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Weakly Quasi-First-Countable Spaces and Box Products

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Abstract. A space X is said to be weakly quasi-first-countable if and only if for all $x \in X$, there exists countably many countable families of decreasing subsets containing x such that a set O is open if and only if for any $x \in O$, O contains a member of each family associated to x. For a space X, we denote the countable σ-product of X endowed with the box topology by $\sigma B(X)$. We prove that if X is first-countable and locally compact, then $\sigma B(X)$ is weakly quasi-first-countable, which gives a general method to construct weakly quasi-first-countable spaces which are neither weakly first-countable nor quasi-first-countable.

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1. Introduction

In [23], Sirois-Dumais introduced the quasi-first-countable spaces and weakly quasi-first-countable spaces, which are natural generalizations of first-countable spaces and weakly first-countable spaces, respectively. It has been found that the weakly quasi-first-countable spaces play an interesting role in the theory of generalized metric spaces and topological groups, see [15, 21–24]. In [15], Liu and Lin introduced the notion of $\aleph_0$-weak bases, and revealed the elementary character of weakly quasi-first-countable spaces. In fact, every weakly quasi-first-countable space can be seen as the space each point in which has a countable locally $\aleph_0$-weak base. For canonical examples, the Aren's space S_2 is a weakly-first-countable space but not quasi-first-countable, the sequential fan space S_ω is a quasi-first-countable space but not weakly first-countable. So the topological sum $S_2 \oplus S_\omega$ is a weakly quasi-first-countable space which is neither weakly first-countable nor quasi-first-countable. However, this space is quite trivial and we are seeking abundant weakly quasi-first-countable spaces which are neither weakly first-countable nor quasi-first-countable. So we are interested in the following question.

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Question 1.1. Is there a general method of constructing weakly quasi-first-countable spaces which are neither weakly first-countable nor quasi-first-countable?

In the section 2 of this paper, we prove that for each first-countable, locally compact space X, the countable σ-product $\sigma B(X)$ of X endowed with the box topology is weakly quasi-first-countable, by which we obtain a general method of constructing weakly quasi-first-countable spaces which are neither weakly first-countable nor quasi-first-countable. In [22], Shen and Lin proved that a topological group is metrizable if and only if it is weakly quasi-first-countable and contains no closed copies of S_ω. We also give a general method of constructing weakly quasi-first-countable, non-metrizable topological groups. Some further results on $\sigma B(X)$ are given in section 3.

In this paper all spaces are regular T_1 and ω denotes the first infinite ordinal. We recall some basic definitions.

A space X is called a *sequential space* [10] if for each subset U of X, U is open if and only if every sequence L converging to $x \in U$ is eventually in U (that is, $L - U$ is finite). X is called *Fréchet* [10] if for each $x \in \overline{A} \subset X$, there is a sequence in A converging to x. Obviously, every Fréchet space is sequential. The *sequential fan space* S_ω [4] is obtained by identifying all the limit points of ω many convergent sequences; A space X is called an *S_2-space* (*Arens' space*) [1] if $X = \{\infty\} \cup \{x_n : n < \omega\} \cup \{x_n(m) : m, n < \omega\}$ and the topology is defined as follows: Each $x_n(m)$ is isolated; a basic neighborhood of x_n is $\{x_n\} \cup \{x_n(m) : m > k, \text{for some } k < \omega\}$; a basic neighborhood of ∞ is $\{\infty\} \cup (\cup\{V_n : n > k \text{ for some } k < \omega\})$, where V_n is a neighborhood of x_n. Note that S_2 is not Fréchet.

For a space X, Let X^ω be the countable box product of X, that is, the topology of X^ω is generated by the base $\{\prod_{i<\omega} U_i : U_i \text{ is open in } X \text{ for each } i < \omega\}$. For $p \in X^\omega$, we denote the n-th coordinate of p by $p(n)$ for each $n < \omega$. Define the *σ-Box product* of X at a point p as the subspace $\{x \in X^\omega : x(n) \neq p(n) \text{ for at most finitely many } n < \omega\}$ of X^ω, which we denote by $\sigma B(p, X)$. Note that in the literature, the σ-Box product of a space X is sometimes called the direct sum of X.

2. On weakly quasi-first-countable spaces

Definition 2.1. *Let $\mathscr{B} = \cup\{\mathscr{B}_x(n) : x \in X, n < \omega\}$ be a family of subsets of a space X. For each $x \in X, n < \omega$, $\mathscr{B}_x(n)$ is a network at x and closed under finite intersections.*

(1) *$\mathscr{B}$ is said to be an $\aleph_0$-weak base [15] for X if for every subset $U \subset X$, U is open whenever there exists a $B_x(n) \in \mathscr{B}_x(n)$ such that $B_x(n) \subset U$ for each $x \in U$ and $n < \omega$. If also, $\mathscr{B}_x(n)$ is countable for each $x \in X, n < \omega$, then we call X weakly quasi-first-countable [23].*

(2) *$\mathscr{B}$ is said to be a weak base [3] for X if $\mathscr{B}_x(n) = \mathscr{B}_x(1)$ for each $n < \omega$ in the definition of $\aleph_0$-weak bases. X is called weakly first-countable [3] if $B_x(1)$ is also countable for each $x \in X$.*

(3) *$\mathscr{B}$ is said to be an $\aleph_0$-base for X if for each $x \in X, n < \omega$ and $B_n \in \mathscr{B}_x(n)$, $\bigcup_{n<\omega} B_n$ is a neighborhood of x. X is called quasi-first-countable [23] if $\mathscr{B}_x(n)$ is also countable for each $x \in X, n < \omega$.*

It is easy to see the following implications [3, 23].

first-countable	$\Rightarrow$	quasi-first-countable	$\Rightarrow$	Fréchet
$\Downarrow$		$\Downarrow$		$\Downarrow$
weakly first-countable	$\Rightarrow$	weakly quasi-first-countable	$\Rightarrow$	sequential

For canonical examples, S_2 is a weakly first-countable space but not quasi-first-countable, S_ω is a quasi-first-countable space but not weakly first-countable. Moreover, the closed subspaces of a weakly first-countable space (a quasi-first-countable space) are still weakly first-countable (quasi-first-countable). Therefore, any weakly first-countable space (quasi-first-countable space) contains no closed copies of $S_\omega(S_2)$.

Lemma 2.1. [21] *Let X be a space. $\mathscr{B} = \cup\{\mathscr{B}_x(n) : x \in X, n < \omega\}$ is a family of subsets of X, here each $\mathscr{B}_x(n)$ is a network at x in X and $\mathscr{B}_x(n)$ is closed under finite intersections for each $x \in X, n < \omega$. Consider the following two conditions:*

(1) *$\mathscr{B}$ is an $\aleph_0$-weak base for X.*

(2) *For any sequence L converging to x in X, there exist a subsequence L' of L and $n_0 < \omega$ such that L' is eventually in B for each $B \in \mathscr{B}_x(n_0)$.*

We have (1)⇒(2). Moreover, if X is sequential, then (2)⇒(1).

Lemma 2.2. *Let X be a Fréchet, locally compact space. Then $\sigma B(p,X)$ is a sequential space for each $p \in X^\omega$.*

Proof. Let A be a sequentially closed subset of $\sigma B(p,X)$. It is sufficient to show A is a closed subset of $\sigma B(p,X)$. Fix $r \in \sigma B(p,X) - A$, put $A_n = \{x \in A : x(k) = r(k) \text{ for } k > n\}$, then $A = \bigcup_{n<\omega} A_n$.

First we claim that $r(1) \notin \overline{\{x(1) : x \in A_1\}}$. Otherwise we can find a sequence $\{x_m\}_{m<\omega} \subset A_1$ such that $\{x_m(1)\}_{m<\omega}$ converges to $r(1)$ by the Fréchet property of X. Thus $\{x_m\}_{m<\omega}$ converges to r, which is a contradiction with A is sequentially closed. Therefore there is a compact neighborhood U_1 of $r(1)$ such that $U_1 \cap \{x(1) : x \in A_1\} = \emptyset$.

Second, we suppose that for $k < n+1$, we have picked a compact neighborhood U_k of $r(k)$ satisfying that for each $x \in A_n$, $x(k) \notin U_k$ for some $k < n+1$.

Put $B = \{x \in A_{n+1} : x(k) \in U_k \text{ for each } k < n+1\}$. If $r(n+1) \in \overline{\{x(n+1) : x \in B\}}$, then there is a sequence $\{x_m\}_{m<\omega} \subset B$ such that $\{x_m(n+1)\}_{m<\omega}$ converges to $r(n+1)$. Since $\{x_m(n) : m < \omega\} \subset U_n$ and U_n is compact, $\{x_m(n) : m < \omega\}$ has a cluster point $q(n) \in U_n$. So we can find a subsequence $\{x_{m_i}\}_{i<\omega}$ of $\{x_m\}_{m<\omega}$ such that $\{x_{m_i}(n)\}_{i<\omega}$ converges to $q(n)$. Inductively, we can get a subsequence $\{y_m\}_{m<\omega}$ of $\{x_m\}_{m<\omega}$ and $(q(1), q(2), \ldots, q(n)) \in U_1 \times U_2 \times \ldots \times U_n$ such that $\{y_m(n+1)\}_{m<\omega}$ converges to $r(n+1)$ and $\{y_m(k)\}_{m<\omega}$ converges to $q(k)$ for each $k < n+1$. Let $y = (q(1), q(2), \ldots, q(n), r(n+1), r(n+2), \ldots)$, then the sequence $\{y_m\}_{m<\omega}$ converges to y. Since A is sequentially closed, $y \in A_n$. This is a contradiction with the induction hypothesis. So $r(n+1) \notin \overline{\{x(n+1) : x \in B\}}$, and thus we can take a compact neighborhood U_{n+1} of $r(n+1)$ such that for each $x \in A_{n+1}$, $x(k) \notin U_k$ for some $k < n+2$.

Now we have constructed the compact neighborhood U_n of $r(n)$ for each $n < \omega$ satisfying that for each $x \in A_n$, $x(k) \notin U_k$ for some $k < n+1$. So $\prod_{n<\omega} U_n$ is a neighborhood of r and $\prod_{n<\omega} U_n \cap A = \emptyset$. Thus $r \notin \overline{A}$, which implies that A is closed in $\sigma B(p,X)$. Therefore, $\sigma B(p,X)$ is a sequential space. ∎

Lemma 2.3. *Let $\{x_i\}_{i<\omega}$ be a sequence in $\sigma B(p,X)$ for a space X and a point $p \in X^\omega$. If $\{x_i\}_{i<\omega}$ converges to a point $x \in \sigma B(p,X)$, then there are an $i_0 < \omega$ and an $n_0 < \omega$ such that for each $i > i_0$ and $n > n_0$, $x_i(n) = x(n)$.*

Proof. Suppose not, then we can find two subsequences $\{n_k\}_{k<\omega}$ and $\{i_k\}_{k<\omega}$ of ω such that $x_{i_k}(n_k) \neq x(n_k)$ for each $k < \omega$. Now we pick an open neighborhood U_{n_k} of $x(n_k)$ such that $x_{i_k}(n_k) \notin U_{n_k}$ for each $k < \omega$ and let $U_n = X$ for $n \notin \{n_k : k < \omega\}$. It is easy to verify that $U = \prod_{n<\omega} U_n \cap \sigma B(p,X)$ is an open neighborhood of x in $\sigma B(p,X)$ and $U \cap \{x_{i_k} : k < \omega\} = \emptyset$, which is a contradiction with $\{x_i\}_{i<\omega}$ converging to x. Thus, there are an $i_0 < \omega$ and an $n_0 < \omega$ such that for each $i > i_0$ and $n > n_0$, $x_i(n) = x(n)$. ∎

Theorem 2.1. *Let X be a first-countable, locally compact space. Then $\sigma B(p,X)$ is a weakly quasi-first-countable space for each $p \in X^\omega$.*

Proof. For $x \in \sigma B(p,X)$ and each $n < \omega$, we take $\mathscr{P}_n$ be the countable neighborhood base at $x(n)$ in X. Put $\mathscr{B}'_x(n) = \{\prod_{k<\omega} P_k : P_k \in \mathscr{P}_k$ for $k < n+1$ and $P_k = \{x(k)\}$ for $k > n\}$ and $\mathscr{B}_x(n) = \{\cap\mathscr{C} : \mathscr{C}$ is a finite subfamily of $\mathscr{B}'_x(n)\}$. Then $\mathscr{B}_x(n)$ is countable and closed under finite intersections. Also it is easy to see that each $\mathscr{B}_x(n)$ is a network at x.

By Lemma 2.2 and Lemma 2.1, it is sufficient to show $\mathscr{B}_x(n)$ satisfies the condition (2) in Lemma 2.1. Now suppose $\{x_i\}_{i<\omega}$ is a sequence converging to x in $\sigma B(p,X)$. By Lemma 2.3, there is an $i_0 < \omega$ and an $n_0 < \omega$ such that for each $i > i_0$ and $n > n_0$, $x_i(n) = x(n)$. So $\{x_i\}_{i<\omega}$ is eventually in B for each $B \in \mathscr{B}_x(n_0)$. Thus $\sigma B(p,X)$ is weakly quasi-first-countable. ∎

Theorem 2.2. *Let X be a Fréchet, locally compact space and let $p(n)$ be a non-isolated point of X for each $n < \omega$. Then $\sigma B(p,X)$ contains a closed copy of S_2 and a closed copy of S_ω. Consequently, $\sigma B(p,X)$ is neither quasi-first-countable nor weakly first-countable.*

Proof. For each $n < \omega$, since X is Fréchet and $p(n)$ is a non-isolated point in X, we can find a non-trivial sequence $\{p_{mn}\}_{m<\omega} \subset X$ converging to $p(n)$. Without loss of generality, we may assume that $p_{mn} \neq p(n)$ for each $m,n < \omega$.

For each $i,j < \omega$, put $x_{ij}(1) = p_{i1}, x_{ij}(i+1) = p_{j(i+1)}$ and $x_{ij}(k) = p(k)$ for $k \notin \{1, i+1\}$, $L_i = \{x_{ij}\}_{j<\omega}$. For each $i < \omega$, pick $y_i \in \sigma B(p,X)$ as $y_i(1) = p_{i1}$ and $y_i(k) = p(k)$ for $k > 1$, $L_0 = \{y_i\}_{i<\omega}$. Then L_i converges to y_i for each $i < \omega$ and L_0 converges to p. We claim that $L = \{p\} \cup L_0 \cup \bigcup_{i<\omega} L_i$ is a closed copy of S_2. In fact, it is easy to verify that L is a sequentially closed subset of $\sigma B(p,X)$ by Lemma 2.3. By Lemma 2.2, $\sigma B(p,X)$ is sequential. So L is closed in $\sigma B(p,X)$. For given natural numbers $\{j_i : i < \omega\}$, we can similarly show that $\{x_{ij} : j < j_i, i < \omega\}$ is a sequentially closed subset of $\sigma B(p,X)$, thus is a closed subset of L. So L is homeomorphic to S_2. Therefore $\sigma B(p,X)$ contains a closed copy of S_2.

Now for each $i,j < \omega$, we put $x_{ij}(i) = p_{ji}$ and $x_{nk}(k) = p(k)$ for $k \neq i$, $S_i = \{x_{ij}\}_{j<\omega}$. Then S_i converges to p for each $i < \omega$. Similar to the above proof, we can prove that $S = \{p\} \cup \bigcup_{i<\omega} S_i$ is a closed copy of S_ω in $\sigma B(p,X)$. ∎

Corollary 2.1. *Let X be a Fréchet, locally compact homogeneous space, and $p \in X^\omega$. Then the following are equivalent:*

(1) *X is discrete;*
(2) *$\sigma B(p,X)$ contains no closed copies of S_2;*
(3) *$\sigma B(p,X)$ contains no closed copies of S_ω.*

Corollary 2.2. *Let X be a first-countable, locally compact space and let $p(n)$ be a non-isolated point of X for each $n < \omega$. Then $\sigma B(p,X)$ is a weakly quasi-first-countable space which is neither quasi-first-countable nor weakly first-countable. In particular, let X=$\mathbb{R}$ be the real line with the Euclidean topology and $0 = (0,0,\cdots,) \in \mathbb{R}^\omega$. Then $\sigma B(0,\mathbb{R})$ is a weakly quasi-first-countable space which is neither quasi-first-countable nor weakly first-countable..*

In [22], Shen and Lin proved that a topological group is metrizable if and only if it is weakly quasi-first-countable and contains no closed copies of S_ω. Svetlichny [24] proved that every quasi-first-countable topological group is metrizable and gave a weakly quasi-first-countable, non-metrizable topological group. The following corollary gives a general method to construct the weakly quasi-first-countable, non-metrizable topological groups.

Corollary 2.3. *Let G be a first-countable, locally compact topological group, $p(n)$ be a non-isolated point of X for each $n < \omega$. Then $\sigma B(p,G)$ is a weakly quasi-first-countable, non-metrizable topological group.*

3. Generalized metric properties of $\sigma B(p,X)$

In [6], Borges proved that $\sigma B(p,X)$ is stratifiable for each $p \in X^\omega$ if X is stratifiable. Naturally, we are interested in the question that whether the similar results hold for other classical generalized metric properties. We recall some definitions.

Let $\mathscr{P}$ be a cover of a space X. Then $\mathscr{P}$ is called a *network* [2] for X if for any open set U and any $x \in U$, $x \in P \subset U$ for some $P \in \mathscr{P}$; $\mathscr{P}$ is called a *k-network* [11] for X if for any compact set K and for any open set U such that $K \subset U$, $K \subset \cup\mathscr{P}' \subset U$ for some finite $\mathscr{P}' \subset \mathscr{P}$; $\mathscr{P}$ is called a *cs-network* [11] for X if for any convergent sequence L and for any open set U such that $L \subset U$, there exists a $P \in \mathscr{P}$ such that $P \subset U$ and L is eventually in P. X is called a *σ-space* [19] (*$\aleph$-space* [20]) if it has a σ-locally finite network (k-network). X is called a *cosmic space* [18] (*$\aleph_0$-space* [18]) if it has a countable network (k-network). It is well-known to all that a space X is an $\aleph$-space ($\aleph_0$-space) if and only if X has a σ-locally finite (countable) *cs*-network [9, 12].

Lemma 3.1. *Let $\mathscr{P}$ be a cover of a space X and $p \in X^\omega$. Put $\mathscr{B}_i = \{\prod_{k<\omega} P_k : P_k \in \mathscr{P}$ for $k < i+1$ and $P_k = \{p(k)\}$ for $k > i\}$ for each $i < \omega$ and $\mathscr{B} = \bigcup_{i<\omega} \mathscr{B}_i$. Then we have the following claims.*

(1) *If $\mathscr{P}$ is countable, then $\mathscr{B}$ is countable.*
(2) *If $\mathscr{P}$ is σ-locally finite in X, then $\mathscr{B}$ is σ-locally finite in $\sigma B(p,X)$.*
(3) *If $\mathscr{P}$ is a network for X, then $\mathscr{B}$ is a network for $\sigma B(p,X)$.*
(4) *If $\mathscr{P}$ is a cs-network for X, then $\mathscr{B}$ is a cs-network for $\sigma B(p,X)$.*

Proof. (1) is obvious.

(2) Suppose that $\mathscr{P} = \bigcup_{n<\omega} \mathscr{P}_n$ and each $\mathscr{P}_n$ is locally finite in X, then we have $\mathscr{B}_i = \bigcup_{n_1,n_2,\cdots,n_i<\omega} \mathscr{P}_{n_1} \times \mathscr{P}_{n_2} \times \cdots \times \mathscr{P}_{n_i} \times \prod_{k>i}\{p(k)\}$ is σ-locally finite in $\sigma B(p,X)$ for each $i < \omega$. Therefore, $\mathscr{B}$ is σ-locally finite in $\sigma B(p,X)$.

(3) For each open subset U of $\sigma B(p,X)$ and $x \in U$, there is an $i_0 < \omega$ such that $x(k) = p(k)$ for each $k > i_0$. Now we pick open subsets $U_1, U_2, \cdots, U_{i_0}$ of X such that $x \in \prod_{k<i_0+1} U_k \times \prod_{k>i_0}\{p(k)\} \subset U$. Since $\mathscr{P}$ is a network for X, there is a $P_k \in \mathscr{P}$ such that $x(k) \in P_k \subset U_k$ for each $k < \omega$. Put $B = \prod_{k<i_0+1} P_k \times \prod_{k>i}\{p(k)\}$. Then $B \in \mathscr{B}_{i_0} \subset \mathscr{B}$ and $x \in B \subset U$.

(4) Let $\{x_m\}_{m<\omega}$ be a sequence converging to x and U be an open neighborhood of x in $\sigma B(p,X)$. By Lemma 2.3, there is an $i_0 < \omega$ such that for each $m > i_0$ and $k > i_0$, $x_m(k) = x(k) = p(k)$. Similar to the proof of Claim 3, we pick open subsets $U_1, U_2, \cdots, U_{i_0}$ of X such that $x \in \prod_{k<i_0+1} U_k \times \prod_{k>i_0}\{p(k)\} \subset U$.

For each $k < i_0+1$, $\{x_m(k)\}_{m<\omega}$ converges to $x(k)$, so there is a $P_k \in \mathscr{P}$ such that $P_k \subset U_k$ and $\{x_m(k)\}_{m<\omega}$ is eventually in P_k. Put $B = \prod_{k<i_0+1} P_k \times \prod_{k>i}\{p(k)\} \in \mathscr{B}$. Then $B \subset U$ and $\{x_m\}_{m<\omega}$ is eventually in B. Therefore $\bigcup_{n<\omega} \mathscr{P}_n$ is a cs-network for $\sigma B(p,X)$. ∎

Theorem 3.1. *Let X be a σ-space (resp. $\aleph$-space, cosmic space, $\aleph_0$-space). Then $\sigma B(p,X)$ is a σ-space (resp. $\aleph$-space, cosmic space, $\aleph_0$-space) for each $p \in X^\omega$.*

Definition 3.1. *A space is called semi-stratifiable [7] if there is a function G which assigns to each $n < \omega$ and closed set $H \subset X$, an open set $G(n,H)$ containing H such that*

(i) $H = \bigcap_{n<\omega} G(n,H)$;

(ii) $H \subset K$ *implies* $G(n,H) \subset G(n,K)$.

If also,

(iii) $H = \bigcap_{n<\omega} \overline{G(n,H)}$ *(for each compact C and closed H with $C \cap H = \emptyset$, $C \cap G(n,H) = \emptyset$ for some $n < \omega$), then X is stratifiable [5] (k-semi-stratifiable [16]).*

Let $\mathscr{P}$ be a collection of pairs of subsets of X. $\mathscr{P}$ is called *cushioned* [17] if $\overline{\cup\{P_1 : (P_1, P_2) \in \mathscr{P}'\}} \subset \cup\{P_2 : (P_1,P_2) \in \mathscr{P}'\}$ for each $\mathscr{P}' \subset \mathscr{P}$. $\mathscr{P}$ is called a *pair-network* [13] for X if for any open set U and any $x \in U$, $x \in P_1 \subset P_2 \subset U$ for some $(P_1,P_2) \in \mathscr{P}$. Similarly, we can give the definitions of pair-*cs*-networks for a space. Kofner proved that a space X is a semi-stratifiable space if and only if X has a σ-cushioned pair-network [13] ; Foged proved that a space X is a k-semi-stratifiable space if and only if X has a σ-cushioned pair-*cs*-network (pair-cs^*-network) [8, 14].

Theorem 3.2. *Let X be a semi-stratifiable space (k-semi-stratifiable space). Then $\sigma B(p,X)$ is a semi-stratifiable space (k-semi-stratifiable space) for each $p \in X^\omega$.*

Proof. Let $\mathscr{P} = \bigcup_{n<\omega} \mathscr{P}_n$ be a σ-cushioned pair-network (pair-cs-network) for X, where each $\mathscr{P}_n$ is cushioned in X. Put $\mathscr{B}_i = \{(\prod_{k<i+1} P_{k1} \times \prod_{k>i}\{p(k)\}, \prod_{k<i+1} P_{k2} \times \prod_{k>i}\{p(k)\}) : (P_{k1},P_{k2}) \in \mathscr{P}\}$ and $\mathscr{B} = \bigcup_{i<\omega} \mathscr{B}_i$. Similar to the proof of the claim 3 and claim 4 in theorem 3.1, we can verify that $\mathscr{B}$ is a pair-network (pair-cs-network) for $\sigma B(p,X)$.

Now we prove that each $\mathscr{B}_i$ is σ-cushioned in $\sigma B(p,X)$. It is sufficient to show that for each $i < \omega$, $\{(\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) : (P_{k1},P_{k2}) \in \mathscr{P}\}$ is a σ-cushioned family in X^i. Since $\{(\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) : (P_{k1},P_{k2}) \in \mathscr{P}\} = \bigcup_{(n_1,n_2,\cdots,n_i)\in\omega^i} \{(\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) : (P_{k1}, P_{k2}) \in \mathscr{P}_{n_k}$ for each $k < i+1\}$, we only need to show that for each $(n_1,n_2,\cdots,n_i) \in \omega^i$, $\{(\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) : (P_{k1},P_{k2}) \in \mathscr{P}_{n_k}$ for each $k < i+1\}$ is cushioned. Let C be a subfamily of $\{(\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) : (P_{k1},P_{k2}) \in \mathscr{P}_{n_k}$ for each $k < i+1\}$. We prove that $\overline{\cup\{\prod_{k<i+1} P_{k1} : (\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) \in C\}} \subset \cup\{\prod_{k<i+1} P_{k2} : (\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) \in C\}$. Suppose $x \in X^i - \cup\{\prod_{k<i+1} P_{k2} : (\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) \in C\}$. We put $U_k = X - \cup\{P_{k2} : x(k) \in X - P_{k2}, (\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) \in C\}$. Since $\mathscr{P}_{n_k}$ is cushioned, $x(k) \in U_k \subset X - \overline{\cup\{P_{k1} : x(k) \in X - P_{k2}, (\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) \in C\}}$ for each $k < i+1$. Let $V_k = X - \overline{\cup\{P_{k1} : x(k) \in X - P_{k2}, (\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) \in C\}}$. We claim that $\prod_{k<i+1} V_k$ is an open neighborhood of x in X^i and $\prod_{k<i+1} V_k \cap (\cup\{\prod_{k<i+1} P_{k1} : (\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) \in C\}) = \emptyset$. In fact, for each $(\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) \in C$, there is a $k < i+1$ such that $x_k \notin P_{k2}$. So $V_k \cap P_{k1} = \emptyset$, thus $\prod_{k<i+1} V_k \cap \prod_{k<i+1} P_{k1} = \emptyset$. Therefore, $x \notin \overline{\cup\{\prod_{k<i+1} P_{k1} : (\prod_{k<i+1} P_{k1}, \prod_{k<i+1} P_{k2}) \in C\}}$. Subsequently, $\mathscr{B}$ is a σ-cushioned pair-network (pair-cs-network) for $\sigma B(p,X)$. ∎

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