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Coefficients of Bi-Univalent Functions with Positive Real Part Derivatives

[1]JAY M. JAHANGIRI, [2]SAMANEH G. HAMIDI AND [3]SUZEINI ABD. HALIM
[1]Department of Mathematical Sciences, Kent State University, Burton, Ohio 44021-9500, U. S. A.
[2,3]Institute of Mathematical Sciences, Faculty of Science, University of Malaya, 50603 Kuala Lumpur, Malaysia
[1]jjahangi@kent.edu, [2]s.hamidi_61@yahoo.com, [3]suzeini@um.edu.my

Abstract. We consider analytic bi-univalent functions whose derivatives have positive real part on the unit disk. Using the Faber polynomial expansions, we obtain upper bounds for the coefficients of such functions. In certain cases, our estimates improve some of those existing coefficient bounds.

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1. Introduction

Let $\mathscr{A}$ denote the class of functions f which are analytic on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ and normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathscr{P}$ be the class of functions $\phi(z) = 1 + \sum_{n=1}^{\infty} \phi_n z^n$ that are analytic on $\mathbb{D}$ and satisfy the condition $Re(\phi(z)) > 0$ on $\mathbb{D}$. By the Caratheodory Lemma (e.g. see [8, p. 41]) we have $|\phi_n| \leq 2$.

It is well known that every univalent function $f \in \mathscr{A}$ has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \qquad (z \in \mathbb{D}),$$

and

$$f(f^{-1}(w)) = w, \qquad (|w| < 1/4),$$

according to Koebe One Quarter Theorem, [8, p. 31].

A function $f \in \mathscr{A}$ is said to be bi-univalent on $\mathbb{D}$ if f and its inverse $g = f^{-1}$ are both univalent on $\mathbb{D}$.

For $0 \leq \alpha < 1$ and $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$ we let $R(p;\alpha)$ be the class of functions $f \in \mathscr{A}$ so that f and its inverse map $g = f^{-1}$ satisfy the following

$$\mathrm{Re}(f'(z))^p > \alpha; \qquad z \in \mathbb{D}, \tag{1.2}$$

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and

$$\mathrm{Re}(g'(w))^p > \alpha; \qquad w \in \mathbb{D}. \tag{1.3}$$

The functions $f \in \mathscr{A}$ whose derivative $f' \in \mathscr{P}$ are known to be univalent and close-to-convex on $\mathbb{D}$, [8, p. 47].

Finding bounds for the coefficients of classes of bi-univalent functions dates back to 1967 (see Lewin [13]). But the interest on the bounds for the coefficients of classes of bi-univalent functions picked up by the publications [5, 6, 11, 15]. Srivastava *et al.* [15] investigated the bounds for the coefficients $|a_2|$ and $|a_3|$ of the bi-univalent function $f \in \mathscr{A}$ if their derivatives are subordinate to some function in $\mathscr{P}$. Ali *et al.* [5] remarked that for the bi-univalent functions, finding the bounds for $|a_n|$ when $n \geq 4$ is an open problem. Here in this paper we use Faber polynomial coefficient techniques to provide bounds for the general coefficients $|a_n|$ under certain conditions and also obtain estimates for the first two coefficients $|a_2|$ and $|a_3|$ of the bi-univalent functions $f \in R(p;\alpha)$. The bounds provided in this article prove to be better than those estimates determined by Srivastava *et al.* [15].

2. Main results

Using the Faber polynomial expansion of functions $f \in \mathscr{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as, [3, Theorem 6.1, p. 209],

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) w^n, \tag{2.1}$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2\right] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4\right] + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$ [4]. In particular, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned} \frac{1}{2} K_1^{-2} &= -a_2, \\ \frac{1}{3} K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4} K_3^{-4} &= -(5a_2^3 - 5a_2 a_3 + a_4). \end{aligned}$$

In general, an expansion of K_n^p is as [3, p. 183],

$$K_n^p = p a_n + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + \ldots + \frac{p!}{(p-n))!n!} D_n^n, \tag{2.2}$$

where $D_n^p = D_n^p(a_2, a_3, \ldots)$ and by [16] or [2],

$$D_n^m\ (a_1, a_2, \ldots, a_n) = \sum_{m=1}^{\infty} \frac{m!(a_1)^{\mu_1} \ldots (a_n)^{\mu_n}}{\mu_1! \ldots \mu_n!},$$

where $a_1 = 1$ and the sum is taken over all nonnegative integers $\mu_1, \ldots, \mu_n$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + \ldots + \mu_n = m, \\ \mu_1 + 2\mu_2 + \ldots + n\mu_n = n. \end{cases}$$

Evidently: $D_n^n(a_1, a_2, \ldots, a_n) = a_1^n$, [1].

Gong [9] and Schiffer [14] demonstrated the significance of the Faber polynomials [7] in mathematical sciences, especially in geometric function theory. The recent publications of [1–4, 15] dealing with the Taylor expansions of inverse function $g = f^{-1}$, beautifully fits our case for the bi-univalent functions. As a result, we are able to state and prove the following:

Theorem 2.1. *For $0 \leq \alpha < 1$ and $p \in \mathbb{N}$ let $f \in R(p;\alpha)$ be given by* (1.1). *If $a_k = 0$ for $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{2(1-\alpha)}{np}; \qquad n \geq 3.$$

Proof. The main crux of the proof relies on the observation that if $\phi(z) = 1 + \sum_{n=1}^{\infty} \phi_n z^n$ is analytic in $\mathbb{D}$ and $p \in \mathbb{N}$ then

$$(\phi(z))^p = 1 + \sum_{n=1}^{\infty} K_n^p(\phi_1, \phi_2, \ldots, \phi_n) z^n$$

(see [1, Equation (4), p. 449]). If f is of the form (1.1), then

$$f'(z) = 1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} z^n.$$

Therefore, for $(f'(z))^p$, we have (see [1, Equation (4)])

$$(f'(z))^p = 1 + \sum_{n=1}^{\infty} K_n^p(2a_2, 3a_3, \ldots, (n+1)a_{n+1}) z^n. \tag{2.3}$$

Similarly, for $g = f^{-1}$ given by (2.1) we have

$$g'(w) = 1 + \sum_{n=2}^{\infty} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) w^{n-1} = 1 + \sum_{n=1}^{\infty} b_n w^n. \tag{2.4}$$

Consequently, for $(g'(w))^p$ we have

$$(g'(w))^p = 1 + \sum_{n=1}^{\infty} K_n^p(b_1, b_2, \ldots, b_n) w^n. \tag{2.5}$$

On the other hand, the inequalities (1.2) and (1.3) imply the existence of two positive real part functions $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathscr{P}$ and $q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \in \mathscr{P}$ so that

$$\begin{aligned} (f'(z))^p &= \alpha + (1-\alpha) p(z) \\ &= 1 + (1-\alpha) c_1 z + (1-\alpha) c_2 z^2 + \ldots \end{aligned} \tag{2.6}$$

and

$$(g'(w))^p = \alpha + (1-\alpha) q(w)$$

$$= 1 + (1-\alpha)d_1 w + (1-\alpha)d_2 w^2 + \dots. \tag{2.7}$$

Now, comparing the corresponding coefficients of the Equations (2.3) and (2.6) yield

$$K_{n-1}^{p}(2a_2, 3a_3, \dots, na_n) = (1-\alpha)c_{n-1}. \tag{2.8}$$

Similarly, from (2.5) and (2.7) we obtain

$$K_{n-1}^{p}(b_1, b_2, \dots, b_{n-1}) = (1-\alpha)d_{n-1}. \tag{2.9}$$

If $a_k = 0$ for $2 \le k \le n-1$, then the Equations (2.8) and (2.9) in conjunction with the relation (2.2) yield

$$npa_n = (1-\alpha)c_{n-1},$$

and

$$pb_{n-1} = -npa_n = (1-\alpha)d_{n-1}.$$

Taking the absolute values of either of the above two equations and using the Caratheodory Lemma we obtain

$$|a_n| \le \frac{(1-\alpha)|c_{n-1}|}{np} = \frac{(1-\alpha)|d_{n-1}|}{np} \le \frac{2(1-\alpha)}{np}, \qquad n \ge 3.$$

∎

Relaxing the coefficient restrictions imposed in Theorem 2.1, we see the unpredictable behavior of the early coefficients of functions f in $R(p;\alpha)$ illustrated in the following two theorems.

Theorem 2.2. *For* $0 \le \alpha < 1$ *and* $p \ge 2$ *let* $f \in R(p;\alpha)$ *be given by* (1.1). *Then*

(i) $|a_2| \le \frac{1-\alpha}{p}$,

(ii) $|a_3 - a_2^2| \le \frac{2(1-\alpha)}{3p}$.

Proof. Substituting $n = 2$ in Equations (2.8) and (2.9), we obtain $2pa_2 = (1-\alpha)c_1$ and $-2pa_2 = (1-\alpha)d_1$. From either one of the two equations, it follows that

$$|a_2| = \frac{(1-\alpha)|c_1|}{2p} = \frac{(1-\alpha)|d_1|}{2p} \le \frac{1-\alpha}{p}.$$

Next, from Equations (2.8), (2.9) and (2.2) for $n = 3$, we obtain

$$2p(p-1)a_2^2 + 3pa_3 = (1-\alpha)c_2, \tag{2.10}$$

and

$$\frac{p(p-1)}{2}b_1^2 + pb_2 = 2p(p+2)a_2^2 - 3pa_3 = (1-\alpha)d_2. \tag{2.11}$$

Subtracting (2.11) from (2.10), we deduce

$$6p(a_3 - a_2^2) = (1-\alpha)(c_2 - d_2).$$

By taking absolute values of both sides and applying the Caratheodory Lemma, we obtain

$$|a_3 - a_2^2| \le \frac{2(1-\alpha)}{3p}.$$

∎

Theorem 2.3. *For* $0 \le \alpha < 1$ *let* $f \in R(1;\alpha)$ *be given by* (1.1). *Then*

(i)

$$|a_2| \le \begin{cases} \sqrt{\frac{2(1-\alpha)}{3}}, & 0 \le \alpha < \frac{1}{3}; \\ (1-\alpha), & \frac{1}{3} \le \alpha < 1, \end{cases}$$

(ii)

$$|a_3| \leq \begin{cases} \frac{4}{3}(1-\alpha), & 0 \leq \alpha < \frac{1}{3}, \\ \frac{1}{3}(1-\alpha)(5-3\alpha), & \frac{1}{3} \leq \alpha < \frac{2}{3}, \\ \frac{2}{3}(1-\alpha-3|a_2|^2), & \frac{2}{3} \leq \alpha < 1. \end{cases}$$

(iii)

$$|a_3 - a_2^2| \leq \frac{2}{3}(1-\alpha) - |a_2|^2 \quad \textit{if} \qquad \frac{1}{3} \leq \alpha < 1.$$

Proof. To verify the estimate for $|a_2|$, it is sufficient to substitute $n = 2$ and $n = 3$ in Equations (2.8) and (2.9) with $p = 1$, which respectively yield

$$\begin{cases} 2a_2 = (1-\alpha)c_1, \\ -2a_2 = (1-\alpha)d_1, \end{cases} \tag{2.12}$$

and

$$\begin{cases} 3a_3 = (1-\alpha)c_2, \\ 3(2a_2^2 - a_3) = (1-\alpha)d_2. \end{cases} \tag{2.13}$$

From either one of the relations in (2.12) we obtain

$$|a_2| = \frac{(1-\alpha)|c_1|}{2} = \frac{(1-\alpha)|d_1|}{2} \leq (1-\alpha). \tag{2.14}$$

On the other hand, adding the two relations in (2.13) gives

$$6a_2^2 = (1-\alpha)(c_2 + d_2)$$

or

$$|a_2| = \sqrt{\frac{(1-\alpha)|c_2 + d_2|}{6}} \leq \sqrt{\frac{2(1-\alpha)}{3}}. \tag{2.15}$$

We note that for $0 \leq \alpha < 1/3$,

$$\sqrt{\frac{2(1-\alpha)}{3}} < (1-\alpha).$$

Next, subtracting the two relations in (2.13) yields

$$6a_3 = (1-\alpha)(c_2 - d_2) + 3(2a_2^2)$$

or

$$6|a_3| \leq (1-\alpha)(|c_2| + |d_2|) + 6|a_2|^2. \tag{2.16}$$

Using the Caratheodory Lemma and the estimate (2.15) for $0 \leq \alpha < 1/3$, from (2.16) we obtain

$$|a_3| \leq \frac{1}{6}(1-\alpha)(2+2) + \left(\sqrt{\frac{2(1-\alpha)}{3}}\right)^2 = \frac{4(1-\alpha)}{3}.$$

Using the Caratheodory Lemma and the estimate (2.14) for $\alpha \geq \frac{1}{3}$, from (2.16) we obtain

$$|a_3| \leq \frac{1}{6}(1-\alpha)(2+2) + (1-\alpha)^2 = \frac{1}{3}(1-\alpha)(5-3\alpha).$$

Now, the second equation in (2.13) can be rewritten as

$$3a_3 = 6a_2^2 - (1-\alpha)d_2,$$

which upon substitution of $a_2 = -\frac{1-\alpha}{2}d_1$ we obtain

$$3a_3 = \frac{3}{2}(1-\alpha)^2 d_1^2 - (1-\alpha)d_2 = -(1-\alpha)\left[d_2 - \frac{3}{2}(1-\alpha)d_1^2\right].$$

Taking the absolute values, we obtain

$$3|a_3| \le (1-\alpha)\left|d_2 - \frac{3}{2}(1-\alpha)d_1^2\right|.$$

Applying the fact that $|d_2 + \mu d_1^2| \le 2 + \mu |d_1|^2$ if $\mu \ge -\frac{1}{2}$, which is due to the first author [12], and upon noticing that $-\frac{3}{2}(1-\alpha) \ge -\frac{1}{2}$ for $\alpha \ge \frac{2}{3}$ we obtain

$$3|a_3| \le (1-\alpha)\left[2 - \frac{3}{2}(1-\alpha)|d_1|^2\right].$$

Now upon re-substitution of $a_2 = -\frac{1-\alpha}{2}d_1$ we obtain

$$3|a_3| \le (1-\alpha)\left[2 - 6\frac{|a_2|^2}{1-\alpha}\right] = 2\left(1-\alpha-3|a_2|^2\right)$$

or

$$|a_3| \le \frac{2(1-\alpha-3|a_2|^2)}{3}; \quad \frac{2}{3} \le \alpha < 1.$$

Once again, the second equation in (2.13) can be rewritten as

$$3a_3 - 3a_2^2 = 3a_2^2 - (1-\alpha)d_2,$$

which upon substitution of $a_2 = -\frac{1-\alpha}{2}d_1$ in its right hand side and taking the absolute values, we obtain

$$3\left|a_3 - a_2^2\right| \le (1-\alpha)\left|d_2 - \frac{3}{4}(1-\alpha)d_1^2\right|.$$

Since $-\frac{3}{4}(1-\alpha) \ge -\frac{1}{2}$ if $\alpha \ge \frac{1}{3}$, we get

$$3\left|a_3 - a_2^2\right| \le (1-\alpha)\left(2 - \frac{3}{4}(1-\alpha)|d_1|^2\right).$$

Now, upon re-substitution of $a_2 = -\frac{1-\alpha}{2}d_1$ in the right hand side of the above inequality, it turns to

$$3\left|a_3 - a_2^2\right| \le (1-\alpha)\left(2 - \frac{3}{1-\alpha}|a_2|^2\right)$$

or

$$|a_3 - a_2^2| \le \frac{2}{3}(1-\alpha) - |a_2|^2 \ \text{ if } \ \frac{1}{3} \le \alpha < 1. \qquad \blacksquare$$

Remark 2.1. The bounds $|a_2| \le 1-\alpha$ for $\frac{1}{3} \le \alpha < 1$ and $|a_3| \le \frac{4}{3}(1-\alpha)$ for $0 \le \alpha < \frac{1}{3}$ given in Theorem 2.3 above are much better than those corresponding bounds given by Srivastava *et al.* in [15, p. 1191, Theorem 2].

Finally, we give an example of a function satisfying the conditions (1.2) and (1.3).

Example 2.1. Let $f(z) = z + \frac{1-\alpha}{np} z^n$. Then $f'(z) = 1 + \frac{1-\alpha}{p} z^{n-1}$ and

$$\left(f'(z)\right)^p = 1 + \sum_{k=1}^{p} \binom{p}{k} \frac{(1-\alpha)^k}{p^k} z^{k(n-1)}.$$

Set

$$\left(f'(z)\right)^p - \alpha = (1-\alpha)\left(1 + \sum_{k=1}^{p} \binom{p}{k} \frac{(1-\alpha)^{k-1}}{p^k} z^{k(n-1)}\right) = (1-\alpha)\left(1 + \sum_{k=1}^{p} A_k z^{k(n-1)}\right).$$

We note that A_k is a convex null sequence because $\lim_{k\to\infty} A_k = 0$, $1 - A_1 \geq 0$ and $A_k - A_{k+1} \geq 0$. Therefore $\operatorname{Re}\left[(f'(z))^p - \alpha\right] > 0$ or $\operatorname{Re}\left(f'(z)\right)^p > \alpha$.

On the other hand, according to the equations (2.4) and (2.5), for the inverse map $g = f^{-1}$ we obtain $g(w) = w - \frac{1-\alpha}{np} w^n$ and

$$\left(g'(w)\right)^p - \alpha = (1-\alpha)\left(1 + \sum_{k=1}^{p} (-1)^k \binom{p}{k} \frac{(1-\alpha)^{k-1}}{p^k} w^{k(n-1)}\right).$$

Similarly, $\operatorname{Re}\left[(g'(w))^p - \alpha\right] > 0$ since $\frac{(g'(w))^p - \alpha}{1-\alpha}$ is dominated by $1 + \sum_{k=1}^{p} A_k w^{k(n-1)}$ and A_k is a convex null sequence (e.g. see Goodman [10, Chapter 7]).

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