BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **37**(3) (2014), 865–879

Relative Projective Dimensions

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Abstract. In (n,d)-ring and n-coherent ring theory, n-presented modules plays an important role. In this paper, we firstly give some new characterizations of n-presented modules and n-coherent rings. Then, we introduce the concept of $(n,0)$-projective dimension, which measures how far away a finitely generated module is from being n-presented and how far away a ring is from being Noetherian, for modules and rings. This dimension has nice properties when the ring in question is n-coherent. Some known results are extended or obtained as corollaries.

2010 Mathematics Subject Classification: 16D10, 16E40

Keywords and phrases: Noetherian rings, n-coherent rings, $(n,0)$-projective module (dimension), $(n,0)$-injective module, n-presented module.

1. Introduction

Throughout this paper all rings are associative with identity and modules are unitary. $rD(R)$ stands for the right global dimension of a ring R. $pd(M)$, $id(M)$ and $fd(M)$ denote the projective, injective and flat dimension of an R-module M, respectively.

Let $n \geq 0$ be an integer. Following [2, 3, 15], we call a right R-module P *n-presented* if there exists an exact sequence of right R-modules

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to P \to 0$$

where each F_i is finitely generated free (equivalently projective), $i = 0, 1, \cdots, n$. An R-module is 0-presented (resp. 1-presented) if and only if it is finitely generated (resp. finitely presented). Every m-presented R-module is n-presented for $m \geq n$. A ring R is called *right n-coherent* [3] in case every n-presented right R-module is $(n+1)$-presented. It is easy to see that R is right 0-coherent (resp. 1-coherent) if and only if R is right Noetherian (resp. coherent), and every n-coherent ring is m-coherent for $m \geq n$.

Communicated by Ang Miin Huey.
Received: March 15, 2012; *Revised:* June 26, 2012.

Let n and d be non-negative integers and M a right R-module. M is called (n,d)-*injective* [16] if ${Ext_R}^{d+1}(N,M)=0$ for any n-presented right R-module N. M is said to be (n,d)-*projective* [11] if $Ext_R^{d+1}(M,N)=0$ for any (n,d)-injective R-module N. It is easy to see that both (n,d)-injective modules and (n,d)-projective modules are closed under direct summands and finite direct sums. $(1,0)$-injective (resp. $(1,0)$-projective) modules are also called FP-*injective* (resp. FP-*projective*) modules. It is clear that every (n,d)-injective (resp. (m,d)-projective) module is (m,d)-injective (resp. (n,d)-projective) for $m\geq n$.

In (n,d)-ring and n-coherent ring theory (see [2,3,11,16]), n-presented modules plays an important role. For modules and rings, Mao and Ding [10] defined a dimension, called an FP-projective dimension; Ng [12] introduced the concept of finitely presented dimension. In this paper, we introduce a kind of n-presented dimension of modules and rings.

Let $n\geq 1$ be a fixed integer. In Section 2, we introduce the concept of $(n,0)$-*projective dimension* $npd(M)$ for a right R-module M, and the concept of *right* $(n,0)$-*projective dimension* for a ring R, which measures how far away a finitely generated right R-module M is from being n-presented, and how far away a ring is from being right Noetherian, respectively. It is shown that a finitely generated right R-module M is n-presented if and only if it is $(n,0)$-projective if and only if $npD(M)=0$ (Theorem 2.1); R is an n-coherent ring if and only if every $(n,0)$-injective right R-module is $(n,1)$-injective if and only if every $(n,1)$-projective right R-module is $(n,0)$-projective (Theorem 2.2); R is a right Noetherian ring if and only if $rnpD(R)=0$ if and only if every right R-module is $(n,0)$-projective if and only if for a short exact sequence $0\to A\to B\to C\to 0$ of right R-modules, if both B and C are finitely generated, then A is also finitely generated (Corollary 2.3).

Let $n\geq 1$ be a fixed integer and R a right n-coherent ring. In Section 3, we prove that $rnpD(R)$ = $\sup\{npd(M)$: M is a cyclic right R-module$\}$ = $\sup\{id(M)$: M is an $(n,0)$-injective right R-module$\}$ (Theorem 3.1). As corollaries we obtain that R is right Noetherian if and only if $rnpD(R)<\infty$ and every injective right R-module is $(n,0)$-projective if and only if every $(n,0)$-injective right R-module has an $(n,0)$-projective cover with the unique mapping property if and only if every $(n,0)$-injective right R-module has an injective envelope with the unique mapping property (Corollary 3.3). If $rnpD(R)\leq m$, then we have that R is a right m-coherent ring (Proposition 3.5). Let S and T be rings. If $S\oplus T$ is an right n-coherent ring, then we get that $rnpD(S\oplus T)$ = $\sup\{rnpD(S), rnpD(T)\}$ (Theorem 3.2). Let R be a commutative n-coherent ring and P any prime ideal of R, then $npD(R_P)\leq npD(R)$, where R_P is the localization of R at P (Theorem 3.3).

2. Definition and general results

Let R be a ring and $m\geq 0$ an integer. Mao and Ding [10] defined the FP-*projective dimension* $fpd(M)$ of a right R-module M as $\inf\{$m: $Ext_R^{m+1}(M,N)=0$ for any FP-injective right R-module $N\}$, if no such m exists, set $fpd(M)=\infty$; and the *right* FP-*projective dimension* $rfpD(R)$ of R as $\sup\{fpd(M)$: M is a finitely generated right R-module$\}$. We generalize it as follows.

Definition 2.1. *Let $m\geq 0$, $n\geq 1$ be integers, and R a ring. For a right R-module M, set $npd(M)$ = $inf\{m$: $Ext_R^{m+1}(M,N)=0$ for any $(n,0)$-injective right R-module $N\}$, called the $(n,0)$-projective dimension of M. If no such m exists, set $npd(M)=\infty$.*

Put $rnpD(R)=\sup\{npd(M)$: M is a finitely generated right R-module$\}$, and call $rnpD(R)$ the *right* $(n,0)$-*projective dimension* of R. The left (n,0)-projective dimension $lnpD(R)$ of

R may be defined similarly. If R is a commutative ring, we drop the unneeded letters r and l.

We list the following lemma proved in [11, Lemma 3.3] for convenient using.

Lemma 2.1. [11, Lemma 3.3] *Let R be a ring, $n \geq 0$ an integer and $0 \to A \to B \to C \to 0$ a short exact sequence of right R-modules. If C is $(n+1,0)$-projective and B is $(n,0)$-projective, then A is $(n,0)$-projective.*

It is clear that an n-presented right R-module is $(n,0)$-projective. In general, the converse is not true. Glaz (see [8, Theorem 2.1.10]) proved that a finitely generated right R-module is finitely presented if and only if it is FP-projective. We generalize it as the following.

Theorem 2.1. *Let $n \geq 0$ be a fixed integer and R a ring. Then the following are equivalent for a finitely generated right R-module P.*

(1) *P is n-presented.*
(2) *P is $(n,0)$-projective.*
(3) *$npd(P)$ = 0.*

Proof. (1) $\Rightarrow$ (2) is obvious, and (2) $\Leftrightarrow$ (3) holds by definition.

(2) $\Rightarrow$ (1). We use induction on n. The case $n = 0$ is clear, and the case $n = 1$ has been proven in [8, Theorem 2.1.10]. Assume $n > 1$, and P is $(n,0)$-projective. Then P is $(n-1,0)$-projective. So P is $(n-1,0)$-presented by the induction hypothesis. Therefore there exists an exact sequence of right R-modules

$$F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to F_0 \to P \to 0$$

where each F_i is finitely generated projective (hence $(m,0)$-projective, for any non-negative integer m), i = 0, 1, ..., $n-1$. Write $K_1 = \ker(F_0 \to P)$, $K_m = \ker(F_{m-1} \to F_{m-2})$, m = 2, 3, ..., $n-1$. Then we have the following short exact sequences

$$0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow P \longrightarrow 0,$$
$$0 \longrightarrow K_2 \longrightarrow F_1 \longrightarrow K_1 \longrightarrow 0,$$
$$\vdots$$
$$0 \longrightarrow K_{n-1} \longrightarrow F_{n-2} \longrightarrow K_{n-2} \longrightarrow 0.$$

Note that P is $(n,0)$-projective and F_0 is $(n-1,0)$-projective, we obtain K_1 is $(n-1,0)$-projective by Lemma 2.1. It follows that K_2 is $(n-2,0)$-projective again by Lemma 2.1. Continuing this way, we see that K_{n-1} is $(1,0)$-projective. Clearly, K_{n-1} is finitely generated. Thus K_{n-1} is finitely presented by [8, Theorem 2.1.10], and hence there exists an exact sequence $F_n^{'} \to F_{n-1}^{'} \to K_{n-1} \to 0$ with $F_n^{'}$ and $F_{n-1}^{'}$ finitely generated projective. So we get an exact sequence

$$F_n^{'} \to F_{n-1}^{'} \to F_{n-2} \to \cdots \to F_1 \to F_0 \to P \to 0.$$

It follows that P is n-presented, as required. ∎

The following corollary is well-known.

Corollary 2.1. *Let $n \geq 0$ be a fixed integer and R a ring. Then the following statements hold:*

(1) *Every finitely generated projective right R-module is n-presented.*

(2) *For a short exact sequence* $0 \to A \to B \to C \to 0$ *of right* R*-modules, if both* A *and* C *are* n*-presented, then* B *is also* n*-presented.*
(3) *If* $B \cong A \oplus C$*, then* B *is* n*-presented if and only if both* A *and* C *are* n*-presented.*

Proof.

(1) Note that every projective right R-module is $(n,0)$-projective. Thus (1) follows from Theorem 2.1.
(2) Since A and C are n-presented, we have both A and C are finitely generated and $(n,0)$-projective. Hence B is also finitely generated and $(n,0)$-projective. Therefore B is n-presented by Theorem 2.1.
(3) If $B \cong A \oplus C$, then it is easy to see that B is finitely generated and $(n,0)$-projective if and only if both A and C are finitely generated and $(n,0)$-projective. Thus (3) holds by Theorem 2.1, and we complete the proof. ∎

Corollary 2.2. *Let* R *be a ring,* $n \geq 0$ *an integer and* $0 \to K \to P \to M \to 0$ *a short exact sequence of right* R*-modules, where* P *is finitely generated projective. Then* K *is* n*-presented if and only if* M *is* $(n+1,0)$*-presented.*

Proof. If K is n-presented, then clearly M is $(n+1)$-presented. Conversely, if M is $(n+1)$-presented (hence $(n+1,0)$-projective), then it is easy to see that K is finitely generated. On the other hand, K is $(n,0)$-projective by Lemma 2.1. It follows that K is n-presented from Theorem 2.1. ∎

Theorem 2.2. *Let* R *be a ring, and* $n \geq 0$ *a fixed integer. Then the following are equivalent:*

(1) R *is a right* n*-coherent ring.*
(2) *Every* $(n+1,0)$*-injective right* R*-module is* $(n,0)$*-injective.*
(3) *Every* $(n,0)$*-projective right* R*-module is* $(n+1,0)$*-projective.*
(4) *For a short exact sequence* $0 \to A \to B \to C \to 0$ *of right* R*-modules with* B *finitely generated projective, if* C *is* n*-presented, then* A *is also* n*-presented.*
(5) *For a short exact sequence* $0 \to A \to B \to C \to 0$ *of right* R*-modules, if both* B *and* C *are* n*-presented, then* A *is also* n*-presented.*

If $n \geq 1$*, then the above conditions are also equivalent to:*

(6) *Every* $(n,0)$*-injective right* R*-module is* $(n,1)$*-injective*
(7) *Every* $(n,1)$*-projective right* R*-module is* $(n,0)$*-projective.*

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$. are obvious.

$(3) \Rightarrow (1)$. Let M be an n-presented right R-modules. Then M is finitely generated and $(n,0)$-projective by Theorem 2.1. Note that M is $(n+1,0)$-projective by (3). Thus M is $(n+1)$-presented again by Theorem 2.1.

$(4) \Rightarrow (1)$. Let M be any n-presented right R-module. Then there exists a short exact sequence $0 \to K \to P \to M \to 0$ of right R-modules with P finitely generated projective and K n-presented by (4). Hence M is $(n+1)$-presented by Corollary 2.2, and (1) follows.

$(1) \Rightarrow (5)$. If C is n-presented, then C is $(n+1)$-presented by (1). The rest proof is similar to that of Corollary 2.2.

$(5) \Rightarrow (4)$. By (5), it suffices to show that B is n-presented. But this follows from Corollary 2.1.

Now suppose $n \geq 1$.

$(4) \Rightarrow (6)$. Let M be an $(n,0)$-injective right R-module and C any n-presented right R-module. Then we get a short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules with B finitely generated projective. By (4), A is n-presented. Thus,

$$Ext_R^2(C,M) \cong Ext_R^1(A,M) = 0.$$

Therefore, M is $(n,1)$-injective.

$(6) \Rightarrow (7)$ is easy.

$(7) \Rightarrow (1)$. Let P be an n-presented right R-module. We get a short exact sequence $0 \to K \to F \to P \to 0$ of right R-modules with F finitely generated projective and K finitely generated. For any $(n,1)$-injective right R-module M, we have

$$Ext_R^1(K,M) \cong Ext_R^2(P,M) = 0.$$

So K is $(n,1)$-projective and hence $(n,0)$-projective by (7). Thus, K is n-presented by Theorem 2.1. Therefore, P is $(n+1)$-presented and (1) holds. ∎

It is well known that a ring R is right Noetherian if and only if every right R-module is FP-projective if and only if $rfpD(R) = 0$ (see [10, Proposition 2.6]). Now, we have the following.

Corollary 2.3. *Let $n \geq 1$ be a fixed integer. Then the following are equivalent for a ring R:*

(1) *R is right Noetherian.*
(2) *$rnpD(R) = 0$.*
(3) *Every finitely generated right R-module is n-presented.*
(4) *Every $(n,0)$-injective right R-module is injective.*
(5) *Every right R-module is $(n,0)$-projective.*
(6) *Every finitely generated right R-module is $(n,0)$-projective.*
(7) *Every cyclic right R-module is $(n,0)$-projective.*
(8) *For a short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, if both B and C are finitely generated, then A is also finitely generated.*

If R is right n-coherent, then the above conditions are also equivalent to:

(9) *Every $(n,0)$-injective right R-module is $(n,0)$-projective.*

Proof. $(1) \Leftrightarrow (3) \Rightarrow (4)$ and $(5) \Rightarrow (6) \Rightarrow (7)$ are trivial.

$(4) \Rightarrow (5)$ Let M be any right R-module and N any $(n,0)$-injective right R-module. Then $Ext_R^1(M,N) = 0$ since N is injective by (4). Hence M is $(n,0)$-projective.

$(7) \Rightarrow (4)$. Let N be any $(n,0)$-injective right R-module, and I any right ideal of R. By (7), R/I is $(n,0)$-projective. So $Ext_R^1(R/I,N) = 0$. That is, N is injective.

$(2) \Leftrightarrow (6)$ holds by definition, $(3) \Leftrightarrow (6)$ holds by Theorem 2.1, $(1) \Leftrightarrow (8)$ holds by Theorem 2.2, and $(4) \Leftrightarrow (9)$ has been proven in [11, Proposition 4.10]. ∎

Corollary 2.4. *Let $n \geq 1$ be an integer and R a ring. If $rnpD(R) \leq 1$, then $rnpD(R) = rfpD(R)$.*

Proof. This follows from the fact that $rnpD(R) = 0$ if and only if $rfpD(R) = 0$ by Corollary 2.3 and [10, Proposition 2.6]. ∎

Remark 2.1.

(1) From Theorem 2.1 and Corollary 2.3, we see that $npd(M)$ measures how far away a finitely generated right R-module M is from being n-presented, and $rnpD(R)$ measures how far away a ring is from being right Noetherian.

(2) It is clear that $fpd(M) \leq npd(M) \leq pd(M)$, and $rfpD(R) \leq rnpD(R) \leq rD(R)$. Since $rfpD(R) = rD(R)$ if and only if R is von Neumann regular [10, Remarks 2.2], we have $rfpD(R) = rnpD(R) = rD(R)$ if and only if R is von Neumann regular. It is also easy to see that $rnpD(R) = rD(R)$ if and only if R is a right $(n,0)$-ring (see [16, Definition 2.5]).
(3) It is known that a right Noetherian ring need not be left Noetherian, so $rnpD(R) \neq lnpD(R)$ in general.
(4) The equivalence of (1) through (3) in Theorem 2.2 has been proven in [11, Theorem 4.1]. Here we prove the equivalence in a different way.
(5) If $n = 1$, then Theorem 2.2 is just some characterizations of coherent rings.

Recall that a ring R is called right self-$(n,0)$-injective in case R_R is $(n,0)$-injective. Stenström proved that if R is right coherent and right self-FP-injective, then every flat right R-module is FP-injective (see [14, Lemma 4.1]). We generalize it as the following

Proposition 2.1. *Let $n \geq 1$ be a fixed integer. If R is a right n-coherent and right self-$(n,0)$-injective ring, then every flat right R-module is $(n,0)$-injective.*

Proof. Let M be a flat right R-module. Then, by [9, Theorem 4.85], we get a pure short exact sequence $0 \to K \to F \to M \to 0$ where $F \cong \bigoplus_I R$ for a set I. Since R is right n-coherent and right self-$(n,0)$-injective, we have F is $(n,0)$-injective by [16, Lemma 2.9]. Hence we obtain the following exact sequence

$$0 \to Hom_R(N,K) \to Hom_R(N,F) \to Hom_R(N,M) \to Ext_R^1(N,K) \to Ext_R^1(N,F) = 0$$

for any n-presented (hence finitely presented) right R-module N. It follows that $Ext_R^1(N,K) = 0$, and so K is $(n,0)$-injective. Note that R is right n-coherent, we have M is $(n,0)$-injective by [11, Theorem 4.1], as desired. ∎

3. $(n,0)$-projective dimensions over n-coherent rings

Proposition 3.1. *Let $n \geq 1$, $m \geq 0$ be integers. If R is a right n-coherent ring, then the following are equivalent for a right R-module M:*

(1) *$npd(M) \leq m$.*
(2) *$Ext_R^{m+1}(M,N) = 0$ for any $(n,0)$-injective right R-module N.*
(3) *$Ext_R^{m+j}(M,N) = 0$ for any $(n,0)$-injective right R-module N and $j \geq 1$.*
(4) *There exists an exact sequence $0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, where each P_i is $(n,0)$-projective.*
(5) *If $\cdots \to P_{m-1} \to P_{m-2} \to \cdots \to P_1 \to P_0 \to M \to 0$ is a projective resolution of M, then $ker(P_{m-1} \to P_{m-2})$ is $(n,0)$-projective.*

Proof. $(1) \Rightarrow (2)$. We use induction on m. The case $m = 0$ is clear. Let $m \geq 1$. If $npd(M) = m$, then (2) holds by definition. Suppose $npd(M) \leq m-1$. For any $(n,0)$-injective right R-module N, the short exact sequence $0 \to N \to E \to L \to 0$ with E injective induces an exact sequence

$$Ext_R^m(M,L) \to Ext_R^{m+1}(M,N) \to Ext_R^{m+1}(M,E) = 0.$$

Since R is n-coherent, we get L is $(n,0)$-injective by [11, Theorem 4.1]. So $Ext_R^m(M,L) = 0$ by the induction hypothesis. It follows that $Ext_R^{m+1}(M,N) = 0$, as desired.

$(2) \Rightarrow (3)$. Using induction on j, the proof is similar to that of $(1) \Rightarrow (2)$.

(3) $\Rightarrow$ (1), and (2) $\Rightarrow$ (5) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (2). Write $K_1 = \ker(P_0 \to M)$, $K_i = \ker(P_{i-1} \to P_{i-2})$, $i = 2, 3, \ldots, m-1$. Then we have the following short exact sequences

$$0 \longrightarrow K_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$
$$0 \longrightarrow K_2 \longrightarrow P_1 \longrightarrow K_1 \longrightarrow 0,$$
$$\vdots$$
$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow K_{m-1} \longrightarrow 0.$$

From the bottom exact sequence, we get the exactness of the sequence

$$0 = Ext_R^1(P_m, N) \to Ext_R^2(K_{m-1}, N) \to Ext_R^2(P_{m-1}, N)$$

for any $(n,0)$-injective right R-module N. Since P_{m-1} is $(n,0)$-projective, using an argument similar to that of (1) $\Rightarrow$ (2), we get $Ext_R^2(P_{m-1}, N) = 0$. Hence $Ext_R^2(K_{m-1}, N) = 0$. Continuing this way, we obtain $Ext_R^{m+1}(M, N) = 0$. Thus (2) holds. ∎

Proposition 3.2. *Let R be a right n-coherent ring* $(n \geq 1)$ *and* $0 \to A \to B \to C \to 0$ *a short exact sequence of right R-modules. Then the following are true:*

(1) *If two of* $npd(A)$, $npd(B)$ *and* $npd(C)$ *are finite, so is the third.*
(2) $npd(A) \leq \sup\{npd(B), npd(C) - 1\}$.
(3) $npd(B) \leq \sup\{npd(A), npd(C)\}$.
(4) $npd(C) \leq \sup\{npd(B), npd(A) + 1\}$.
(5) *If B is* $(n,0)$*-projective and* $0 < npd(A) < \infty$, *then* $npd(C) = npd(A) + 1$.

Proof. Easy to verify by Proposition 3.1. ∎

Corollary 3.1. *Let R be a right n-coherent ring* $(n \geq 1)$, *A, B and C right R-modules. If* $B \cong A \oplus C$, *then* $npd(B) = \sup\{npd(A), npd(C)\}$.

Proof. Since $B \cong A \oplus C$, we get two short exact sequences $0 \to A \to B \to C \to 0$ and $0 \to C \to B \to A \to 0$. By Proposition 3.2 (3), it is enough to show that $npd(B) \geq \sup\{npd(A), npd(C)\}$. Suppose $npd(B) < \sup\{npd(A), npd(C)\}$, then $npd(B) < npd(A)$ or $npd(B) < npd(C)$. We may assume $npd(B) < npd(A)$. By Proposition 3.2 (2), $npd(C) \leq \sup\{npd(B), npd(A) - 1\}$. So $npd(C) \leq npd(A) - 1$, that is, $npd(C) < npd(A)$. In addition, also by Proposition 3.2 (2), we have $npd(A) \leq \sup\{npd(B), npd(C) - 1\}$. Hence $npd(A) \leq npd(C) - 1$, since $npd(B) < npd(A)$, and so $npd(A) < npd(C)$, a contradiction. ∎

Let M be a right R-module. Recall that a a homomorphism $\phi : M \to F$ where F is a right $(n,0)$-injective R-module, is called an $(n,0)$-injective *preenvelope* [5] of M if for any homomorphism $f : M \to F'$ with F' is $(n,0)$-injective, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Moreover, if the only such g are automorphism of F when $F' = F$ and $f = \phi$, then the $(n,0)$-injective preenvelope ϕ is called an $(n,0)$-injective *envelope*. A monomorphic $(n,0)$-injective preenvelope ϕ is said to be *special* [6, Definition 7.1.6] if coker ϕ is $(n,0)$-projective. $(n,0)$-projective (pre)covers and special $(n,0)$-projective precovers can be defined dually. It is proved that every right R-module has a special $(n,0)$-projective precover and a special $(n,0)$-injective preenvelope (see [11, Theorem 3.9]).

Theorem 3.1. *Let R be a right n-coherent ring* $(n \geq 1)$, *then the following are identical:*

(1) $rnpD(R)$

(2) $\sup\{npd(M)$*: M is a cyclic right R-module*$\}$
(3) $\sup\{npd(M)$*: M is any right R-module*$\}$
(4) $\sup\{npd(M)$*: M is an* $(n,0)$*-injective right R-module*$\}$
(5) $\sup\{id(M)$*: M is an* $(n,0)$*-injective right R-module*$\}$

Proof. $(1) \leq (2)$. We may assume $\sup\{npd(M)$: M is a cyclic right R-module$\} = m < \infty$. Let A be any finitely generated right R-module. We use induction on the number of generators of A. If A has l generators, let A' be a submodule generated by one of these generators. Then both A/A' and A' are finitely generated on less then l generators. Let N be any $(n,0)$-injective right R-module. Consider the short exact sequence $0 \to A' \to A \to A/A' \to 0$ which induces an exact sequence

$$Ext_R^{m+1}(A/A',N) \to Ext_R^{m+1}(A,N) \to Ext_R^{m+1}(A',N)$$

where

$$Ext_R^{m+1}(A/A',N) = Ext_R^{m+1}(A',N) = 0$$

by induction hypothesis. Thus $Ext_R^{m+1}(A,N) = 0$. So $npd(A) \leq m$.

$(2) \leq (3)$ is clear.

$(3) \leq (4)$. We may assume $\sup\{npd(M)$: M is an $(n,0)$-injective right R-module$\} = m < \infty$. Let A be any right R-module, then A has a special $(n,0)$-injective preenvelope by [11, Theorem 3.9], that is, there exists a short exact sequence $0 \to A \to E \to L \to 0$ with E $(n,0)$-injective and L $(n,0)$-projective. Therefore, $npd(A) \leq npd(E) \leq m$ by Proposition 3.2.

$(4) \leq (5)$. We may assume $\sup\{id(M)$: M is an $(n,0)$-injective right R-module$\} = m < \infty$. Let A and B be any $(n,0)$-injective right R-modules. Then $Ext_R^{m+1}(A,B) = 0$ since $id(B) \leq m$. So $npd(A) \leq m$ by Proposition 3.1.

$(5) \leq (1)$. We may assume $rnpD(R) = m < \infty$. Let M be an $(n,0)$-injective right R-module. Then $Ext_R^{m+1}(R/I,M) = 0$ for any right ideal I of R since $npd(R/I) \leq m$ by hypothesis. Hence $id(M) \leq m$, this completes the proof. ∎

Corollary 3.2. *Let* $n \geq 1$ *be a fixed integer. Then the following are equivalent for a right n-coherent ring R:*

(1) $rnpD(R) \leq m$.
(2) $npd(M) \leq m$ *for any* $(n,0)$*-injective right R-module M.*
(3) $npd(M) \leq m$ *for any injective right R-module M, and* $rnpD(R) < \infty$.
(4) $id(M) \leq m$ *for any* $(n,0)$*-injective right R-module M.*
(5) $id(M) \leq m$ *for all right R-module M that are both* $(n,0)$*-injective and* $(n,0)$*-projective, and* $rnpD(R) < \infty$.

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (4)$ holds by Theorem 3.1. $(2) \Rightarrow (3)$ and $(4) \Rightarrow (5)$ are clear.

$(5) \Rightarrow (4)$. Let M be any $(n,0)$-injective right R-module. By (5) and Theorem 3.1 (4), $npd(M) = m$ for a non-negative integer m. Note that every right R-module has a special $(n,0)$-projective precover by [11, Theorem 3.9], we obtain an exact sequence

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

where each P_t is both $(n,0)$-projective and $(n,0)$-injective, $t = 0, 1, \ldots, m$. Hence $id(P_t) \leq m$ by (5), $t = 0, 1, \ldots, m$. So $id(M) \leq m$.

(3) $\Rightarrow$ (2). Let M be any $(n,0)$-injective right R-module. By (3) and Theorem 3.1 (5), $id(M) = t$ for a non-negative integer t. Hence we get an injective resolution of M:

$$0 \to M \to E^0 \to E^1 \cdots \to E^{t-1} \to E^t \to 0.$$

By (3), $npd(E^i) \leq m$, i = 0, 1, ..., t. Hence we have $npd(M) \leq m$ by Proposition 3.2, as desired. ∎

Recall that an injective envelope $\phi : M \to E(M)$ of M has the *unique mapping property* [4] if for any homomorphism $f : M \to A$ with A injective, there is a unique homomorphism $g : E(M) \to A$ such that $g\phi = f$. The concept of an $(n,0)$-projective cover with the unique mapping property can be defined similarly.

Corollary 3.3. *Let $n \geq 1$ be a fixed integer. Then the following are equivalent for a right n-coherent ring R:*

(1) *R is right Noetherian.*
(2) *$rnpD(R) < \infty$ and every injective right R-module is $(n,0)$-projective.*
(3) *Every $(n,0)$-injective right R-module is $(n,0)$-projective.*
(4) *Every $(n,0)$-injective right R-module has an $(n,0)$-projective cover with the unique mapping property.*
(5) *Every $(n,0)$-injective right R-module has an injective envelope with the unique mapping property.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) holds by Corollary 3.2 and Corollary 2.3.

(1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5). Let M be any $(n,0)$-injective right R-module. Then M is $(n,0)$-projective and injective, since R is right Noetherian by (1). Thus (4) and (5) follows.

(4) $\Rightarrow$ (3). For any $(n,0)$-injective right R-module M, let $g : P \to M$ be the $(n,0)$-projective cover of M with the unique mapping property, where P is $(n,0)$-projective. Write K = ker$g$. Then K is $(n,0)$-injective by [6, Corollary 7.2.3] and [11, Theorem 3.9]. Hence there exists an $(n,0)$-projective cover $f : P^{'} \to K$ of K by (4). So, we obtain the following exact commutative diagram:

$$\begin{array}{ccccccccc} & & & & P^{'} & & & & \\ & & & \stackrel{f}{\swarrow} & \downarrow if & \searrow 0 & & & \\ 0 & \longrightarrow & K & \stackrel{i}{\longrightarrow} & P & \stackrel{g}{\longrightarrow} & M & \longrightarrow & 0 \end{array}$$

Since $g(if) = 0$, we have $if = 0$ by (4). Whence $K = \mathrm{Im}f \subseteq \ker(i) = 0$, that is, M is $(n,0)$-projective.

(5) $\Rightarrow$ (1). Let M be any $(n,0)$-injective right R-module. By Corollary 2.3, we need only to show that M is injective. Let $f : M \to E$ be the injective envelope of M with the unique mapping property. Write L = coker$f$. Since R is n-coherent, L is $(n,0)$-injective by [11, Theorem 4.1]. So there exists an injective envelope $g : L \to E^{'}$ of L by (5). Therefore we get the following exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \stackrel{f}{\longrightarrow} & E & \stackrel{\pi}{\longrightarrow} & L & \longrightarrow & 0 \\ & & & \stackrel{0}{\searrow} & \downarrow g\pi & \swarrow g & & & \\ & & & & E^{'} & & & & \end{array}$$

Since $(g\pi)f = 0$, we have $g\pi = 0$ by (5). Hence $L = \mathrm{Im}\pi \subseteq \ker(g) = 0$. So M is injective. This completes the proof. ∎

Recall that a short exact sequence $0 \to A \to B \to C \to 0$ is said to be *n-pure* [11] if $Hom(M,B) \to Hom(M,C) \to 0$ is exact for any n-presented module M. A submodule N of M is called an n-pure submodule if the sequence $0 \to N \to M \to M/N \to 0$ is n-pure.

Proposition 3.3. *Let $n \geq 1$ be a fixed integer and R a right n-coherent ring. Observe the following statements:*

(1) $rnpD(R) \leq 1$.
(2) *For any n-pure submodule N of an injective right R-module E, the quotient E/N is injective (i.e., $id(N) \leq 1$).*
(3) *Every submodule of an $(n,0)$-projective right R-module is $(n,0)$-projective.*
(4) *Every right ideal of R is $(n,0)$-projective.*
(5) *For any pure submodule N of an injective right R-module E, the quotient E/N is injective.*
(6) *Every submodule of an FP-projective right R-module is FP-projective.*
(7) *Every right ideal of R is FP-projective.*

Then: $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$ *and* $(2) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7)$.

Proof. $(1) \Rightarrow (2)$. Let N be an n-pure submodule of an injective right R-module E. Then it is easy to see that N is $(n,0)$-injective. Hence $id(N) \leq 1$ by Theorem 3.1 (5). So the short exact sequence $0 \to N \to E \to E/N \to 0$ implies that E/N is injective.

$(2) \Rightarrow (3)$. Let L be any $(n,0)$-injective right R-module. Then it is clear that L is an n-pure submodule of its injective envelope $E(L)$, and hence $id(L) \leq 1$ by (2). If N is a submodule of an $(n,0)$-projective right R-module M, then the exactness of the sequence

$$0 = Ext_R^1(M,L) \to Ext_R^1(N,L) \to Ext_R^2(M/N,L) = 0$$

implies that $Ext_R^1(N,L) = 0$, and so N is $(n,0)$-projective.

$(4) \Rightarrow (1)$. Let I be an ideal of R. The exact sequence $0 \to I \to R \to R/I \to 0$ implies that $npd(R/I) \leq 1$ by Proposition 3.1. So (1) holds by Theorem 3.1 (2).

$(2) \Rightarrow (5)$. It is easy to verify that every pure right R-module is n-pure. So (5) follows.

$(5) \Rightarrow (6)$ is similar to that of $(2) \Rightarrow (3)$, $(3) \Rightarrow (4)$ and $(6) \Rightarrow (7)$ are trivial. ∎

It is known that if R is a right coherent ring, then $fd(M) = pd(M)$ for any finitely present right R-module M (see [7, Lemma 5]). Mao and Ding (see [10, Proposition 4.1]) proved that if R is also self-FP-injective, then $fd(M) = pd(M)$ for any FP-projective right R-module M. Here we have the following

Proposition 3.4. *Let n be a fixed positive integer. If R is a right n-coherent and right self-$(n,0)$-injective ring, then $fd(M) = pd(M)$ for any $(n,0)$-projective right R-module M.*

Proof. It is enough to show that $fd(M) \geq pd(M)$. We may assume that $fd(M) = m < \infty$. Then there exists an exact sequence

$$0 \to F_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with P_0, P_1, $\cdots$, P_{m-1} projective and F_m flat. Consider the short exact sequence $0 \to K \to P \to F_m \to 0$ where P is projective. By [9, Theorem 4.85], the short exact sequence above is pure, and hence n-pure. By Proposition 2.1, P is $(n,0)$-injective. So K is $(n,0)$-injective by [11, Proposition 3.6]. Since M is $(n,0)$-projective, so is F_m. Thus the exactness of the sequence

$$0 \to Hom_R(F_m,K) \to Hom_R(P,K) \to Hom_R(K,K) \to Ext_R^1(F_m,K) = 0$$

implies that the sequence $0 \to K \to P \to F_m \to 0$ is split exact, and so F_m is projective, that is, $pd(M) \leq m$. This completes the proof. ∎

Proposition 3.5. *Let $n \geq 1$ be a fixed integer and R a right n-coherent ring. If $rnpD(R) \leq m$, then R is a right m-coherent ring.*

Proof. The case $m = 0$ holds by Corollary 2.3. Suppose $m \geq 1$. Let M be an m-presented right R-module, then M has a free resolution

$$F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

with each F_i finitely generated free. Write $K_m = \ker(F_{m-1} \to F_{m-2})$, then

$$Ext_R^1(K_m, N) \cong Ext_R^{m+1}(M, N) = 0$$

for any FP-injective right R-module N, since $rnpD(R) \leq m$ and every FP-injective right R-module is $(n, 0)$-injective. Note that K_m is finitely generated. We obtain K_m is finitely presented by Theorem 2.1. This implies that M is $(m+1)$-presented, and so R is a right m-coherent ring. ∎

To prove the next main result, we need four lemmas.

Lemma 3.1. *Let $f\colon R \to S$ be a surjective ring homomorphism. If M_S is a right S-module (hence a right R-module) and A_R is a right R-module, then the following statements hold:*

(1) $M \otimes_R S_S \cong M_S$.
(2) *If A_R is a finitely generated right R-module, then $A \otimes_R S_S$ is a finitely generated right S-module.*
(3) *M_S is a finitely generated right S-module if and only if M_R is a finitely generated right R-module.*

Proof. (1). Easy.

(2). Clearly, S is a cyclic R-module. Suppose $x_1, x_2, \cdots, x_n$ are generators of A. Then it is easy to verify that $x_1 \otimes 1_S, x_2 \otimes 1_S, \cdots, x_n \otimes 1_S$ are generators of $A \otimes_R S_S$, where 1_S denotes the identity of S. Thus $A \otimes_R S_S$ is a finitely generated right S-module.

(3). If M_S is a finitely generated right S-module, and suppose $x_1, x_2, \cdots, x_n$ are generators of M, then $M = x_1S + x_2S + \cdots + x_nS$. So $M = x_1R + x_2R + \cdots + x_nR$ since $f\colon R \to S$ is surjective. Hence M_R is a finitely generated right R-module. The converse holds by (1) and (2). ∎

Lemma 3.2. *Let $f\colon R \to S$ be a surjective ring homomorphism, n a non-negative integer, and M a right S-module. If both S_R and ${}_RS$ are projective, then M_S is an n-presented right S-module if and only if M_R is an n-presented right R-module. (Note that the case $n = 1$ has been proven in* [10, Lemma 3.13]*.)*

Proof. The case $n = 0$ follows by Lemma 3.1. So next we assume $n > 0$.

"⇒". Suppose M is an n-presented right S-module. Then there exists an exact sequence

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

of right S-modules with K finitely generated, and P_i finitely generated projective, $i = 0, 1, \cdots, n-1$. By Lemma 3.1, each P_i and K are finitely generated right R-modules. Since S_R is projective, we have each P_i is a projective right R-module. So, M is an n-presented right R-module.

" $\Leftarrow$". Assume M is an n-presented right R-module. Then there exists an exact sequence

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

of right R-modules with K finitely generated, and P_i finitely generated projective, i = 0, 1, $\cdots$, $n-1$. Since ${}_RS$ is projective, the sequence

$$0 \to K \otimes_R S_S \to P_{n-1} \otimes_R S_S \to \cdots \to P_1 \otimes_R S_S \to P_0 \otimes_R S_S \to M \otimes_R S_S \to 0$$

is exact. By Lemma 3.1, $M \otimes_R S_S \cong M_S$, and both $K \otimes_R S_S$ and each $P_i \otimes_R S_S$ are finitely generated S-modules. Since each P_i is a projective right R-module, we have each $P_i \otimes_R S_S$ is a projective right S-module. So M is an n-presented right S-module. ∎

Let n and d be non-negative integers. Recall that a left R-module A is called (n,d)-*flat* [16], in case $Tor^R_{d+1}(B,A) = 0$ for any n-presented right R-module B.

Lemma 3.3. *Let f: $R \to S$ be a surjective ring homomorphism, M_S a right S-module and ${}_SA$ a left S-module. If both S_R and ${}_RS$ are projective, then the following statements hold for any non-negative integers n and d:*

(1) *M_S is an (n,d)-injective right S-module if and only if M_R is an (n,d)-injective right R-module.*
(2) *${}_SA$ is an (n,d)-flat left S-module if and only if ${}_RA$ is an (n,d)-flat left R-module.*
(3) *If R is a right n-coherent ring, then S is a right n-coherent ring.*

Proof. (1). "$\Rightarrow$". Suppose M_S is an (n,d)-injective right S-module. Let N_R be any n-presented right R-module. Then, using an argument similar to that in Lemma 3.2, we get that $N \otimes_R S_S$ is an n-presented right S-module. By [13, Theorem 11.65], we have

$$Ext^{d+1}_R(N_R,M_R) \cong Ext^{d+1}_S(N \otimes_R S_S,M_S) = 0.$$

Therefore M_R is an (n,d)-injective right R-module.

"$\Leftarrow$". Assume M_R is an (n,d)-injective right R-module. Let N_S be any n-presented right S-module. Then $N \otimes_R S_S \cong N_S$ by Lemma 3.1 and N_R is an n-presented right R-module by Lemma 3.2. Again by [13, Theorem 11.65], we have

$$Ext^{d+1}_S(N_S,M_S) \cong Ext^{d+1}_S(N \otimes_R S_S,M_S) \cong Ext^{d+1}_R(N_R,M_R) = 0.$$

Therefore M_S is an (n,d)-injective right S-module.

(2). "$\Rightarrow$". If ${}_SA$ is an (n,d)-flat left S-module. Let B_R be any n-presented right R-module. Then $B \otimes_R S_S$ is an n-presented right S-module. By [13, Corollary 11.63], we have

$$Tor^R_{d+1}(B_R,{}_RA) \cong Tor^S_{d+1}(B \otimes_R S_S,{}_SA) = 0.$$

Therefore ${}_RA$ is an (n,d)-flat left R-module.

"$\Leftarrow$". If ${}_RA$ is an (n,d)-flat left R-module. Let B_S be any n-presented right R-module. Then $B \otimes_R S_S \cong B_S$ by Lemma 3.1 and B_R is an n-presented right R-module by Lemma 3.2. By [13, Corollary 11.63], we have

$$Tor^S_{d+1}(B_S,{}_SA) \cong Tor^S_{d+1}(B \otimes_R S_S,{}_SA) \cong Tor^R_{d+1}(B_R,{}_RA) = 0.$$

Therefore ${}_SA$ is an (n,d)-flat left S-module.

(3). Let M_S be an n-presented right R-module, then M_R is an n-presented right R-module by Lemma 3.2. Thus M_R is an $(n+1)$-presented right R-module since R is a right n-coherent ring. Therefore M_S is an $(n+1)$-presented right S-module again by Lemma 3.2, and so S is a right n-coherent ring. ∎

We list the following lemma proved in [10, Lemma 3.14] for convenient using.

Lemma 3.4. [10, Lemma 3.14]. *Let R and S be rings. Every right $(R\oplus S)$-module has a unique decomposition that $M = A\oplus B$, where $A = M(R,0)$ is a right R-module and $B = M(0,S)$ is a right S-module via $xr = x(r,0)$ for $x\in A$, $r\in R$, and $ys = y(0,s)$ for $y\in B$, $s\in S$.*

We are now in a position to prove the following main result.

Theorem 3.2. *Let S and T be rings, and $n\geq 1$ a fixed integer. If $S\oplus T$ is a right n-coherent ring, then*

$$rnpD(S\oplus T) = \sup\{rnpD(S), rnpD(T).$$

Proof. For convenience, we write $R = S\oplus T$. Since R is a right n-coherent ring, we have both S and T are right n-coherent rings by Lemma 3.3.

We first show that $rnpD(R)\leq \sup\{rnpD(S), rnpD(T)\}$. We may assume $\sup\{rnpD(S), rnpD(T)\} = m < \infty$. Let M be a right (R)-module and N any $(n,0)$-injective right (R)-module. Then $N = A\oplus B$, where A is a right S-module and B is a right T-module by Lemma 3.4. Note that both A and B are $(n,0)$-injective right (R)-modules. Hence A is an $(n,0)$-injective right S-module and B is an $(n,0)$-injective right T-module by Lemma 3.3. By [13, Theorem 11.65], we have

$$\begin{aligned} Ext_R^{m+1}(M,N) &\cong Ext_R^{m+1}(M,A)\oplus Ext_R^{m+1}(M,B)\\ &\cong Ext_S^{m+1}(M\otimes_R S_S,A)\oplus Ext_T^{m+1}(M\otimes_R T_T,B) = 0,\end{aligned}$$

and hence $rnpD(R)\leq \sup\{rnpD(S), rnpD(T)\}$.

Next we prove that $rnpD(R)\geq \sup\{rnpD(S), rnpD(T)\}$. We may assume $rnpD(R) = m<\infty$. Let M be a right S-module and N any $(n,0)$-injective right S-module. Then N is an $(n,0)$-injective right (R)-module by Lemma 3.3. By Lemma 3.1, $M\otimes_R S_S\cong M_S$. Again by [13, Theorem 11.65], we have

$$Ext_S^{m+1}(M,N)\cong Ext_S^{m+1}(M\otimes_R S_S,N)\cong Ext_R^{m+1}(M,N) = 0.$$

Therefore $rnpD(R)\geq rnpD(S)$. Similarly for $rnpD(R)\geq rnpD(T)$, and hence $rnpD(R)\geq \sup\{rnpD(S), rnpD(T)\}$. This completes the proof. ∎

Remark 3.1. Let R_1, R_2, $\cdots$, R_m be rings and n a positive integer. The theorem above shows that $rnpD(\bigoplus_{i=1}^m R_i) = \sup\{rnpD(R_1), rnpD(R_2), \cdots, rnpD(R_m)\}$ if $\bigoplus_{i=1}^m R_i$ is an n-coherent ring. In particular, we obtain the known result that $\bigoplus_{i=1}^m R_i$ is right Noetherian if and only if each R_i is right Noetherian. But in general $rnpD(\bigoplus_{i=1}^\infty R_i)\neq \sup_{i\geq 1}\{rnpD(R_i)\}$. For example, Z_2 is a field of two elements, but $\bigoplus_{i=1}^\infty Z_2$ is not Noetherian.

Lemma 3.5. *Assume n and d are non-negative integers, R is a commutative ring, and P is any prime ideal of R. Let R_P denote the localization of R at P, M is an R_P-module (M may be viewed as an R-module), and A is an R-module. Then the following statements hold:*

(1) *If A is an n-presented R-module, then A_P is an n-presented R_P-module.*
(2) *If M is an (n,d)-injective R_P-module, then M is an (n,d)-injective R-module.*
(3) *If M is an (n,d)-flat R_P-module, then M is an (n,d)-flat R-module.*
(4) *If A is an (n,d)-projective R-module, then A_P is an (n,d)-projective R_P-module.*

Proof. (1). Suppose A is an n-presented R-module. Then there exists an exact sequence of R-modules

$$F_n\to F_{n-1}\to\cdots\to F_1\to F_0\to A\to 0$$

where each F_i is finitely generated projective, $i = 0, 1, \cdots, n$. It gives rise to the exactness of the sequence

$$(F_n)_{\mathrm{P}} \to (F_{n-1})_{\mathrm{P}} \to \cdots \to (F_1)_{\mathrm{P}} \to (F_0)_{\mathrm{P}} \to A_{\mathrm{P}} \to 0$$

of R_{P}-modules. By [6, Remark 2.2.5], each $(F_i)_{\mathrm{P}}$ is a finitely generated projective R_{P}-module, $i = 0, 1, \cdots, n$. Hence A_{P} is an n-presented R_{P}-module.

(2). Assume M is an (n,d)-injective R_{P}-module. Let N be any n-presented R-module, then N_{P} is an n-presented R_{P}-module by (1). Note that R_{P} is a flat R-module and $R_{\mathrm{P}} \otimes_R N \cong N_{\mathrm{P}}$. By [13, Theorem 11.65], we have

$$Ext_R^{d+1}(N,M) \cong Ext_{R_{\mathrm{P}}}^{d+1}(R_{\mathrm{P}} \otimes_R N, M) \cong Ext_{R_{\mathrm{P}}}^{d+1}(N_{\mathrm{P}}, M) = 0.$$

Therefore M is an (n,d)-injective R-module.

(3). Similar to that of (2).

(4). Suppose A is an (n,d)-projective R-module. Let B be any (n,d)-injective R_{P}-module, then B is an (n,d)-injective R-module by (2). Note that $A_{\mathrm{P}} \cong R_{\mathrm{P}} \otimes_R A$. By [13, Theorem 11.65], we have

$$Ext_{R_{\mathrm{P}}}^1(A_{\mathrm{P}}, B) \cong Ext_{R_{\mathrm{P}}}^1(R_{\mathrm{P}} \otimes_R A, B) \cong Ext_R^1(A,B) = 0.$$

Therefore A_{P} is an (n,d)-projective R_{P}-module. ∎

Corollary 3.4. *Let R be a commutative ring and P any prime ideal of R. If M is an R_{P}-module, then the following statements hold:*

(1) *M is an injective R_{P}-module if and only if M is an injective R-module.*
(2) *M is a flat R_{P}-module if and only if M is a flat R-module.*

Proof. (1). If M is an injective R_{P}-module, then M is an injective R-module by Lemma 3.5. If M is an injective R-module, then M_{P} is an injective R_{P}-module by [13, Theorem 3.76]. Note that $M \cong M_{\mathrm{P}}$ as R_{P}-modules. Thus (1) follows.

(2). Similar to that of (1). ∎

Theorem 3.3. *Let $n \geq 1$ be a fixed integer and R a commutative n-coherent ring. If P is any prime ideal of R, then $npD(R_{\mathrm{P}}) \leq npD(R)$.*

Proof. We may assume $npD(R) = t < \infty$. Let M be any R_{P}-module. Note that M may be viewed as an R-module. Thus $npd(M_R) \leq t$. If $t = 0$, then M is an $(n,0)$-projective R-module. Since $M \cong M_{\mathrm{P}}$ as R_{P}-modules, we have M is an $(n,0)$-projective R_{P}-module by Lemma 3.5, and so the theorem follows. Next we assume $t \geq 1$. By Proposition 3.1 (5), There exists an exact sequence

$$0 \to K \to F_{t-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

of R-modules, where each F_i is a projective R-module, $i = 1, 2, \cdots, t-1$, and K is an $(n,0)$-projective R-module. The above sequence induces an R_{P}-module exact sequence

$$0 \to K_{\mathrm{P}} \to (F_{t-1})_{\mathrm{P}} \to \cdots \to (F_1)_{\mathrm{P}} \to (F_0)_{\mathrm{P}} \to M_{\mathrm{P}} \to 0.$$

By [6, Remark 2.2.5], each $(F_i)_{\mathrm{P}}$ is a projective R_{P}-module, $i = 1, 2, \cdots, t-1$. Note that K_{P} is an $(n,0)$-projective R_{P}-module by Lemma 3.5. Thus, for any $(n,0)$-injective R_{P}-module N, we have

$$Ext_{R_{\mathrm{P}}}^{t+1}(M_{\mathrm{P}}, N) \cong Ext_{R_{\mathrm{P}}}^1(K_{\mathrm{P}}, N) = 0$$

and so $npd(M_{\mathrm{P}})_{R_{\mathrm{P}}} \leq t$ by definition. Since $M \cong M_{\mathrm{P}}$ as R_{P}-modules, $npd(M) \leq t$. Therefore $npD(R_{\mathrm{P}}) \leq npD(R)$, and we complete the proof. ∎

Remark 3.2.

(1) The theorem above shows the well-known result that any localization of a Noetherian ring is again Noetherian. But in general $npD(R) \neq \sup\{npD(R_P): P$ is a prime ideal of $R\}$. For example, take R to be the direct product of countably many copies of $\mathbf{Z}_2$, then R is not Noetherian. Thus $npD(R) > 0$. However, $npD(R_P) = 0$ for any prime ideal of R.

(2) Let R be a commutative ring and P any prime ideal of R. Corollary 3.4 shows that if M is an R_P-module, then M is a flat (resp. injective) R_P-module if and only if M is a flat (resp. injective) R-module. But, in general, a projective R_P-module need not be a projective R-module. For example, R_P is a projective R_P-module, but R_P need not be a projective R-module.

Acknowledgement. This research was supported by the Scientific Research Fund of Education Department of Guangxi Province (No. 201203YB224).

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