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)h(y)}{|x-y|^{n-1}}\prod_{j=1}^{m}((b_j)_{B_i}-b_j(y))\,\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}}\\
\le&\sum_i\prod_{j=1}^m\left|b_j(x)-(b_j)_{B_i}\right|\mu_\Omega(h_i)(x)+\sum_i\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\left|(b(x)-b_{B_i})_\sigma\right|\mu_\Omega\left(h_i\,(b_{B_i}-b)_{\sigma'}\right)(x)\\
&+\mu_\Omega\left(\sum_i h_i\prod_{j=1}^m((b_j)_{B_i}-b_j)\right)(x).
\end{aligned}
$$

So, we can write I_2 as

$$
\begin{aligned}
I_2\le\ &\omega\left(\left\{x\in\mathbb{R}^n\setminus E:\sum_i\prod_{j=1}^m\left|b_j(x)-(b_j)_{B_i}\right|\mu_\Omega(h_i)(x)>\frac{\lambda}{6}\right\}\right)\\
&+\omega\left(\left\{x\in\mathbb{R}^n\setminus E:\sum_i\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\left|(b(x)-b_{B_i})_\sigma\right|\mu_\Omega\left(h_i\,(b_{B_i}-b)_{\sigma'}\right)(x)>\frac{\lambda}{6}\right\}\right)\\
&+\omega\left(\left\{x\in\mathbb{R}^n\setminus E:\mu_\Omega\left(\sum_i h_i\prod_{j=1}^m((b_j)_{B_i}-b_j)\right)(x)>\frac{\lambda}{6}\right\}\right)\\
=\ &I_{21}+I_{22}+I_{23}. \qquad (3.7)
\end{aligned}
$$

For I_{21}, using Chebyshev's inequality and Minkowski's inequality, we have

$$
\begin{aligned}
I_{21}&=\omega\left(\left\{x\in\mathbb{R}^n\setminus E:\sum_i\prod_{j=1}^m\left|b_j(x)-(b_j)_{B_i}\right|\mu_\Omega(h_i)(x)>\frac{\lambda}{6}\right\}\right)\\
&\le C\lambda^{-1}\sum_i\int_{\mathbb{R}^n\setminus 4B_i}\prod_{j=1}^m\left|b_j(x)-(b_j)_{B_i}\right|\mu_\Omega(h_i)(x)\omega(x)\mathrm{d}x\\
&\le C\lambda^{-1}\sum_i\int_{\mathbb{R}^n\setminus 4B_i}\prod_{j=1}^m\left|b_j(x)-(b_j)_{B_i}\right|\\
&\quad\times\left(\int_0^{|x-x_i|+2r_i}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)h_i(y)}{|x-y|^{n-1}}\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}}\omega(x)\mathrm{d}x\\
&\quad+C\lambda^{-1}\sum_i\int_{\mathbb{R}^n\setminus 4B_i}\prod_{j=1}^m\left|b_j(x)-(b_j)_{B_i}\right|\\
&\quad\times\left(\int_{|x-x_i|+2r_i}^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)h_i(y)}{|x-y|^{n-1}}\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}}\omega(x)\mathrm{d}x\\
&=I_{211}+I_{212}. \qquad (3.8)
\end{aligned}
$$

For $x \in \mathbb{R}^n \setminus 4B_i$ and $y \in B_i$, there has $|x-y| \le |x-x_i| + r_i$ and $|x-y| \sim |x-x_i| \sim |x-x_i| + 2r_i$, and then

$$\int_{|x-y|}^{|x-x_i|+2r_i} \frac{\mathrm{d}t}{t^3} = \frac{1}{2}\left(\frac{1}{|x-y|^2} - \frac{1}{(|x-x_i|+2r_i)^2}\right) \le \frac{Cr_i}{|x-y|^3}.$$

Noting that $\operatorname{supp} h_i \subset B_i$, it follows from the Minkowski's inequality that

$$\begin{aligned}
I_{211} &\le C\lambda^{-1} \sum_i \int_{\mathbb{R}^n\setminus 4B_i} \prod_{j=1}^m \left|b_j(x) - (b_j)_{B_i}\right| \\
&\quad \times \left(\int_{B_i} \frac{|\Omega(x-y)||h_i(y)|}{|x-y|^{n-1}} \left(\int_{|x-y|}^{|x-x_i|+2r_i} \frac{\mathrm{d}t}{t^3} \right)^{\frac{1}{2}} \mathrm{d}y \right) \omega(x)\mathrm{d}x \\
&\le C\lambda^{-1} \sum_i r_i^{1/2} \int_{\mathbb{R}^n\setminus 4B_i} \prod_{j=1}^m \left|b_j(x) - (b_j)_{B_i}\right| \left(\int_{B_i} \frac{|\Omega(x-y)||h_i(y)|}{|x-y|^{n+1/2}} \mathrm{d}y \right) \omega(x)\mathrm{d}x \\
&\le C\lambda^{-1} \sum_i r_i^{1/2} \int_{B_i} |h_i(y)| \sum_{k=1}^{\infty} \left(\int_{2^{k+1}B_i\setminus 2^k B_i} \frac{|\Omega(x-y)|}{|x-y|^{n+1/2}} \prod_{j=1}^m \left|b_j(x) - (b_j)_{B_i}\right| \omega(x)\mathrm{d}x \right) \mathrm{d}y \\
&\le C\lambda^{-1} \sum_i r_i^{1/2} \int_{B_i} |h_i(y)| \sum_{k=1}^{\infty} \left(\left(\int_{2^{k+1}B_i\setminus 2^k B_i} \frac{|\Omega(x-y)|^r}{|x-y|^{n+1/2}} \mathrm{d}x \right)^{1/r} \right. \\
&\quad \left. \times \left(\int_{2^{k+1}B_i\setminus 2^k B_i} \frac{\omega^{r'}(x)}{|x-y|^{n+1/2}} \prod_{j=1}^m \left|b_j(x) - (b_j)_{B_i}\right|^{r'} \mathrm{d}x \right)^{1/r'} \right) \mathrm{d}y
\end{aligned} \tag{3.9}$$

Noting that $2^{k-1} r_i \le |x-y| \le 2^{k+2} r_i$ whenever $y \in B_i$ and $x \in 2^{k+1}B_i \setminus 2^k B_i$, we have

$$\begin{aligned}
\left(\int_{2^{k+1}B_i\setminus 2^k B_i} \frac{|\Omega(x-y)|^r}{|x-y|^{n+1/2}} \mathrm{d}x \right)^{1/r} &\le \left(\int_{2^{k-1}r_i \le |x-y| \le 2^{k+2} r_i} \frac{|\Omega(x-y)|^r}{|x-y|^{n+1/2}} \mathrm{d}x \right)^{1/r} \\
&\le \left(\int_{2^{k-1}r_i}^{2^{k+2}r_i} \rho^{n-1} \left(\int_{S^{n-1}} \frac{|\Omega(x')|^r}{\rho^{n+1/2}} \mathrm{d}x' \right) \mathrm{d}\rho \right)^{1/r} \\
&\le C(2^k r_i)^{-\frac{1}{2r}} \|\Omega\|_{L^r(S^{n-1})}.
\end{aligned} \tag{3.10}$$

And noting that $\omega^{r'} \in A_1$ and $\|b_j\|_{\mathrm{BMO}} \le C\|b_j\|_{\mathrm{Osc}_{\exp L^{\tau_j}}}$ for $\tau_j \ge 1$ $(1 \le j \le m)$, by the Hölder's inequality, Minkowski's inequality, the properties of $\mathrm{BMO}(\mathbb{R}^n)$ functions and Lemma 2.5, we have

$$\begin{aligned}
&\left(\int_{2^{k+1}B_i\setminus 2^k B_i} \frac{\omega^{r'}(x)}{|x-y|^{n+1/2}} \prod_{j=1}^m \left|b_j(x) - (b_j)_{B_i}\right|^{r'} \mathrm{d}x \right)^{1/r'} \\
&\le C(2^{k+1}r_i)^{-(n+\frac{1}{2})/r'} \left(\int_{2^{k+1}B_i} \omega^{r'}(x) \prod_{j=1}^m \left|b_j(x) - (b_j)_{B_i}\right|^{r'} \mathrm{d}x \right)^{1/r'} \\
&\le C(2^{k+1}r_i)^{-(n+\frac{1}{2})/r'} \left(\frac{|2^{k+1}B_i|}{|2^{k+1}B_i|} \int_{2^{k+1}B_i} \omega^{r'}(x) \prod_{j=1}^m \left|b_j(x) - (b_j)_{B_i}\right|^{r'} \mathrm{d}x \right)^{1/r'}
\end{aligned}$$

$$\leq C\left(2^k r_i\right)^{-\frac{1}{2r'}}(k+1)^m \inf_{y\in B_i}\omega(y). \tag{3.11}$$

This, together with (3.9) and (3.10), gives

$$I_{211} \leq C\|\Omega\|_{L^r(S^{n-1})}\lambda^{-1}\sum_i r_i^{1/2}\int_{B_i}|h_i(y)|\left(\sum_{k=1}^{\infty}(k+1)^m(2^k r_i)^{-\frac{1}{2}}\right)\omega(y)\mathrm{d}y$$

$$\leq C\lambda^{-1}\sum_i\int_{B_i}|h_i(y)|\left(\sum_{k=1}^{\infty}(k+1)^m 2^{-k/2}\right)\omega(y)\mathrm{d}y \leq C\lambda^{-1}\sum_i\int_{B_i}|h_i(y)|\omega(y)\mathrm{d}y. \tag{3.12}$$

Next, let us consider I_{212}. Write $K(x,y,x_i) = (\Omega(x-y))/(|x-y|^{n-1}) - (\Omega(x-x_i))/(|x-x_i|^{n-1})$ for simplicity. Noting that for any $y\in B_i$, any $x\in\mathbb{R}^n\setminus 4B_i$ and t with $|x-x_i|+2r_i\leq t$, there has $|x-y|\leq|x-x_i|+r_i<t$, then by the cancellation condition of h_i, we have

$$\begin{aligned}
I_{212} &\leq C\lambda^{-1}\sum_i\int_{\mathbb{R}^n\setminus 4B_i}\prod_{j=1}^m\left|b_j(x)-(b_j)_{B_i}\right|\left(\int_{B_i}|K(x,y,x_i)||h_i(y)|\right.\\
&\quad\left.\times\left(\int_{|x-x_i|+2r_i}^{\infty}\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}}\mathrm{d}y\right)\omega(x)\mathrm{d}x\\
&\leq C\lambda^{-1}\sum_i\int_{\mathbb{R}^n\setminus 4B_i}\prod_{j=1}^m\left|b_j(x)-(b_j)_{B_i}\right|\left(\int_{B_i}\frac{|K(x,y,x_i)||h_i(y)|}{|x-x_i|}\mathrm{d}y\right)\omega(x)\mathrm{d}x\\
&\leq C\lambda^{-1}\sum_i\int_{B_i}|h_i(y)|\sum_{k=1}^{\infty}(2^k r_i)^{-1}\left(\int_{2^{k+1}B_i\setminus 2^k B_i}|K(x,y,x_i)|\right.\\
&\quad\left.\times\prod_{j=1}^m\left|b_j(x)-(b_j)_{B_i}\right|\omega(x)\mathrm{d}x\right)\mathrm{d}y
\end{aligned}$$

By the Hölder's inequality, Lemma 2.1 and Lemma 2.5, there has

$$\begin{aligned}
&\int_{2^{k+1}B_i\setminus 2^k B_i}|K(x,y,x_i)|\prod_{j=1}^m\left|b_j(x)-(b_j)_{B_i}\right|\omega(x)\mathrm{d}x\\
&\leq\left(\int_{2^{k+1}B_i\setminus 2^k B_i}|K(x,y,x_i)|^r\mathrm{d}x\right)^{1/r}\left(\int_{2^{k+1}B_i\setminus 2^k B_i}\prod_{j=1}^m\left|b_j(x)-(b_j)_{B_i}\right|^{r'}\omega^{r'}(x)\mathrm{d}x\right)^{1/r'}\\
&\leq C(k+1)^m 2^k r_i\left(2^{-k}+\int_{\frac{|y-x_i|}{2^{k+1}r_i}}^{\frac{|y-x_i|}{2^k r_i}}\frac{\omega_r(\delta)}{\delta}\mathrm{d}\delta\right)\inf_{y\in B_i}\omega(y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_{212} &\leq C\lambda^{-1}\sum_i\int_{B_i}|h_i(y)|\omega(y)\sum_{k=1}^{\infty}(k+1)^m\left(2^{-k}+\int_{\frac{|y-x_i|}{2^{k+1}r_i}}^{\frac{|y-x_i|}{2^k r_i}}\frac{\omega_r(\delta)}{\delta}\mathrm{d}\delta\right)\mathrm{d}y\\
&\leq C\lambda^{-1}\sum_i\int_{B_i}|h_i(y)|\omega(y)\left(\sum_{k=1}^{\infty}(k+1)^m 2^{-k}+\int_0^1\frac{\omega_r(\delta)}{\delta}\left(\log\frac{1}{\delta}\right)^m\mathrm{d}\delta\right)\mathrm{d}y
\end{aligned}$$

$$\leq C\lambda^{-1}\sum_i \int_{B_i} |h_i(y)|\omega(y)\mathrm{d}y. \tag{3.13}$$

Note that $h_i(y) = f(y) + f_{B_i}$ when $y \in B_i$, it follows from (3.6), (3.8), (3.12) and (3.13) that

$$I_{21} \leq C\lambda^{-1}\sum_i \int_{B_i} |h_i(y)|\omega(y)\mathrm{d}y \leq C\lambda^{-1}\sum_i \int_{B_i} (|f(y)| + |f_{B_i}|)\omega(y)\mathrm{d}y \leq C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}.$$

To estimate I_{23}, noting that $\Omega \in L^r(S^{n-1})$ for some $r > 1$ and $\omega^{r'} \in A_1$, using Lemma 2.2, (2.6), (2.7), Lemma 2.5, (2.5) and (3.4), we have

$$\begin{aligned}
I_{23} &\leq \omega\left(\left\{x \in \mathbb{R}^n : \mu_\Omega\left(\sum_i h_i \prod_{j=1}^m ((b_j)_{B_i} - b_j)\right)(x) > \frac{\lambda}{6}\right\}\right) \\
&\leq C\lambda^{-1}\int_{\mathbb{R}^n}\sum_i |h_i(x)|\omega(x)\prod_{j=1}^m \left|(b_j)_{B_i} - b_j(x)\right|\mathrm{d}x \\
&\leq C\lambda^{-1}\sum_i\left(\int_{B_i}|f(x)|\omega(x)\prod_{j=1}^m\left|(b_j)_{B_i} - b_j(x)\right|\mathrm{d}x + \int_{B_i}|f_{B_i}|\omega(x)\prod_{j=1}^m\left|(b_j)_{B_i} - b_j(x)\right|\mathrm{d}x\right) \\
&\leq C\lambda^{-1}\sum_i \omega(B_i)\|f\|_{L(\log L)^m, B_i, \omega}\prod_{j=1}^m \left\||b_j - (b_j)_{B_i}|\right\|_{\exp L, B_i, \omega} \\
&\quad + C\lambda^{-1}\sum_i \frac{1}{|B_i|}\int_{B_i}|f(y)|\mathrm{d}y\int_{B_i}\omega(x)\prod_{j=1}^m\left|(b_j)_{B_i} - b_j(x)\right|\mathrm{d}x \\
&\leq C\lambda^{-1}\sum_i\left(\omega(B_i)\|f\|_{L(\log L)^m, B_i, \omega} + \int_{B_i}|f(y)|\mathrm{d}y \inf_{y\in B_i}\omega(y)\right) \\
&\leq C\lambda^{-1}\sum_i\left(\omega(B_i)\inf\left\{\lambda + \frac{\lambda}{\omega(B_i)}\int_{B_i}\Phi_m\left(\frac{|f(y)|}{\lambda}\right)\omega(y)\mathrm{d}y\right\} + \int_{B_i}|f(y)|\omega(y)\mathrm{d}y\right) \\
&\leq C\sum_i\left(\omega(B_i) + \int_{B_i}\Phi_m\left(\frac{|f(y)|}{\lambda}\right)\omega(y)\mathrm{d}y\right) + C\lambda^{-1}\int_{\mathbb{R}^n}|f(y)|\omega(y)\mathrm{d}y \\
&\leq C\sum_i\left(\lambda^{-1}\int_{B_i}|f(y)|\omega(y)\mathrm{d}y + \int_{B_i}\frac{|f(y)|}{\lambda}\left(1 + \log^+\frac{|f(y)|}{\lambda}\right)^m\omega(y)\mathrm{d}y\right) \\
&\quad + C\lambda^{-1}\int_{\mathbb{R}^n}|f(y)|\omega(y)\mathrm{d}y \\
&\leq C\int_{\mathbb{R}^n}\frac{|f(y)|}{\lambda}\left(1 + \log^+\frac{|f(y)|}{\lambda}\right)^m\omega(y)\mathrm{d}y.
\end{aligned}$$

Now, let us turn to estimate for I_{22}. Using the Minkowski's inequality, we have

$$\begin{aligned}
I_{22} &= \omega\left(\left\{x \in \mathbb{R}^n \setminus E : \sum_i\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\left|(b(x) - b_{B_i})_\sigma\right|\mu_\Omega\left(h_i\left(b_{B_i} - b\right)_{\sigma'}\right)(x) > \frac{\lambda}{6}\right\}\right) \\
&\leq C\lambda^{-1}\sum_i\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\int_{\mathbb{R}^n\setminus 4B_i}\left|(b(x) - b_{B_i})_\sigma\right|\mu_\Omega\left(h_i\left(b_{B_i} - b\right)_{\sigma'}\right)(x)\omega(x)\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
&\le C\lambda^{-1}\sum_i\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\int_{\mathbb{R}^n\setminus 4B_i}\left|(b(x)-b_{B_i})_\sigma\right|\left(\int_0^{|x-x_i|+2r_i}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)h_i(y)}{|x-y|^{n-1}}\right.\right.\\
&\qquad\left.\left.\times(b_{B_i}-b(y))_{\sigma'}\,\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac12}\omega(x)\mathrm{d}x\\
&\quad+C\lambda^{-1}\sum_i\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\int_{\mathbb{R}^n\setminus 4B_i}\left|(b(x)-b_{B_i})_\sigma\right|\left(\int_{|x-x_i|+2r_i}^{\infty}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)h_i(y)}{|x-y|^{n-1}}\right.\right.\\
&\qquad\left.\left.\times(b_{B_i}-b(y))_{\sigma'}\,\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac12}\omega(x)\mathrm{d}x\\
&=C\lambda^{-1}\sum_i(I_{221}+I_{222}).
\end{aligned}$$

For I_{221} and I_{222}, similar to the estimates for I_{21} and I_{23}, we can get

$$\begin{aligned}
I_{221}&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}r_i^{1/2}\int_{\mathbb{R}^n\setminus 4B_i}\left|(b(x)-b_{B_i})_\sigma\right|\left(\int_{B_i}\frac{|\Omega(x-y)||h_i(y)|}{|x-y|^{n+1/2}}\right.\\
&\qquad\left.\times(b_{B_i}-b(y))_{\sigma'}\,\mathrm{d}y\right)\omega(x)\mathrm{d}x\\
&\le C\left(\omega(B_i)\inf\left\{\lambda+\frac{\lambda}{\omega(B_i)}\int_{B_i}\Phi_m\left(\frac{|f(y)|}{\lambda}\right)\omega(y)\mathrm{d}y\right\}+\int_{B_i}|f(y)|\omega(y)\mathrm{d}y\right).\\
I_{222}&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\int_{\mathbb{R}^n\setminus 4B_i}\left|(b(x)-b_{B_i})_\sigma\right|\left(\int_{B_i}\frac{|K(x,y,x_i)||h_i(y)|}{|x-x_i|}\right.\\
&\qquad\left.\times(b_{B_i}-b(y))_{\sigma'}\,\mathrm{d}y\right)\omega(x)\mathrm{d}x\\
&\le C\left(\omega(B_i)\inf\left\{\lambda+\frac{\lambda}{\omega(B_i)}\int_{B_i}\Phi_m\left(\frac{|f(y)|}{\lambda}\right)\omega(y)\mathrm{d}y\right\}+\int_{B_i}|f(y)|\omega(y)\mathrm{d}y\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
I_{22}&\le C\lambda^{-1}\sum_i\left(\omega(B_i)\inf\left\{\lambda+\frac{\lambda}{\omega(B_i)}\int_{B_i}\Phi_m\left(\frac{|f(y)|}{\lambda}\right)\omega(y)\mathrm{d}y\right\}+\int_{B_i}|f(y)|\omega(y)\mathrm{d}y\right)\\
&\le C\int_{\mathbb{R}^n}\frac{|f(y)|}{\lambda}\left(1+\log^+\frac{|f(y)|}{\lambda}\right)^m\omega(y)\mathrm{d}y.
\end{aligned}$$

From (3.7) and the above estimates for I_{21}, I_{22} and I_{23}, we have

$$I_2\le C\int_{\mathbb{R}^n}\frac{|f(y)|}{\lambda}\left(1+\log^+\frac{|f(y)|}{\lambda}\right)^m\omega(y)\mathrm{d}y.$$

This finishes the proof of Theorem 1.2.

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