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Cayley Graphs of Ideals in a Commutative Ring

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Abstract. Let R be a commutative ring. We associate a digraph to the ideals of R whose vertex set is the set of all nontrivial ideals of R and, for every two distinct vertices I and J, there is an arc from I to J, denoted by $I \to J$, whenever there exists a nontrivial ideal L such that $J = IL$. We call this graph the ideal digraph of R and denote it by $\overrightarrow{I\Gamma}(R)$. Also, for a semigroup H and a subset S of H, the Cayley graph $\mathrm{Cay}(H,S)$ of H relative to S is defined as the digraph with vertex set H and edge set $E(H,S)$ consisting of those ordered pairs (x,y) such that $y = sx$ for some $s \in S$. In fact the ideal digraph $\overrightarrow{I\Gamma}(R)$ is isomorphic to the Cayley graph $\mathrm{Cay}(\mathfrak{I}^*,\mathfrak{I}^*)$, where $\mathfrak{I}$ is the set of all ideals of R and $\mathfrak{I}^*$ consists of nontrivial ideals. The undirected ideal (simple) graph of R, denoted by $I\Gamma(R)$, has an edge joining I and J whenever either $J = IL$ or $I = JL$, for some nontrivial ideal L of R. In this paper, we study some basic properties of graphs $\overrightarrow{I\Gamma}(R)$ and $I\Gamma(R)$ such as connectivity, diameter, graph height, Wiener index and clique number. Moreover, we study the Hasse ideal digraph $\overrightarrow{H\Gamma}(R)$, which is a spanning subgraph of $\overrightarrow{I\Gamma}(R)$ such that for each two distinct vertices I and J, there is an arc from I to J in $\overrightarrow{H\Gamma}(R)$ whenever $I \to J$ in $\overrightarrow{I\Gamma}(R)$, and there is no vertex L such that $I \to L$ and $L \to J$ in $\overrightarrow{I\Gamma}(R)$.

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1. Introduction

The investigation of graphs associated to algebraic structures is very important. Many fundamental papers devoted to graphs assigned to a ring have appeared recently, see for example [1–4, 6, 9, 23, 27]. Most properties of a ring are connected to a behavior of its ideals. Besides, ideals play crucial roles in the study of ring constructions, see [15] and [16]. This is why it is useful and interesting to associate graphs to ideals of a ring, as for example in [11] and [24].

Among all types of graphs related to various algebraic structures, Cayley graphs have attracted serious attention in the literature, since they have many useful applications, see

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[17–19, 21, 22, 25, 26, 28] for examples of recent results and further references. Also, for a semigroup H and a subset S of H, the Cayley graph $\mathrm{Cay}(H,S)$ of H relative to S is defined as the digraph with vertex set H and edge set $E(H,S)$ consisting of those ordered pairs (x,y) such that $y = sx$ for some $s \in S$ (cf. [20]). By the ordered pair (x,y), we mean that $x \to y$. In this note we introduce a digraph which can be considered as a generalization of Cayley graphs of semigroups.

In this paper, we associate a digraph to the ideals of a commutative ring. Let $\mathfrak{I}$ be the set of all ideals of R and $\mathfrak{I}^*$ be the set of all nontrivial ideals of R. The ideal digraph of R, denoted by $\overrightarrow{I\Gamma}(R)$, is a digraph whose vertex set is the set $\mathfrak{I}^*$ and, for every two distinct vertices I and J, there is an arc from I to J whenever there exists a nontrivial ideal L such that $J = IL$. In fact the ideal digraph $\overrightarrow{I\Gamma}(R)$ is the Cayley graph $\mathrm{Cay}(\mathfrak{I}^*, \mathfrak{I}^*)$. The ideal graph of R, denoted by $I\Gamma(R)$, is a simple graph whose vertex set is the set of all nontrivial ideals of R and two distinct vertices I and J are adjacent if and only if either $J = IL$ or $I = JL$, for some nontrivial ideal L of R.

In sections two and three, we study some basic properties of graphs $\overrightarrow{I\Gamma}(R)$ and $I\Gamma(R)$ such as connectivity, diameter, graph height, Wiener index and clique number. In the last section, we study the Hasse ideal digraph $\overrightarrow{H\Gamma}(R)$, which is a spanning subgraph of $\overrightarrow{I\Gamma}(R)$ that, for each two distinct vertices I and J, there is an arc from I to J in $\overrightarrow{H\Gamma}(R)$ whenever $I \to J$ in $\overrightarrow{I\Gamma}(R)$, and there is no vertex L such that $I \to L$ and $L \to J$ in $\overrightarrow{I\Gamma}(R)$.

Throughout this paper, all rings are assumed to be commutative with non-zero identity. By $\mathrm{Max}(R)$, $U(R)$, $\mathrm{Nil}(R)$ and $Z(R)$, we denote the set of maximal ideals, unit elements, nilpotent elements and zero-divisors of R, respectively. A non-zero ideal I of R is said to be *minimal* if there is no nontrivial ideal of R properly contained in I. We denote the set of all minimal ideals of R by $\mathrm{Min}(R)$. Also, R_P stands for the localization of the ring R in the prime ideal P (cf. [14]).

Now we recall some definitions and notations on graphs. We use the standard terminology of graphs following [10]. Let $G = (V,E)$ be a graph, where V is the set of vertices and E is the set of edges. The graph $H = (V_0, E_0)$ is a *subgraph* of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called a *spanning subgraph* of G if its vertex set is V. The *distance* between two distinct vertices a and b in G, denoted by $\mathrm{d}(a,b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we set $\mathrm{d}(a,b) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(a,b) : a \text{ and } b \text{ are distinct vertices of } G\}$. The *girth* of G is the length of the shortest cycle in G, denoted by $\mathrm{gr}(G)$ ($\mathrm{gr}(G) := \infty$ if G has no cycles). Also, for two distinct vertices a and b in G, the notation $a - b$ means that a and b are adjacent. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. We say that G is *empty* if no two vertices of G are adjacent. For a vertex x in G, the *degree* of x is the number of vertices adjacent to x and it is denoted by $\deg(x)$. A vertex x is an *isolated vertex*, if $\deg(x) = 0$. A *clique* of a graph is a complete subgraph of it and the number of vertices in a largest clique of G is called the *clique number* of G and is denoted by $\omega(G)$. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. For a positive integer r, an *r-partite graph* is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. The *Wiener index* of G, denoted by $W(G)$, is the sum of the length of all shortest paths connecting every two distinct vertices of G. In other words, $W(G) = \sum_{a,b \in V(G)} d(a,b)$. (cf. [12], [13]). A graph is *Hamiltonian* if it contains a cycle which visits each vertex exactly once and also returns to the starting vertex. An *Eulerian graph* is a graph which has a path that visits each edge exactly once which starts and ends on the same vertex. By [10, Theorem 4.1], a connected non-empty graph is Eulerian if and only if the degree of each vertex is even. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [10, p. 153]).

A digraph Γ is called *weakly connected* if the undirected underlying simple graph obtained by replacing all directed edges of Γ with undirected edges is a connected graph. A digraph is called *strongly connected* if it contains a directed path from u to v and a directed path from v to u for every pair of distinct vertices u and v. The *indegree* of a vertex u is the number of arcs incident to u and is denoted by $\deg^-(u)$. Also, the *outdegree* of u is the number of arcs that u incidents to them and is denoted by $\deg^+(u)$. A *directed Hamilton cycle* of Γ is a directed cycle that includes every vertex of Γ. A *directed Euler path* is a directed path that traverses each arc of Γ exactly once with starts and ends on the same vertex. Also, for distinct vertices u and v in Γ, we use the notation $u \to v$ to show that there is an arc from u to v.

Throughout the paper, by a graph we mean a directed graph without multiple edges, but possibly with loops.

2. Graph height of ideal graph

In this section we first study some basic properties of the ideal graph $\overrightarrow{I\Gamma}(R)$. Next we introduce the concept of graph height for $\overrightarrow{I\Gamma}(R)$ and study the graph height of $\overrightarrow{I\Gamma}(R)$.

We begin this section with the following proposition which can be easily gained from the definitions.

Recall that an ideal I of R is *multiplicative* if, for every ideal J of R with $J \subset I$, there exists an ideal L of R such that $IL = J$.

Proposition 2.1.

(i) *Suppose that I and J are distinct vertices in $\overrightarrow{I\Gamma}(R)$. If $I \to J$, then $J \subset I$.*

(ii) *In the ideal graph $\overrightarrow{I\Gamma}(R)$ we have* $\mathrm{gr}(\overrightarrow{I\Gamma}(R)) = \infty$.

(iii) *The ideal graph $\overrightarrow{I\Gamma}(R)$ contains no directed Hamilton cycle or directed Euler path.*

(iv) *For every maximal ideal $\mathfrak{m}$ and every minimal ideal I of R, we have that*

$$\deg^-(M) = \deg^+(I) = 0.$$

(v) *I is a multiplicative ideal if and only if*

$$\deg^+(I) = |\{J \mid J \text{ is a nontrivial ideal of } R \text{ with } J \subset I\}|.$$

Note that, for each positive integer n, one can find a ring R such that $|V(\overrightarrow{I\Gamma}(R))| = 2^n - 2$. To do this, consider the ring $R = \mathbb{F}_1 \times \cdots \times \mathbb{F}_n$, where each $\mathbb{F}_i$ is a field, for $i = 1, \ldots, n$.

Let R be a principal ideal domain, PID briefly, and I be a nontrivial ideal of R. Then there exists an element $a \in R$ such that $I =< a >$. Since R is a unique factorization domain, UFD briefly, we have $a = up_1^{\alpha_1} \cdots p_n^{\alpha_n}$, where $u \in U(R)$, $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$ and $p_1, \ldots, p_n$ are irreducible elements in $R \setminus U(R)$. If J is an ideal of R and $I \subset J$, then $J =< p_1^{\beta_1} \ldots p_n^{\beta_n} >$ with $0 \leq \beta_i \leq \alpha_i$, for $i = 1, \ldots, n$. So, we have the following Proposition.

Proposition 2.2. *Let R be a PID and $a \in R$. If $a = up_1^{\alpha_1} \cdots p_n^{\alpha_n}$ is a decomposition of a, then we have*

$$\deg^-(< a >) = \prod_{i=1}^{n} (\alpha_i + 1) - 2.$$

Proposition 2.3. *Let R be an integral domain that is not a field. Then we have the following statements.*

(i) *If P is a nontrivial finitely generated prime ideal of R, then* $\deg^-(P) = 0$.

(ii) *For every minimal ideal I in $\mathfrak{I}^*$, we have that $J \to I$ in $\overrightarrow{I\Gamma}(R)$ if and only if $I \subset J$.*

Proof. (i) Suppose that P is nonzero prime ideal of an integral domain R. If the ideal J is adjacent to P, then there exists an ideal L of R such that $P = LJ$. Since P is prime and $P \subset J$, we have that $L \subseteq P = LJ \subseteq L$, and so $P = PJ$.

If P is finitely generated, then there exist $a_1, \ldots, a_n \in P$ such that $P =< a_1, \ldots, a_n >$. Since $P = PJ$, for $i = 1, 2, \ldots, n$, a_i has an expansion $a_i = \sum_{j=1}^{n} r_{ij} a_j$, for some $r_{ij} \in J$. This can be rewritten $\sum_{j=1}^{n} (\delta_{ij} - r_{ij}) a_j = 0$, for $1 \leq i \leq n$, where δ_{ij} is the kronecker delta. The coefficients of this system of linear equations can be viewed as a square matrix $(\delta_{ij} - r_{ij})$ of elements of R. Let b_{ij} denote its (i, j)th cofactor, and d its determinate. By multiplying the above equation through by b_{ik} and summing over i, we get $da_k = 0$ for all $1 \leq k \leq n$. Since R is an integral domain, we have that $d = 0$. On the other hand, it is easy to see that $d = 1 + j$ for some $j \in J$, and so $1 \in J$. This means that $J = R$ which is impossible.

(ii) Let I be a nonzero minimal ideal of R. Suppose that $I \subset J$. Then we have $I = IJ$. This implies that $J \to I$.

The converse statement is clear. ∎

Now, we introduce the concept of graph height of a graph.

Definition 2.1. *In a non-empty graph G, the maximum length of the paths in G is called graph height of G. We denote the graph height of the ideal graph $\overrightarrow{I\Gamma}(R)$, by $h_{\overrightarrow{\Gamma}}(R)$, and the graph height of the ideal graph $I\Gamma(R)$, by $h_{\Gamma}(R)$.*

Example 2.1.

(i) Let $R = \mathbb{Z}_n$ and $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, where p_i is a prime number and α_i is a natural number, for $i = 1, \ldots, r$. Then one can easily see that the path

$$(p_1) \to (p_1^2) \to \cdots \to (p_1^{\alpha_1}) \to (p_1^{\alpha_1} p_2) \to \cdots \to (p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{r-1}^{\alpha_{r-1}} p_r^{\alpha_r - 1})$$

has the maximum length among all paths in $\overrightarrow{I\Gamma}(R)$. Hence we have that $h_{\overrightarrow{\Gamma}}(R) = \sum_{i=1}^{r} \alpha_i - 1$.

(ii) Suppose that K is a field and $R = K[x]/(x^n)$, where $n \geq 2$. Clearly

$$(x) \to (x^2) \to \cdots \to (x^{n-1})$$

is one of the paths in $\overrightarrow{I(\Gamma(R))}$ of maximum length. Therefore $h_{\overrightarrow{\Gamma}}(R) = n - 2$.

Proposition 2.4. *Assume that $h_{\overrightarrow{\Gamma}}(R) = 0$. Then, for every non-zero proper ideal I of R, we have either $I^2 = I$ or $I^2 = 0$.*

Proof. Suppose that $h_{\Gamma}(R) = 0$ and I is a nonzero proper ideal of R. If $I^2 \neq I$ and $I^2 \neq 0$, then we have the path $I \to I^2$ in $\overrightarrow{I\Gamma}(R)$. This means that $h_{\overrightarrow{\Gamma}}(R) \geq 1$, which is impossible. ∎

We say that R is a *multiplicative ring* if all nontrivial ideals of R are multiplicative.

Proposition 2.5. *Let R be a multiplicative ring. Then $h_{\overrightarrow{\Gamma}}(R) < \infty$ if and only if R is Artinian.*

Proof. First suppose that $h_{\overrightarrow{\Gamma}}(R) < \infty$. Since R is multiplicative, every chain of ideals in R with respect to inclusion induces a path in $\overrightarrow{I\Gamma}(R)$. This implies that R is Artinian.

The converse statement is clear. ∎

Proposition 2.6. *Assume that $h_{\overrightarrow{\Gamma}}(R) < \infty$. If $a \in R \setminus \mathrm{Nil}(R)$, then there exists $r \in R$ such that $1 - ra \in Z(R)$.*

Proof. If $a \in U(R)$, then by setting $r := a^{-1}$ the result holds. Now suppose that a is a non-unit element in $R \setminus \mathrm{Nil}(R)$ and consider the path $(a) \to (a^2) \to \cdots$ in $\overrightarrow{I\Gamma}(R)$. Since $h_{\Gamma}(R) < \infty$, we have $(a^n) = (a^{n+1})$, for some positive integer n. Hence $a^n(1 - ra) = 0$, for some $r \in R$. Now, one can easily see that $1 - ra \in Z(R)$. ∎

The following corollaries immediately follow from Proposition 2.6.

Corollary 2.1. *If R is a ring with $h_{\overrightarrow{\Gamma}}(R) < \infty$, then we have that $R = Z(R) \cup U(R)$.*

Corollary 2.2. *For every integral domain R that is not a field, $h_{\overrightarrow{\Gamma}}(R) = \infty$.*

Proposition 2.7. *If $h_{\overrightarrow{\Gamma}}(R) < \infty$, then, for each nontrivial ideal I of R, we have either I is nilpotent or I^n is idempotent, for some positive integer n.*

Proof. Suppose that $h_{\overrightarrow{\Gamma}}(R) < \infty$ and that I is a nontrivial ideal of R. Consider the path $I \to I^2 \to \cdots$. Since $h_{\overrightarrow{\Gamma}}(R) < \infty$, for some positive integer n, we have $I^n = 0$ or $I^n = I^{n+r}$, for all positive integers r. So the result holds. ∎

We end this section with the following corollary.

Proposition 2.8. *Let R be a ring with finite number of maximal ideals. Then $h_{\overrightarrow{\Gamma}}(R) < \infty$ if and only if, for every maximal ideal $\mathfrak{m}$ of R, $h_{\overrightarrow{\Gamma}}(R_{\mathfrak{m}}) < \infty$.*

Proof. First suppose that $h_{\overrightarrow{\Gamma}}(R) < \infty$. Let $\mathfrak{m}$ be a maximal ideal of R such that $h_{\overrightarrow{\Gamma}}(R_{\mathfrak{m}}) = \infty$. So there exists an infinite path in $\overrightarrow{I\Gamma}(R_{\mathfrak{m}})$. This implies that $\overrightarrow{I\Gamma}(R)$ contains a path of length infinity, which is a contradiction.

Conversely, suppose that, for every maximal ideal $\mathfrak{m}$, $h_{\overrightarrow{\Gamma}}(R_{\mathfrak{m}}) < \infty$. Since the number of maximal ideals of R is finite, we have the following inequality

$$h_{\overrightarrow{\Gamma}}(R) = \max\{h_{\overrightarrow{\Gamma}}(R_{\mathfrak{m}}) \mid \mathfrak{m} \in \mathrm{Max}(R)\} < \infty. \quad ∎$$

3. Connectedness, clique number and the Wiener index of $I\Gamma(R)$

In this section we study some basic properties of the undirected ideal graph $I\Gamma(R)$. We begin this section with the following proposition.

Proposition 3.1. *If R is an integral domain that is not a field, then $I\Gamma(R)$ is connected with*

$$\mathrm{diam}(I\Gamma(R)) \leq 2.$$

In particular $I\Gamma(R)$ is complete if and only if R is a multiplicative ring and the set of nontrivial ideals of R is totally ordered with respect to the inclusion.

Proof. Suppose that I and J are two distinct nontrivial ideals of R. Since R is an integral domain, we have $IJ \neq 0$. Therefore we have the path $I - IJ - J$ in $I\Gamma(R)$. So $\mathrm{diam}(I\Gamma(R)) \leq 2$. Moreover, one can easily see that $I\Gamma(R)$ is complete if and only if R is multiplicative and the set of nontrivial ideals of R is totally ordered with respect to the inclusion. ∎

Corollary 3.1. *If the ideal graph $I\Gamma(R)$ is complete, then R is a local ring.*

Note that if R is not an integral domain, then we may have that the graph $I\Gamma(R)$ is connected. For example, $\mathbb{Z}_4$ is not an integral domain and $I\Gamma(\mathbb{Z}_4)$ is connected.

In the following theorem, we find a lower bound for the clique number of $I\Gamma(R)$.

Theorem 3.1. *In the ideal graph $I\Gamma(R)$ we have the following inequality.*

$$\omega(I\Gamma(R)) \geq h_{\overrightarrow{\Gamma}}(R) + 1$$

Proof. Let $I_1 \to I_2 \to \cdots \to I_{n+1}$ be a path of length n in $\overrightarrow{I\Gamma}(R)$, where $n = h_{\overrightarrow{\Gamma}}(R)$. Then one can easily see that, for each i with $1 \leq i \leq n-1$, we have $I_i \to I_{i+2}$. This implies that, for each i, j with $1 \leq i < j \leq n$, we have $I_i \to I_j$. Hence the vertices in the set $\{I_1, I_2, \cdots, I_{n+1}\}$ form a clique for $I\Gamma(R)$. Therefore $\omega(I\Gamma(R)) \geq h_{\overrightarrow{\Gamma}}(R) + 1$. ∎

Example 3.1. Suppose that $R = K[x]/(x^n)$, where K is a field and $n \geq 2$. Then, in view of Theorem 3.1 in conjunction with Example 2.1(ii), we have $\omega(I\Gamma(R)) \geq n - 1$.

The following corollary immediately follows from Theorem 3.1.

Corollary 3.2. *If $I\Gamma(R)$ is planar, then we have that $h_{\overrightarrow{\Gamma}}(R) \leq 3$.*

Now we have the question that if $h_{\overrightarrow{\Gamma}}(R) \leq 3$, then is it true that the graph $I\Gamma(R)$ is planar?

Recall that an *independent set* of an undirected graph G is a subset of the vertices of G such that no two vertices in the subset represent an edge of G. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent set.

Now suppose that the vertices I and J are adjacent in $I\Gamma(R)$. Then $I \subset J$ or $J \subset I$. Thus the set of all maximal ideals and the set of all minimal ideals form independent sets for $I\Gamma(R)$. Therefore we have the following proposition.

Proposition 3.2. *In the ideal graph $I\Gamma(R)$ we have the following inequality.*

$$\alpha(I\Gamma(R)) \geq \max\{|\mathrm{Max}(R)|, |\mathrm{Min}(R)|\}$$

Proposition 3.3. *If all ideals of R are multiplicative and prime, then $I\Gamma(R)$ is complete.*

Proof. Suppose that P and Q are two nontrivial ideals of R. Clearly PQ is a prime ideal. Hence we have $P \subseteq PQ$ or $Q \subseteq PQ$. Thus $P \subseteq Q$ or $Q \subseteq P$. Therefore, by Proposition 3.1, $I\Gamma(R)$ is complete. ∎

Recall that an undirected graph is called a *forest* if it contains no cycle.

Proposition 3.4. *Assume that $I\Gamma(R)$ is a forest. Then we have that $R = Z(R) \cup U(R)$.*

Proof. Suppose that $a \in R$ and consider the set of ideals $A = \{< a^i >| \, i \geq 1\}$ of R. If $|A| \geq 3$, then one can easily find a triangle in $I\Gamma(R)$ which is impossible. So we have that $|A| \leq 2$. Thus there exists $r \in R$ such that $a^i = ra^{i+1}$, and so $a^i(1 - ra) = 0$. Suppose that $a \notin U(R)$. Hence $1 - ra \neq 0$, and it is not hard to see that a is a zero-divisor. ∎

In the following theorem, we study the cases where $I\Gamma(\mathbb{Z}_n)$ is Eulerian.

Theorem 3.2. *Assume that $n = p_1^{r_1} \dots p_m^{r_m}$, where p_i's are prime numbers, $m \geq 1$ and $r_i > 0$, for $i = 1, \dots, m$. Then $I\Gamma(\mathbb{Z}_n)$ is Eulerian if and only if, for every integer $p_1^{t_1} p_2^{t_2} \dots p_s^{t_s}$, with $1 \leq s \leq m$ and $1 \leq t_i \leq r_i$, one of the following conditions hold.*

- (i) *$m \geq 3$. In this situation one of the following holds:*
  - (a) *For all $1 \leq i \leq m$ either $r_i = 1$ or r_i is even.*
  - (b) *If there exists an odd r_i with $s + 1 \leq i \leq m$, then there exists an odd t_i such that $1 \leq i \leq s$.*
  - (c) *If there exist some odd integers r_i and t_k, with $1 \leq i, k \leq s$, and for all j with $s + 1 \leq j \leq m$, r_j is even, then there exists $1 \leq k' \leq s$ such that $r_{k'}$ is odd and $t_{k'}$ is even, or $r_{k'}$ is even and $t_{k'}$ is odd.*
- (ii) *$m = 2$ and, r_1 and r_2 are even.*
- (iii) *$m = 1$ and, r_1 is an even number with $r_1 \geq 4$.*

Proof. Let I be a nontrivial ideal of $\mathbb{Z}_n$. Without loss of generality, we may assume that $I = < p_1^{t_1} p_2^{t_2} \dots p_s^{t_s} >$, where $1 \leq s \leq m$ and $1 \leq t_i \leq r_i$. One can easily check that the degree of I satisfy in the following equality.

$$\deg(I) = \prod_{i=1}^{s}(r_i - t_i + 1) \prod_{i=s+1}^{m} (r_i + 1) - 2 + \prod_{i=1}^{s}(t_i + 1) - 2$$

Now one can easily check that $\deg(I)$ is an even number if and only if one of the conditions (i), (ii) or (iii) happens. Hence the result holds. ∎

In the rest of this section, we calculate the Wiener index $W(I\Gamma(R))$, where R is a finite direct product of fields. To this end, the following lemma is useful.

Lemma 3.1. *Let $R = \mathbb{F}_1 \times \dots \times \mathbb{F}_n$, where $\mathbb{F}_i$ is a field for $i = 1, \dots, n$. Then there is a classification of $|V(I\Gamma(R))|$ to $n - 1$ classes such that, for each ideal I in the i^{th} class, $\deg^-(I) = 2^{n-i} - 2$ and $\deg^+(I) = 2^i - 2$.*

Proof. It is well known that every ideal of R has the form $\bar{\mathbb{F}}_1 \times \dots \times \bar{\mathbb{F}}_n$, where $\bar{\mathbb{F}}_i = 0$ or $\mathbb{F}_i$, for all $i = 1, \dots, n$. Let V_i be the set of all nontrivial ideals of R with i nonzero components. Now suppose that $I = \bar{\mathbb{F}}_1 \times \dots \times \bar{\mathbb{F}}_n$ is an arbitrary element in V_i. Then it is clear that, for each ideals J and L with $0 \neq J \subset I$ and $I \subset L \neq R$, I is adjacent to J and that L is adjacent to I. Thus $\deg^-(I) = 2^{n-i} - 2$ and $\deg^+(I) = 2^i - 2$. ∎

Theorem 3.3. *Let $R = F_1 \times \dots \times F_n$, where F_i is a field for $i = 1, \dots, n$. If $n \geq 3$, then*

$$W(I\Gamma(R)) = 2(4^n - 3^n) + 2^n - 4.$$

Proof. According to Lemma 3.1, we classified the set of vertices and we have $V(I\Gamma(R)) = \bigcup_{i=1}^{n-1} V_i$ such that $|V_i| = \binom{n}{r}$. Suppose that I is an arbitrary element in V_r. Without loss of generality, we may assume that $I = F_1 \times \dots \times F_r \times 0 \times \dots \times 0$. In view of Lemma 3.1, $\deg(I) = \deg^-(I) + \deg^+(I) = 2^{n-r} + 2^r - 4$. Now suppose that $J \in V(I\Gamma(R))$ such that I and J are not adjacent. So we have the following two cases:

Case 1. $J = 0 \times \cdots \times 0 \times F_{r+1} \times \cdots \times F_n$. Hence there is no vertex L in $I\Gamma(R)$ which is adjacent to I and J. Thus $d(I,J) \geq 3$. On the other hand, if $L_1 := 0 \times \cdots \times 0 \times F_r \times \cdots \times F_n$ and $L_2 := 0 \times \cdots \times 0 \times F_r \times 0 \times \cdots \times 0$, then there exists a path $I - L_2 - L_1 - J$ in $I\Gamma(R)$. This implies that $d(I,J) = 3$.
Case 2. $J \neq 0 \times \cdots \times 0 \times F_{r+1} \times \cdots \times F_n$. Since $I \cap J \neq 0$, there is a path $I - I \cap J - J$ in $I\Gamma(R)$. This means that $d(I,J) = 2$. Hence

$$\begin{aligned} W(I) &= 2^{n-r} + 2^r - 4 + 3 + 2\,(2^n - 2 - (2^{n-r} + 2^r - 4) - 1) \\ &= 2^{n+1} - 2^{n-r} - 2^r + 3. \end{aligned}$$

Thus

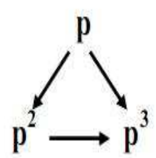
$$\begin{aligned} W(I\Gamma(R)) &= \sum_{r=1}^{n-1} \binom{n}{r} (2^{n+1} - 2^{n-r} - 2^r + 3) \\ &= (2^{n+1} + 3) \sum_{r=1}^{n-1} \binom{n}{r} - \sum_{r=1}^{n-1} \binom{n}{r} (2^{n-r} + 2^r) \\ &= (2^{n+1} + 3)(2^n - 2) - 2(3^n - 2^n - 1) \\ &= 2(4^n - 3^n) + 2^n - 4. \end{aligned}$$

∎

4. Hasse ideal graph

The *Hasse ideal graph*, which is denoted by $\overrightarrow{H\Gamma}(R)$, is a spanning subgraph of $\overrightarrow{I\Gamma}(R)$ that, for each two distinct vertices I and J, there is an arc from I to J in $\overrightarrow{H\Gamma}(R)$ whenever $I \to J$ in $\overrightarrow{I\Gamma}(R)$, and there is no vertex L such that $I \to L$ and $L \to J$ in $\overrightarrow{I\Gamma}(R)$. We denote the undirected Hasse ideal graph by $H\Gamma(R)$.

Example 4.1. In the following figures, we present the graphs $\overrightarrow{H\Gamma}(\mathbb{Z}_{p^r})$ and $\overrightarrow{I\Gamma}(\mathbb{Z}_{p^r})$, where p is a prime number and $r = 4, 5$.



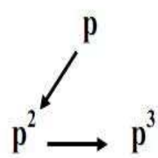
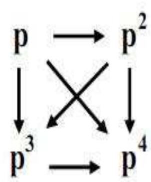


Figure 1. $\overrightarrow{I\Gamma}(\mathbb{Z}_{p^4})$ Figure 2. $\overrightarrow{H\Gamma}(\mathbb{Z}_{p^4})$



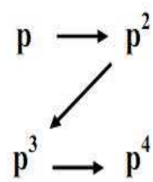


Figure 3. $\overrightarrow{I\Gamma}(\mathbb{Z}_{p^5})$ Figure 4. $\overrightarrow{H\Gamma}(\mathbb{Z}_{p^5})$

Proposition 4.1. *In the Hasse ideal graph $H\Gamma(R)$ we have* $\mathrm{gr}(H\Gamma(R)) \geq 4$.

Proof. Suppose to the contrary that $H\Gamma(R)$ contains a cycle $I - J - K - I$. Clearly, if there exists a directed path $I \to J \to K$ in $\overrightarrow{H\Gamma}(R)$, then we have $I \to K$. This means that I and K are not adjacent in $H\Gamma(R)$, which is impossible. Hence we have $I \to J \leftarrow K$ in $\overrightarrow{H\Gamma}(R)$. In this situation we have that either I and J, or J and K are not adjacent in $H\Gamma(R)$, which is a contradiction. Thus we have $\mathrm{gr}(H\Gamma(R)) \geq 4$. ∎

Theorem 4.1. *The Hasse ideal graph $H\Gamma(\mathbb{Z}_n)$ is a forest if and only if $n = pq,\ p^2q$ or p^r, where p and q are prime numbers and $r \geq 1$.*

Proof. If $n = pq,\ p^2q$ or p^r, then one can easily see that $H\Gamma(\mathbb{Z}_n)$ has no cycle. Thus $H\Gamma(Z_n)$ is a forest.

Conversely, suppose that $H\Gamma(\mathbb{Z}_n)$ is a forest. If $n \notin \{pq, p^2q, p^r\}$, then we have the following three cases:

Case 1. p^rq divides n, where $r \geq 3$. In this case, we have the cycle $p - p^2 - p^2q - pq - p$ in $H\Gamma(\mathbb{Z}_n)$.

Case 2. p^rq^s divides n, where $r, s \geq 2$. So we have the cycle $p - p^2 - p^2q - pq - p$.

Case 3. pqt divides n, where t is a prime number distinct from p and q. In this case one can easily find the cycle $t - tq - q - pq - p - tp - t$ in $H\Gamma(\mathbb{Z}_n)$.

Thus if $n \notin \{pq, p^2q, p^r\}$, then $H\Gamma(\mathbb{Z}_n)$ is not a forest, which is impossible. Therefore we have that $n = pq,\ p^2q$ or p^r. ∎

In the following, we study the Hasse ideal graph $\overrightarrow{H\Gamma}(\mathbb{Z}_n)$. For $a \in \mathbb{Z}_n$, we use a to denote the ideal generated by a.

Proposition 4.2. *Put $R := \mathbb{Z}_n$. If $n = p^2$, for some prime number p, then $H\Gamma(R) = K_1$. Otherwise, $H\Gamma(R)$ is a bipartite graph.*

Proof. Suppose that $n = p_1 \cdots p_r$, where $r \geq 2$ and p_i's are (not necessarily distinct) prime numbers, for $i = 1, \ldots, r$. If $n = p^2$, then clearly $H\Gamma(Z_n) = K_1$. Now assume that $n \neq p^2$. We set

$$V_1 := \{(p_{i_1} \ldots p_{i_t}) \mid 1 \leq i_1, \ldots, i_t \leq r,\ t \text{ is odd}\}$$

and

$$V_2 := \{(p_{i_1} \ldots p_{i_t}) \mid 1 \leq i_1, \ldots, i_t \leq r,\ t \text{ is even}\}.$$

Now one can easily check that $H(\Gamma(\mathbb{Z}_n))$ is a bipartite graph with parts V_1 and V_2. ∎

Corollary 4.1. *Suppose that $R = \mathbb{Z}_n$ and n is not a prime number. Then we have the following statements.*

(i) $\mathrm{gr}(H\Gamma(R))$ *is an even number or it is infinity.*
(ii) *If $n = p^2$, then $\chi(H\Gamma(R)) = 1 = \omega(H\Gamma(R))$.*
(iii) *If $n \neq p^2$, then $\chi(H\Gamma(R)) = 2 = \omega(H\Gamma(R))$.*

Example 4.2. Let $R = \mathbb{Z}_n$ and $n = p^rq^s$, where p, q are two prime numbers and $r, s \geq 2$. We present $H\Gamma(Z_{p^rq^s})$ in Figure 5.

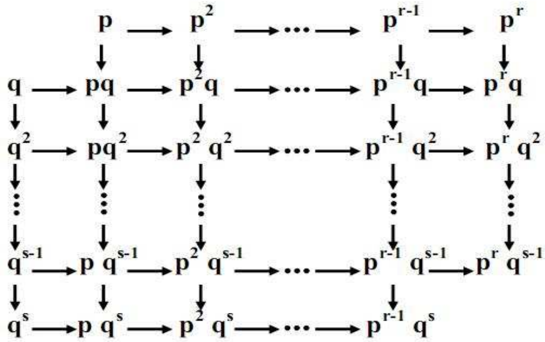


Figure 5. $\overrightarrow{H\Gamma}(Z_{p^r q^s})$

In the rest of this section, we investigate cut vertices in $H\Gamma(R)$. Recall that a vertex x is a cut vertex in a graph G, whenever the number of connected components in $G \setminus \{x\}$ is more than the number of connected components in G.

Proposition 4.3. *Let R be a valuation ring. If $h_\Gamma(R) \geq 2$, then $H\Gamma(R)$ contains some cut vertices.*

Proof. Since ideals of a valuation ring with the inclusion relation form a chain, the graph $H\Gamma(R)$ is a union of paths. It is clear that every path with length more than 2 has some cut vertices. Therefore, since $h_\Gamma(R) \geq 2$, the result holds. ∎

Definition 4.1. *A prime ideal P of a ring R is said to be a divided prime ideal if $P \subset < x >$ for every $x \in R \setminus P$ (cf. [5, 7, 8]).*

Proposition 4.4. *Let R be a multiplicative ring and I be a divided prime ideal which is not maximal or minimal. Then I is a cut vertex in $H\Gamma(R)$.*

Proof. Since I is not maximal or minimal and R is multiplicative, there are ideals J_1 and J_2 such that $J_1 \to I \to J_2$. Now suppose that I is not a cut vertex. Then there is an ideal L such that one of the following cases holds:

(i) $J_2 \leftarrow L \leftarrow J_1$
(ii) $J_2 \leftarrow L, J_1 \leftarrow L$
(iii) $L \leftarrow J_2, L \leftarrow J_1$

By using our hypothesis, I is a divided ideal. Hence $I \subset L$ or $I \supset L$. Without loss of generality, we may assume that $I \subset L$, and so $L \to I$. On the other hand $I \to J_2$. Thus in the Hasse ideal graph, L can not be adjacent to J_2. Therefore the situations (i) and (ii) provide contradictions. Now in the case (iii), we have that $J_1 \to L$. Also, since $L \to I$, J_1 can not be adjacent to I. On the other hand $J_1 \to I$ which is the required contradiction. This means that I is a cut vertex. ∎

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