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Locally Maximal Regular Subsemibands of $\mathscr{SOP}_n$

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Abstract. In this paper we describe locally maximal regular subsemibands of finite singular orientation-preserving transformation semigroups and completely obtain their classification.

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1. Introduction

Let $[n] = \{1, 2, \ldots, n\}$ ordered in the standard way. We denote by $\mathscr{T}_n$ the semigroup of all full transformations of $[n]$, and by $\mathscr{S}ing_n$ its subsemigroup of all singular transformations of $[n]$. We say that a transformation α in $\mathscr{S}ing_n$ is order-preserving if, for all $x, y \in [n]$, $x \le y$ implies $x\alpha \le y\alpha$. We denote by $\mathscr{O}_n$ the subsemigroup of $\mathscr{S}ing_n$ of all order-preserving singular transformations. Let

$$\mathscr{O}_n^k = \{a^{-k} f a^k : f \in \mathscr{O}_n\}, \quad k \in [n],$$

where $a = (123\cdots n)$ is the fixed generator of the cyclic group $\mathscr{Z}_n$. Catarino and Higgins [2] proved that $\mathscr{O}_n^k$ and $\mathscr{O}_n$ are isomorphic subsemigroups of $\mathscr{T}_n$.

Let $\alpha \in \mathscr{T}_n$, we say that α is *orientation-preserving* if the sequence $(1\alpha, 2\alpha, \ldots, n\alpha)$ is cyclic, that is, there exists no more than one subscript i such that $i\alpha > (i+1)\alpha$. The notion of an orientation-preserving transformation was introduced by McAlister in [16] and, independently, by Catarino and Higgins in [2]. We denote by $\mathscr{OP}_n$ the subsemigroup of $\mathscr{T}_n$ of all orientation-preserving full transformations of $[n]$, and by $\mathscr{SOP}_n$ the subsemigroup of $\mathscr{S}ing_n$ of all orientation-preserving singular transformations of $[n]$.

A semigroup S is called *idempotent-generated* or *semiband* if it is generated by its idempotents. The latter term was introduced by F. Pastijn [17].

Let S be a semigroup. The set of all subsemigroups or subsemigroups with particular properties of S is partially ordered with respect to inclusions, and the maximal elements of

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this set are called *maximal subsemigroups* or *maximal subsemigroups with particular properties* of S. The history of the problem to classify (or describe) all maximal subsemigroups or maximal subsemigroups with particular properties of a given semigroup goes back at least to [11]. Various special subsemigroups of $\mathscr{T}_n$ have been studied by many authors (see for example [1–4, 6–10, 13, 18–33]). In recently years the problem was studied for several classes of transformation semigroups in [1, 3–6, 9, 12, 14, 15, 18–33]. In particular, Zhao, Bo and Mei [28] classified completely locally maximal subsemibands of $\mathscr{SOP}_n$. Recently, Zhao [29] characterized completely maximal subsemibands of $\mathscr{SOP}_n$. Further, Zhao [30] completely described maximal regular subsemibands of $\mathscr{SOP}_n$. In this paper, we aim to give more insight into the subsemigroup structure of the semigroup $\mathscr{SOP}_n$ by characterizing the locally maximal regular subsemibands of $\mathscr{SOP}_n$.

Remark 1.1. In the paper it will always be clear from context when additions are modular.

For convenience, we introduce the following notation. From Catarino and Higgins [2], Green's equivalences in $\mathscr{SOP}_n$ can be characterized as:

$$\alpha\mathscr{L}\beta \Leftrightarrow \mathrm{Im}(\alpha)=\mathrm{Im}(\beta),$$
$$\alpha\mathscr{R}\beta \Leftrightarrow \mathrm{Ker}(\alpha)=\mathrm{Ker}(\beta),$$
$$\alpha\mathscr{J}\beta \Leftrightarrow |\mathrm{Im}(\alpha)|=|\mathrm{Im}(\beta)|.$$

Thus $\mathscr{SOP}_n$ has $n-1$ $\mathscr{J}$-classes: $J_1, J_2, \ldots, J_{n-1}$, where

$$J_r=\{\alpha\in\mathscr{SOP}_n : |\mathrm{Im}(\alpha)|=r\}.$$

Obviously, we have $\mathscr{SOP}_n=\bigcup_{r=1}^{n-1} J_r$. Let

$$K(n,r)=\{\alpha\in\mathscr{SOP}_n : |\mathrm{Im}(\alpha)|\leq r\}=J_1\cup J_2\cdots\cup J_r,$$

where $1\leq r\leq n-1$. The sets $K(n,r)$ are the two-sided ideals of $\mathscr{SOP}_n$. We want to focus on the class J_{n-1} at the top of the semigroup $\mathscr{SOP}_n$. As in [29], we use the notation

$$L_k=\{\alpha\in\mathscr{SOP}_n : \mathrm{Im}(\alpha)=[n]\setminus\{k\}\},$$
$$R_{(i,i+1)}=\{\alpha\in\mathscr{SOP}_n : \text{the unique non-singleton class of } \mathrm{Ker}(\alpha) \text{ is } \{i,i+1\}\}$$

for $\mathscr{L}$-classes and $\mathscr{R}$-classes in J_{n-1}. Hence J_{n-1} has n $\mathscr{L}$-classes $L_1, L_2, \ldots, L_n$ and n $\mathscr{R}$-classes $R(1,2), R(2,3), \ldots, R(n-1,n), R(n,1)$.

Gomes and Howie [10] used the notation $[i\to i-1]$ for the decreasing idempotent e defined by $ie=i-1, xe=x$ $(x\neq i)$. They also used the notation $[i\to i+1]$ for the increasing idempotent f defined by $if=i+1, xf=x$ $(x\neq i)$.

As usual, we denote by $E(S)$ the set of all idempotents of a subset S of $\mathscr{SOP}_n$. Employing above notation the set $E(J_{n-1})$ consists of n decreasing idempotents $[i\to i-1]$ $(i\in[n])$ and n increasing idempotents $[i\to i+1]$ $(i\in[n])$. Let $E^+_{n-1}=\{[i\to i+1] : i\in[n]\}$ and $E^-_{n-1}=\{[i\to i-1] : i\in[n]\}$ be the increasing and decreasing idempotent sets, respectively. Then $E(J_{n-1})=E^+_{n-1}\cup E^-_{n-1}$. Note that $[0\to 1]=[n\to 1]$, $[1\to 0]=[1\to n]$, $[n\to n+1]=[n\to 1]$, $[n+1\to n]=[1\to n]$, etc., by Remark 1.1.

With above notation, we have the following simple observations:

Lemma 1.1. *Let* $n\geq 3$. *Then*

$$E(R_{(k,k+1)})=\{[k\to k+1],[k+1\to k]\},\quad k\in[n],$$
$$E(L_k)=\{[k\to k-1],[k\to k+1]\},\quad k\in[n].$$

Lemma 1.2. *Let $n \geq 3$. Then*

$$E(\mathscr{O}_n^k \cap J_{n-1}) = E(J_{n-1}) \backslash \{[k \to k+1], [k+1 \to k]\}, \quad k \in [n].$$

2. Locally maximal regular subsemibands of $\mathscr{SOP}_n$

Let I be a subset of $E(J_{n-1})$. A subsemiband $\langle I \rangle$ of $\mathscr{SOP}_n$ is called *locally maximal regular subsemiband* of $\mathscr{SOP}_n$ if $\langle I \rangle$ is a regular, and any regular subsemiband $\langle J \rangle$ ($J \subseteq E(J_{n-1})$) of $\mathscr{SOP}_n$ properly containing $\langle I \rangle$ must be $\mathscr{SOP}_n$. In this section, we obtain a classification of locally maximal regular subsemibands of $\mathscr{SOP}_n$.

Our main result is

Theorem 2.1. *Let $n \geq 3$. Then each one of the following types* (*A*), (*B*) *is a locally maximal regular subsemiband of $\mathscr{SOP}_n$:*

(A) $S_i = \mathscr{O}_n^i$, $i \in [n]$.
(B) $T_i = \{\alpha \in \mathscr{SOP}_n : i\alpha = i\}$, $i \in [n]$.

Conversely, every locally maximal regular subsemiband of $\mathscr{SOP}_n$ is one of types (A), (B). *$\mathscr{SOP}_n$ has $2n$ locally maximal regular subsemibands.*

To prove Theorem 2.1 we need the following series of lemmas. First, we need the following notation.

As in [2], let $k \in \{0,1,2,\ldots,n-1\}$, define a total order $\leq_k$ on $[n]$ by

$$k+1 \leq_k k+2 \leq_k \cdots \leq_k n \leq_k 1 \leq_k \cdots \leq_k k.$$

We write $i <_k j$ if $i \leq_k j$ and $i \neq j$. Note that $i \leq_k j$ if and only if $i-k \leq j-k$. We say that $A = (a_1, a_2, \ldots, a_t)$ is cyclic with respect to $\leq_k$ if there exist no more than one subscript i such that $a_{i+1} <_k a_i$. Clearly $A = (a_1, a_2, \ldots, a_t)$ is cyclic with respect to $\leq_k$ if and only if there exists $j \in \{0, \ldots, t-1\}$ such that

$$a_{j+1} \leq_k \cdots \leq_k a_t \leq_k a_1 \leq_k \cdots \leq_k a_j.$$

The following lemma was proved by Catarino and Higgins [2, Lemma 1.4].

Lemma 2.1. *Let $A = (a_1, a_2, \ldots, a_t)$ be any sequence of elements from $[n]$. Then the following are equivalent:*

(a) *A is cyclic with respect to $\leq_0 = \leq$.*
(b) *A is cyclic with respect to $\leq_k$ for some k.*
(c) *A is cyclic with respect to $\leq_k$ for all k.*

Let

(2.1) $$M_{i,j} = \langle E(J_{n-1}) \backslash \{[i \to i+1], [j \to j-1]\} \rangle, \quad i,j \in [n],$$

(2.2) $$S_j^k = \{\alpha \in \mathscr{O}_n^k : (\forall x \in [n]) j \leq_k x \Longrightarrow j \leq_k x\alpha\}, \quad j \neq k+1 (mod(n)),$$

(2.3) $$T_j^k = \{\alpha \in \mathscr{O}_n^k : (\forall x \in [n]) x \leq_k j \Longrightarrow x\alpha \leq_k j\}, \quad j \neq k (mod(n)).$$

The following lemma establishes the relationships among $M_{i,j}$, S_j^i and T_i^{j-1}.

Lemma 2.2. *Let $M_{i,j}$, S_j^i and T_i^{j-1} be defined as* (2.1), (2.2) *and* (2.3), *respectively. Then*

$$M_{i,j} \cap J_{n-1} = (S_j^i \cup T_i^{j-1}) \cap J_{n-1}, \; j \neq i+1 \; (mod(n)).$$

Proof. See [29, Lemma 3.5]. ∎

The following lemma is the main result of [28, Theorem 3.5].

Lemma 2.3. *Let S_j^k and T_j^k be defined as* (2.2) *and* (2.3)*, respectively. Then*

(1) $S_j^k = \langle E(\mathscr{O}_n^k \cap J_{n-1}) \backslash \{[j \to j-1]\} \rangle,\ j \neq k+1\ (mod(n))$.

(2) $T_j^k = \langle E(\mathscr{O}_n^k \cap J_{n-1}) \backslash \{[j \to j+1]\} \rangle,\ j \neq k\ (mod(n))$.

Proof. See [28, Theorem 3.5]. ∎

We can use Lemmas 1.2, 2.1 and 2.3 to obtain the following.

Lemma 2.4. *Let $M_{i,i}$ be defined as* (2.1)*. Then*

$$M_{i,i} = \{\alpha \in \mathscr{SOP}_n : i\alpha = i\}.$$

Proof. Let $D_i = \{\alpha \in \mathscr{SOP}_n : i\alpha = i\}$ and $F = E(J_{n-1}) \backslash \{[i \to i+1], [i \to i-1]$. Then $M_{i,i} = \langle F \rangle$. It is easy to verify that D_i is a subsemigroup of $\mathscr{SOP}_n$. Note that $F \subseteq D_i$. Thus $M_{i,i} = \langle F \rangle \subseteq D_i$.

For the reverse containment, let T_i^{i-1} be defined as (2.3). By Lemmas 1.2 and 2.3, we have

$$T_i^{i-1} = \langle E(J_{n-1}) \backslash \{[i-1 \to i],\ [i \to i-1],\ [i \to i+1]\} \rangle.$$

It follows easily that

$$M_{i,i} = \langle F \rangle = \langle T_i^{i-1} \cup \{[i-1 \to i]\} \rangle. \tag{2.4}$$

By the definition of T_i^{i-1}, we easily deduce that

$$T_i^{i-1} = \{\alpha \in \mathscr{O}_n^{i-1} : i\alpha = i\}. \tag{2.5}$$

We now prove that $D_i \subseteq M_{i,i}$. Let $\alpha \in D_i$. We partition into two cases.

Case 1. $|\mathrm{Im}(\alpha)| = 1$. Clearly, $\alpha = \begin{pmatrix} [n] \\ i \end{pmatrix}$ (since $\alpha \in D_i$, we have $i\alpha = i$). Then, by (2.4) and (2.5),

$$\alpha = \begin{pmatrix} [n] \\ i \end{pmatrix} \in \{\alpha \in \mathscr{O}_n^{i-1} : i\alpha = i\} = T_i^{i-1} \subseteq M_{i,i}.$$

Case 2. $|\mathrm{Im}(\alpha)| = r \geq 2$. From [2, Theorem 3.3], we know that α can be expressed as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $A_s = \{a_s, a_s+1, \ldots, a_{s+1}-1\}$, $s = 1, 2, \ldots, r$, the subset $\{a_1, a_2, \ldots, a_r\}$ is an initial points set of kernel classes $A_1, A_2, \ldots, A_r$, $a_1 < a_2 < \cdots < a_r$ and $(b_1, b_2, \ldots, b_r)$ is cyclic. Since $\alpha \in D_i$, we have $i\alpha = i$. Then there exist $k \in \{1, 2, \ldots, r\}$ such that $b_k = i$ and $i \in A_k$. Note that $A_k = \{a_k, a_k+1, \ldots, a_{k+1}-1\}$. We may now partition into two cases according to $a_k = i$ or $a_k \neq i$.

Case 2.1. $a_k = i$. Note that $b_k = i$ and $(b_1, b_2, \ldots, b_r)$ is a cyclic. From Lemma 2.1, we easily deduce that

$$i = b_k \leq_{i-1} \cdots \leq_{i-1} b_r \leq_{i-1} b_1 \leq_{i-1} \cdots \leq_{i-1} b_{k-1}. \tag{2.6}$$

Note that $A_k = \{i, i+1, \ldots, a_{k+1}-1\}$ and $A_{k-1} = \{a_{k-1}, a_{k-1}+1, \ldots, i-1\}$ (since $a_k - 1 = i-1$). Then, by (2.5), (2.6) and the definition of $\mathscr{O}_n^{i-1}$,

$$\alpha = \begin{pmatrix} A_k & A_{k+1} & \cdots & A_r & A_1 & \ldots & A_{k-1} \\ b_k & b_{k+1} & \cdots & b_r & b_1 & \ldots & b_{k-1} \end{pmatrix} \in T_i^{i-1}.$$

Thus, by (2.4), $\alpha \in T_i^{i-1} \subseteq M_{i,i}$.

Case 2.2. $a_k \neq i$. Since $i \in A_k$ and $A_k = \{a_k, a_k+1, \ldots, a_{k+1}-1\}$, we have $i-1 \in A_k$. Let $A_k^* = \{a_k, a_k+1, \ldots, i-1\}$ and $A_k^{**} = \{i, \ldots, a_{k+1}-1\}$. Then $A_k = A_k^* \cup A_k^{**}$. Note that $b_k = i$ and $A_s = \{a_s, a_s+1, \ldots, a_{s+1}-1\}$, $s = 1, 2, \ldots, r$. Let

$$\beta = \begin{pmatrix} A_k^{**} & A_{k+1} & \cdots & A_r & A_1 & \cdots & A_{k-1} & A_k^* \\ i & a_{k+1} & \cdots & a_r & a_1 & \cdots & a_{k-1} & i-1 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} A_k^{**} & A_{k+1} & \cdots & A_r & A_1 & \cdots & A_{k-2} & A_{k-1} \cup A_k^* \\ b_k & b_{k+1} & \cdots & b_r & b_1 & \cdots & b_{k-2} & b_{k-1} \end{pmatrix}.$$

Then clearly $\alpha = \beta[i-1 \to i]\gamma$. Note that $[i-1 \to i] \in M_{i,i}$ (by (2.4)). To prove that $\alpha \in M_{i,i}$, it suffices to prove that $\gamma \in M_{i,i}$ and $\beta \in M_{i,i} \cup \{1_{[n]}\}$, where $1_{[n]}$ be the identity mapping on $[n]$. By (2.5), (2.6) and the definition of $\mathscr{O}_n^{i-1}$, we have $\gamma \in T_i^{i-1}$. Thus, by (2.4), $\gamma \in T_i^{i-1} \subseteq M_{i,i}$. Since $i-1, i \in A_k$, we have $a_k \leq i-1 < i \leq a_{k+1}-1 < a_{k+1}$. Note that $a_1 < a_2 < \cdots < a_r$. It easily follows from Lemma 2.1 that

$$i \leq_{i-1} a_{k+1} \leq_{i-1} \cdots \leq_{i-1} a_r \leq_{i-1} a_1 \leq_{i-1} \cdots \leq_{i-1} a_{k-1} \leq_{i-1} a_k \leq_{i-1} i-1.$$

If $r = n-1$, then clearly $\beta = 1_{[n]}$; if $r \leq n-2$, then, by (2.5) and the definition of $\mathscr{O}_n^{i-1}$, $\beta \in T_i^{i-1}$. Thus, by (2.4), $\beta \in T_i^{i-1} \subseteq M_{i,i}$. ∎

Recall that Catarino and Higgins [2] had already proved that $\mathscr{SOP}_n$ is a subsemiband of $\mathscr{T}_n$. We have proved in [28, Theorem 2.1] the following result.

Lemma 2.5. *For $n \geq 3$, let $A \subseteq E(\mathscr{SOP}_n)$. Then*

$$\langle A \rangle = \mathscr{SOP}_n \quad \textit{if and only if} \quad E_{n-1}^+ \subseteq A \textit{ or } E_{n-1}^- \subseteq A.$$

The following lemma was proved by the author in [28, Lemma 2.3].

Lemma 2.6. *Let $i \in [n]$, $e = [i+1 \to i]$, $f \in E(J_{n-1})$ and $ef \notin \{e, f\}$. Then*

$$ef \in J_{n-1} \quad \textit{if and only if} \quad f = [i+2 \to i+1].$$

Let I be a subset of $E(J_{n-1})$. It is obvious that $I \subseteq E(\langle I \rangle \cap J_{n-1})$. In general, $E(\langle I \rangle \cap J_{n-1}) \subseteq I$ is false. For example, let $I = E_{n-1}^+$, then, by Lemma 2.5, $\langle I \rangle = \mathscr{SOP}_n$ and so $E(\langle I \rangle \cap J_{n-1}) = E(J_{n-1}) = E_{n-1}^+ \cup E_{n-1}^-$. Then clearly $E(\langle I \rangle \cap J_{n-1}) \nsubseteq I$. However, we have the following.

Lemma 2.7. *Let I be a subset of $E(J_{n-1})$. If $\langle I \rangle \subset \mathscr{SOP}_n$. Then*

$$E(\langle I \rangle \cap J_{n-1}) = I.$$

Proof. Clearly, $I \subseteq E(\langle I \rangle \cap J_{n-1})$. Now, we need to prove that $E(\langle I \rangle \cap J_{n-1}) \subseteq I$. Note that $I \subseteq E(J_{n-1})$ and $E(J_{n-1}) = E_{n-1}^+ \cup E_{n-1}^-$. Let $I_1^* = \langle I \rangle \cap E_{n-1}^+$ and $I_2^* = \langle I \rangle \cap E_{n-1}^-$, then $E(\langle I \rangle \cap J_{n-1}) = I_1^* \cup I_2^*$. Let $G_i = I_i^* \backslash I$, $i = 1, 2$. To prove that $E(\langle I \rangle \cap J_{n-1}) \subseteq I$, we only need to prove that $G_i = \emptyset$, $i = 1, 2$. Note that $G_1 \subseteq I_1^* \subseteq E_{n-1}^+$ and $G_2 \subseteq I_2^* \subseteq E_{n-1}^-$. Now, we assume that $G_1 \neq \emptyset$, and so there is some idempotent element $e = [k \to k+1] \in I_1^* \backslash I$. Note that $I_1^* \subseteq \langle I \rangle$ and $I \subseteq E(J_{n-1})$. Obviously, we may suppose that

$$e = e_1 e_2 \cdots e_r, \quad \text{where} \quad e_i \in I,\ i = 1, 2, \ldots, r, r > 1,$$

and

$$e_i e_{i+1} \cdots e_j \neq e_i, e_j, \quad 1 \leq i < j \leq r.$$

Since $e \in J_{n-1}$ *and* $e \notin I$ it follows that eRe_1, eLe_r. By Lemma 1.1, we have $e_1 = [k+1 \to k]$ and $e_r = [k \to k-1]$. By repeated use of Lemma 2.6, we have $e_i = [k+i \to k+i-1]$ $(i = 1,2,\ldots,r)$. Then $e_r = [k+r \to k+r-1] = [k \to k-1]$ and so $r \equiv 0$ $(mod\ n)$. It follows immediately that $E^-_{n-1} = \{[k+i \to k+i-1] : i \in [n]\} = \{e_1, e_2, \ldots, e_r\} \subseteq I$. Thus, by Lemma 2.5, $\langle I \rangle = \mathscr{SOP}_n$, contradicting the assumption that $\langle I \rangle \subset \mathscr{SOP}_n$. Similarly, we can prove that $G_2 = \emptyset$. ∎

Let I and J be nonempty subsets of $E(J_{n-1})$. It is obvious that $I \subseteq J \Rightarrow \langle I \rangle \subseteq \langle J \rangle \Rightarrow K(n,n-2) \cup \langle I \rangle \subseteq K(n,n-2) \cup \langle J \rangle$. In general, $I \subseteq J \Leftarrow \langle I \rangle \subseteq \langle J \rangle$ and $I \subseteq J \Leftarrow K(n,n-2) \cup \langle I \rangle \subseteq K(n,n-2) \cup \langle J \rangle$ are false. For example, let $I = E^+_{n-1}$ and $J = E^-_{n-1}$, then, by Lemma 2.5, $\langle I \rangle = \langle J \rangle = \mathscr{SOP}_n$ and so $K(n,n-2) \cup \langle I \rangle = K(n,n-2) \cup \langle J \rangle = \mathscr{SOP}_n$. Clearly $I \cap J = \emptyset$. However, we can use Lemma 2.7 to obtain the following.

Lemma 2.8. *Let I and J be nonempty subsets of $E(J_{n-1})$. If $\langle J \rangle \subset \mathscr{SOP}_n$. Then*

(i) $I \subseteq J \Leftrightarrow \langle I \rangle \subseteq \langle J \rangle \Leftrightarrow K(n,n-2) \cup \langle I \rangle \subseteq K(n,n-2) \cup \langle J \rangle$.

(ii) $I \subset J \Leftrightarrow \langle I \rangle \subset \langle J \rangle \Leftrightarrow K(n,n-2) \cup \langle I \rangle \subset K(n,n-2) \cup \langle J \rangle$.

Proof. (i) Clearly,

$$I \subseteq J \Rightarrow \langle I \rangle \subseteq \langle J \rangle \Rightarrow K(n,n-2) \cup \langle I \rangle \subseteq K(n,n-2) \cup \langle J \rangle.$$

To prove that

$$I \subseteq J \Leftarrow \langle I \rangle \subseteq \langle J \rangle \Leftarrow K(n,n-2) \cup \langle I \rangle \subseteq K(n,n-2) \cup \langle J \rangle.$$

It suffices to prove that

$$K(n,n-2) \cup \langle I \rangle \subseteq K(n,n-2) \cup \langle J \rangle \Rightarrow I \subseteq J.$$

Suppose that $K(n,n-2) \cup \langle I \rangle \subseteq K(n,n-2) \cup \langle J \rangle$. Then $\langle I \rangle \cap J_{n-1} = (K(n,n-2) \cup \langle I \rangle) \cap J_{n-1} \subseteq (K(n,n-2) \cup \langle J \rangle) \cap J_{n-1} = \langle J \rangle \cap J_{n-1}$. Thus, by Lemma 2.7,

$$I \subseteq E(\langle I \rangle \cap J_{n-1}) \subseteq E(\langle J \rangle \cap J_{n-1}) = J.$$

(ii) By (i), we easily deduce that

$$I = J \Leftrightarrow \langle I \rangle = \langle J \rangle \Leftrightarrow K(n,n-2) \cup \langle I \rangle = K(n,n-2) \cup \langle J \rangle.$$

It follows immediately that

$$I \subset J \Leftrightarrow \langle I \rangle \subset \langle J \rangle \Leftrightarrow K(n,n-2) \cup \langle I \rangle \subset K(n,n-2) \cup \langle J \rangle.$$ ∎

We can use Lemmas 2.4 and 2.8 to obtain the following.

Lemma 2.9. *Let $i \in [n]$, $T_i = \{\alpha \in \mathscr{SOP}_n : i\alpha = i\}$. Then T_i is a locally maximal regular subsemiband of $\mathscr{SOP}_n$.*

Proof. Let $M_{i,i}$ be as defined in (2.1). By Lemma 2.4, we have

$$T_i = \{\alpha \in \mathscr{SOP}_n : i\alpha = i\} = M_{i,i} = \langle E(J_{n-1}) \backslash \{[i \to i+1], [i \to i-1]\} \rangle. \tag{2.7}$$

Then T_i is a subsemiband of $\mathscr{SOP}_n$. Let $\alpha \in T_i$. If $|\mathrm{Im}(\alpha)| = 1$, then clearly α is an idempotent, and so α is regular. If $|\mathrm{Im}(\alpha)| \geq 2$, from [2, Theorem 3.3], we know that α can be expressed as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $A_s = \{a_s, a_s+1, \ldots, a_{s+1}-1\}$, $s = 1,2,\ldots,r$, the subset $\{a_1, a_2, \ldots, a_r\}$ is an initial points set of kernel classes $A_1, A_2, \ldots, A_r$, $a_1 < a_2 < \cdots < a_r$ and $(b_1, b_2, \ldots, b_r)$ is cyclic.

Since $\alpha \in T_i$, we have $i\alpha = i$. Then there exist $k \in \{1,2,\ldots,r\}$ such that $b_k = i$ and $i \in A_k$. Let $C_j = \{b_j, b_j+1, \ldots, b_{j+1}-1\}$, $j = 1,2,\ldots,r$, and let

$$\beta = \begin{pmatrix} C_k & C_{k+1} & \cdots & C_r & C_1 & \cdots & C_{k-1} \\ i & a_{k+1} & \cdots & a_r & a_1 & \cdots & a_{k-1} \end{pmatrix},$$

then $\alpha = \alpha\beta\alpha$ (since $i \in A_k$) and $i\beta = i$ (since $i = b_k \in C_k$). Since $i \in A_k$, we have $a_k \le i \le a_{k+1} - 1 < a_{k+1}$. Note that $a_1 < a_2 < \cdots < a_r$. It easily follows from Lemma 2.1 that

$$i \le_{i-1} a_{k+1} \le_{i-1} \cdots \le_{i-1} a_r \le_{i-1} a_1 \le_{i-1} \cdots \le_{i-1} a_{k-1} \le_{i-1} i-1.$$

Then, by (2.5) and the definition of $\mathscr{O}_n^{i-1}$, $\beta \in T_i^{i-1}$. Thus, by (2.4) and (2.7), $\beta \in T_i^{i-1} \subseteq M_{i,i} = T_i$ and so α is regular (note that $\alpha = \alpha\beta\alpha$). Hence T_i is a locally regular subsemiband of $\mathscr{SOP}_n$.

Let $\langle J \rangle$ ($J \subseteq E(J_{n-1})$) be a locally regular subsemiband of $\mathscr{SOP}_n$ properly containing T_i. We shall prove that $\langle J \rangle = \mathscr{SOP}_n$. If $\langle J \rangle \subset \mathscr{SOP}_n$, then, by Lemma 2.8 (ii) and (2.7),

$$E(J_{n-1}) \backslash \{[i \to i+1], [i \to i-1]\} \subset J,$$

and so

$$E(J_{n-1}) \backslash \{[i \to i+1]\} \subseteq J \textit{ or } E(J_{n-1}) \backslash \{[i \to i-1]\} \subseteq J.$$

Note $E(J_{n-1}) = E^+_{n-1} \cup E^-_{n-1}$. It follows that

$$E^-_{n-1} \subseteq J \textit{ or } E^+_{n-1} \subseteq J.$$

Thus, by Lemma 2.5, $\langle J \rangle = \mathscr{SOP}_n$, contradicting the assumption that $\langle J \rangle \subset \mathscr{SOP}_n$. ∎

A proper subsemigroup S of $\mathscr{SOP}_n$ is called *maximal regular subsemiband* of $\mathscr{SOP}_n$ if S is a regular subsemiband, and any regular subsemiband of $\mathscr{SOP}_n$ properly containing S must be $\mathscr{SOP}_n$. The following lemma is the main result of [30].

Lemma 2.10. *Let $n \ge 3$. Then each maximal regular subsemiband of $\mathscr{SOP}_n$ must be one of the following forms:*

(C) $K(n,n-2) \cup \mathscr{O}_n^i$, $i \in [n]$.
(D) $K(n,n-2) \cup \{\alpha \in \mathscr{O}_n^{i-1} \cup \mathscr{O}_n^i : i\alpha = i\}$, $i \in [n]$.

Proof. See [30, Theorem 4]. ∎

The following lemma gives a necessary condition for a locally regular subsemiband of $\mathscr{SOP}_n$ to be maximal.

Lemma 2.11. *Let I be a nonempty subset of $E(J_{n-1})$. If $\langle I \rangle$ is a locally maximal regular subsemiband of $\mathscr{SOP}_n$ then $T = K(n,n-2) \cup \langle I \rangle$ is a maximal regular subsemiband of $\mathscr{SOP}_n$.*

Proof. From [30, Lemma 13] we know that if S is a regular semigroup and I is an ideal of S, then I is also a regular semigroup. Note that $\mathscr{SOP}_n$ is a regular semigroup (see [2, Theorem 3.1]) and $K(n,n-2)$ is an ideal of $\mathscr{SOP}_n$. Then $K(n,n-2)$ is regular and so $T = K(n,n-2) \cup \langle I \rangle$ is regular (since $\langle I \rangle$ is regular). From [29, Theorem 2.1] we know that $K(n,n-2) = \langle E(J_{n-2}) \rangle$. Then

$$T = K(n,n-2) \cup \langle I \rangle = \langle E(J_{n-2}) \cup I \rangle.$$

Thus $T = K(n,n-2) \cup \langle I \rangle$ is a regular subsemiband of $\mathscr{SOP}_n$.

Let S be a regular subsemiband of $\mathscr{SOP}_n$ properly containing T. Clearly $S = \langle E(S)\rangle$ and $K(n,n-2) \subseteq T \subset S$. We easily deduce that $S = K(n,n-2)\cup S = K(n,n-2)\cup\langle E(S\cap J_{n-1})\rangle$ and so

$$K(n,n-2)\cup\langle I\rangle = T \subset S = K(n,n-2)\cup\langle E(S\cap J_{n-1})\rangle.$$

Note that $E(S\cap J_{n-1}) \subseteq E(J_{n-1})$ and $S \subset \mathscr{SOP}_n$. Then, by Lemma 2.8 (ii), $\langle I\rangle \subset \langle E(S\cap J_{n-1})\rangle$ and so, by the locally maximality of $\langle I\rangle$, $\langle E(S\cap J_{n-1})\rangle = \mathscr{SOP}_n$. Thus $S = \mathscr{SOP}_n$ and so $T = K(n,n-2)\cup\langle I\rangle$ is a maximal regular subsemiband of $\mathscr{SOP}_n$. ∎

Our final lemma is

Lemma 2.12. *Let* $n \geq 3$. *Then*

$$\mathscr{O}_n^k = \langle E(\mathscr{O}_n^k \cap J_{n-1})\rangle.$$

Proof. See [28, Lemma 2.2]. ∎

Now, we can prove Theorem 2.1.

Proof of Theorem 2.1. From Lemma 2.9 we know that T_i is a locally maximal regular subsemiband of $\mathscr{SOP}_n$. By Lemmas 1.2 and 2.12, we have

$$O_n^i = \langle E(J_{n-1})\backslash\{[i\to i+1],[i+1\to i]\}\rangle. \tag{2.8}$$

It is well known that $\mathscr{O}_n$ is regular. From Lemma 4.1 in [2], we know that the mapping $\varphi_i : f \to a^{-i} f a^i$ is an isomorphism between $\mathscr{O}_n$ and $\mathscr{O}_n^i$, where $a = (123\cdots n)$ is the fixed generator of the cyclic group $\mathscr{Z}_n$. Then $\mathscr{O}_n^i$ is regular and so, by (2.8), $S_i = \mathscr{O}_n^i$ is a locally regular subsemiband of $\mathscr{SOP}_n$. Let $\langle J\rangle$ ($J \subseteq E(J_{n-1})$) be a locally regular subsemiband of $\mathscr{SOP}_n$ properly containing $S_i = O_n^i$. We shall prove that $\langle J\rangle = \mathscr{SOP}_n$. If $\langle J\rangle \subset \mathscr{SOP}_n$, then, by Lemma 2.8 (ii) and (2.8),

$$E(J_{n-1})\backslash\{[i\to i+1],[i+1\to i]\} \subset J,$$

and so

$$E(J_{n-1})\backslash\{[i\to i+1]\} \subseteq J \textit{ or } E(J_{n-1})\backslash\{[i+1\to i]\} \subseteq J.$$

Note that $E(J_{n-1}) = E_{n-1}^+ \cup E_{n-1}^-$. It follows that

$$E_{n-1}^- \subseteq J \textit{ or } E_{n-1}^+ \subseteq J.$$

Thus, by Lemma 2.5, $\langle J\rangle = \mathscr{SOP}_n$, contradicting the assumption that $\langle J\rangle \subset \mathscr{SOP}_n$. Hence S_i is a locally maximal regular subsemiband of $\mathscr{SOP}_n$.

Conversely, we shall prove that each locally maximal regular subsemiband of $\mathscr{SOP}_n$ must be of the form S_i or T_i. Let $C_i = K(n,n-2)\cup\ \mathscr{O}_n^i$ and $D_i = K(n,n-2)\cup\{\alpha \in \mathscr{O}_n^{i-1}\cup\mathscr{O}_n^i : i\alpha = i\}$. By (2.8), we have

$$C_i = K(n,n-2)\cup\ \mathscr{O}_n^i = K(n,n-2)\cup\langle E(J_{n-1}\backslash\{[i\to i+1],[i+1\to i]\}\rangle. \tag{2.9}$$

Let $M_{i,i}$, S_i^i and T_i^{i-1} be defined as (2.1), (2.2) and (2.3), respectively. By the definition of S_i^i, T_i^{i-1}, we easily deduce that $S_i^i = \{\alpha \in \mathscr{O}_n^i : i\alpha = i\}$ and $T_i^{i-1} = \{\alpha \in \mathscr{O}_n^{i-1} : i\alpha = i\}$. Then, by Lemma 2.2,

$$\begin{aligned} D_i &= K(n,n-2)\cup\{\alpha\in\mathscr{O}_n^{i-1}\cup\mathscr{O}_n^i : i\alpha = i\} = K(n,n-2)\cup S_i^i\cup T_i^{i-1}\\ &= K(n,n-2)\cup\{(S_i^i\cup T_i^{i-1})\cap J_{n-1}\} = K(n,n-2)\cup(M_{i,i}\cap J_{n-1})\\ &= K(n,n-2)\cup M_{i,i} = K(n,n-2)\cup\langle E(J_{n-1})\backslash\{[i\to i+1],[i\to i-1]\}\rangle. \end{aligned} \tag{2.10}$$

Suppose that $\langle I\rangle$ ($I \subseteq E(J_{n-1})$) is a locally maximal regular subsemiband of $\mathscr{SOP}_n$. Then, by Lemma 2.11, $T = K(n,n-2)\cup\langle I\rangle$ is a maximal regular subsemiband of $\mathscr{SOP}_n$. Thus, by Lemma 2.10, (2.9) and (2.10), there exist $s \in [n]$ such that $T = C_s = K(n,n-2)\cup \langle E(J_{n-1}\backslash\{[s\to s+1],[s+1\to s]\}\rangle$ or there exist $t\in[n]$ such that $T = D_t = K(n,n-2)\cup \langle E(J_{n-1}\backslash\{[t\to t+1],[t\to t-1]\}\rangle$. It follows easily from Lemma 2.8 (i) that

$$\langle I\rangle = \langle E(J_{n-1})\backslash\{[s\to s+1],[s+1\to s]\}\rangle \text{ or } \langle I\rangle = \langle E(J_{n-1})\backslash\{[t\to t+1],[t\to t-1]\}\rangle.$$

Thus, by (2.8) and Lemma 2.4,

$$\langle I\rangle = \langle E(J_{n-1}\rangle\backslash\{[s\to s+1],[s+1\to s]\}\rangle = \mathscr{O}_n^s = S_s$$

or

$$\begin{aligned}\langle I\rangle &= \langle E(J_{n-1})\backslash\{[t\to t+1],[t\to t-1]\}\rangle = M_{t,t}\\ &= \{\alpha\in\mathscr{SOP}_n : t\alpha = t\} = T_t.\end{aligned}$$

It now is obvious that $\mathscr{SOP}_n$ has n locally maximal regular subsemibands of type (A), and n locally maximal regular subsemibands of type (B). Hence $\mathscr{SOP}_n$ has $2n$ locally maximal regular subsemibands. This completes the proof of Theorem 2.1. ∎

Remark 2.1. By Lemma 2.4 and (2.10), we have

$$\begin{aligned}K(n,n-2)\cup\{\alpha\in\mathscr{O}_n^{i-1}\cup\mathscr{O}_n^i : i\alpha = i\} &= K(n,n-2)\cup M_{i,i}\\ &= K(n,n-2)\cup\{\alpha\in\mathscr{SOP}_n : i\alpha = i\}.\end{aligned}$$

From this fact and Lemma 2.10 (the main result of [30]), we immediately obtain the following result, which is a clearer than the main result of [30] (see [30, Theorem 4]).

Theorem 2.2. *Let $n \geq 3$. Then each maximal regular subsemiband of $\mathscr{SOP}_n$ must be one of the following forms:*

(C) $K(n,n-2)\cup\ \mathscr{O}_n^i$, $i\in[n]$.
(D) $K(n,n-2)\cup\{\alpha\in\mathscr{SOP}_n : i\alpha = i\}$, $i\in[n]$.

3. Some related problems

In [26], You described the maximal regular subsemigroup of the ideals of $\mathscr{T}_n$. In turn, the maximal subsemigroup of the ideals of $\mathscr{T}_n$ was given by Yang and Yang [20]. In [27], You and Yang classified the maximal subsemibands of $\mathscr{S}ing_n$. Yang and Yang [25] obtained the classification of maximal regular subsemibands of $\mathscr{S}ing_n$. For the semigroup $\mathscr{T}_n$, it is then natural to ask for the problem concerning the description of (locally) maximal subsemibands or (locally) maximal regular subsemibands of the ideals of $\mathscr{T}_n$ which are open questions.

On the other hand, as the notions of order-preserving transformation and orientation-preserving transformation have been widely considered for several classes of transformation semigroups, it is also natural to consider the semigroups $\mathscr{O}_n$ and $\mathscr{OP}_n$. We also may ask for the problem concerning the description of (locally) maximal subsemibands or (locally) maximal regular subsemibands of the ideals of the two semigroup. Dimitrova and Koppitz [3] determined all the maximal subsemigroups of the ideals of $\mathscr{O}_n$. The same authors [4] classified completely maximal regular subsemigroups of the ideals of $\mathscr{O}_n$. Recently, Zhao [31] classified completely maximal regular subsemibands of the ideals of $\mathscr{O}_n$. Further, Dimitrova, Fernandes and Koppitz [6] described all the maximal subsemigroups of the ideals of $\mathscr{OP}_n$. All the other cases remain as open problems.

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