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# Second Order Duality for Minmax Fractional Programming Problem Involving ($F$,$\alpha,\rho$,d)-Type I Functions

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**Abstract.** In this paper, we focus our study on a minmax fractional programming problem and its second order dual. Weak, strong and strict converse duality theorems are established assuming the involved functions to be second order $(F,\alpha,\rho,d)$- type I.

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## 1. Introduction

We consider the following minmax fractional programming problem:

**(P)** Minimize $\psi(x) = \sup\limits_{y\in Y} \frac{f(x,y)}{h(x,y)}$ subject to $g(x) \leqslant 0$, $x \in R^n$, where $Y$ is a compact subset of $R^l$, $f(\cdot,\cdot) : R^n \times R^l \to R$, $h(\cdot,\cdot) : R^n \times R^l \to R$, are $C^2$ mappings on $R^n \times R^l$ and $g(\cdot) : R^n \to R^m$ is $C^2$ mapping on $R^n$. It is assumed that for each $(x,y)$ in $R^n \times R^l$, $f(x,y) \geqslant 0$ and $h(x,y) > 0$.

There are several researchers interested in a class of minmax programming problem. For detail one can consult [1, 4, 12] and the references cited therein. Particularly, Schmittendorf [19] considered the following minmax problem:

**(P1)** $\min \sup\limits_{y\in Y} f(x,y)$ subject to $g(x) = (g_1(x), g_2(x), \dots, g_r(x))^T \leqslant 0$, where $Y$ is a compact subset of $R^m$, $f(x,y) : R^n \times R^m \to R$ and $g(x) : R^n \to R^r$ are $C^1$ mappings and superscript $T$ denotes the transpose of a column vector.

Under the conditions of convexity, Schmittendorf [19] obtained the necessary and sufficient optimality conditions for (P1). Yadav and Mukherjee [20] employed the optimality conditions presented in [19] to construct two kinds of dual problems and derived duality theorems for convex differentiable minmax fractional programming problem. In [6], Chandra

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and Kumar pointed out that the formulation in [20] has some omissions and inconsistencies, and constructed two modified dual problems and proved duality theorems. Later on, Liu and Wu [15, 16], Liang and Shi [14], Yang and Hou [21] and Ahmad and Husain [3], proposed the convexity/generalized assumption for sufficient optimality conditions in [5], employed the optimality conditions to construct dual problems and established duality theorems.

In [22], Zalmai used a certain infinite dimensional version of Gordan's theorem of the alternative to derive first and second order necessary optimality conditions for a class of minmax programming problems in a Banach space, and discussed several sufficient criteria and duality formulations under generalized invexity assumptions. The second order dual for a nonlinear programming problem was first formulated by Mangasarian [17]. Hanson [10] defined second order type-I functions and obtained second order duality theorems for nonlinear mathematical programming problem under appropriate condition on the involved functions.

Zhang and Mond [23] introduced the concept of second order $(F,\rho)$-convexity and established some duality results concerning with nonlinear multiobjective programming problems. Ahmad and Husain [2] extended $(F,\alpha,\rho,d)$-convex functions which were introduced by Liang *et al.* [13] to second order $(F,\alpha,\rho,d)$-convex functions. Hachimi and Aghezzaf [9] further extended it to second order $(F,\alpha,\rho,d)$-type I functions. Very recently, Gupta and Kailey [8] formulated a pair of second-order multiobjective symmetric dual programs over arbitrary cones and appropriate duality theorems are established under $K$-$\eta$-bonvexity assumptions. Gupta and Dangar [7], considered Mond-Weir type multiobjective second-order symmetric dual models with cone constraints in which the objective function is optimised with respect to an arbitrary closed convex cone and established duality relations under $K$-$\eta$-bonvexity assumptions.

Bector *et al.* [5] discussed second order duality results for minimax programming problems under generalized binvexity. Recently, Husain *et al.* [11] formulated two types of second order dual models for minmax fractional programming problem (P) and established weak, strong and strict converse duality theorems under the assumptions of $\eta$-bonvexity/ generalized $\eta$-bonvexity.

In this paper, inspired from the work of Ahmad and Husain [2], Hachimi and Aghezzaf [9] and Husain *et al.* [11], we establish the second order duality theorems for minmax fractional programming problem (P) under the assumption of generalized second order $(F,\alpha,\rho,d)$-type I functions. The paper is organized as follows. Some definitions and notation are given in Section 2. In Section 3, the duality results are presented. Concluding remarks are presented in Section 4.

## 2. Notation and preliminaries

Let $R^n$ be the $n$-dimensional Euclidean space and $R^n_+$ its non-negative orthant. Let X be a nonempty open subset of $R^n$. For $x, y \in R^n$, we let $x \leqslant y \Leftrightarrow y-x \in R^n_+$; $\quad x < y \Leftrightarrow y-x \in R^n_+ \setminus \{0\}$.

Throughout this paper, we denote by $S = \{x \in X : g(x) \leqslant 0\}$ the set of all feasible solutions of problem (P). For each $(x,y) \in R^n \times R^l$, we define

$$J(x) = \left\{ j \in M = \{1,2,...,m\} : g_j(x) = 0 \right\},$$

$$Y(x) = \left\{ y \in Y : f(x,\ y) = \sup_{z \in Y} f(x,\ z) \right\},$$

and $K(x) = \{(s,\ t,\ \bar{y}) \in N \times R_+^s \times R^{ls} :\ 1 \leqslant\ s\ \leqslant\ n+1,\ t = (t_1, t_2,\ ...,\ t_s) \in R_+^s$

$$\text{with } \sum_{i=1}^{s} t_i = 1\ ,\ \bar{y} = (\bar{y}_1, \bar{y}_2,\ ...,\ \bar{y}_s)\ ,\ \text{with } \bar{y}_i = Y(x)\ \ ,\ i = 1,2,...,s\}.$$

**Definition 2.1.** *A functional $F : X \times X \times R^n \to R$ is said to be sublinear in its third argument if for any $x, \bar{x} \in X$,*

(i) $F(x,\bar{x};a_1+a_2)\ \leqslant\ F(x,\bar{x};a_1) + F(x,\bar{x};a_2) \quad \forall a_1,\ a_2 \in R^n$;
(ii) $F(x,\bar{x};\alpha a) =\ \alpha F(x,\bar{x};a) \quad \forall \alpha \in R_+, \forall a \in R^n$.

By (ii) it is clear that $F(x,\bar{x};\ 0) = 0$.

Now, we let $F$ be a sublinear functional and $d(\cdot,\cdot) : X \times X \to R$. Let $\alpha = (\alpha^1, \alpha^2)$, where $\alpha^1,\ \alpha^2 : X \times X \to R_+ \setminus \{0\}$, $\rho = (\rho^1, \rho^2)$, where $\rho^1 = (\rho_1^1, \rho_2^1, ..., \rho_s^1) \in R^s$ and $\rho^2 = (\rho_1^2, \rho_2^2, ..., \rho_m^2) \in R^m$. Let $f(\cdot,\cdot) : X \times Y(x) \to R$ and $g(\cdot) : X \to R^m$ are twice differentiable functions.

**Definition 2.2.** [4] *For each $j \in M$, $(f, g_j)$ is said to be second-order $(F,\alpha,\rho,d)$-type I at $\bar{x} \in X$ if for all $x \in S$, $p \in R^n$ and $y_i \in Y(x)$, we have*

$$\begin{aligned} & f(x,y_i) - f(\bar{x},y_i) + \frac{1}{2} p^T \nabla^2 f(\bar{x},y_i)\, p \\ & \geqslant F\left(x,\bar{x};\alpha^1(x,\bar{x})\left[\nabla f(\bar{x},y_i) + \nabla^2 f(\bar{x},y_i)\, p\right]\right) + \rho_i^1 d^2(x,\bar{x}),\ \ i = 1,2,...,s, \\ & \quad - g(\bar{x}) + \frac{1}{2} p^T \nabla^2 g(\bar{x})\, p \\ & \geqslant F\left(x,\bar{x};\alpha^2(x,\bar{x})\left[\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x})\, p\right]\right) + \rho_j^2 d^2(x,\bar{x}),\ \ j = 1,2,...,m. \end{aligned}$$

*If the first inequality in the above definition is satisfied under the form*

$$\begin{aligned} & f(x,y_i) - f(\bar{x},y_i) + \frac{1}{2} p^T \nabla^2 f(\bar{x},y_i)\, p \\ & > F\left(x,\bar{x};\alpha^1(x,\bar{x})\left[\nabla f(\bar{x},y_i) + \nabla^2 f(\bar{x},y_i)\, p\right]\right) + \rho_i^1 d^2(x,\bar{x}),\ \ i = 1,2,...,s, \end{aligned}$$

*then we say that for each $j \in M$, $(f, g_j)$ is second-order strictly $(F,\alpha,\rho,d)$-type I at $\bar{x}$.*

**Definition 2.3.** [4] *For each $j \in M$, $(f, g_j)$ is said to be second-order pseudoquasi $(F,\alpha,\rho,d)$-type I at $\bar{x} \in X$ if for all $x \in S$, $p \in R^n$ and $y_i \in Y(x)$, we have*

$$\begin{aligned} & f(x,y_i) < f(\bar{x},y_i) - \frac{1}{2} p^T \nabla^2 f(\bar{x},y_i)\, p \\ & \Rightarrow\ F\left(x,\bar{x};\alpha^1(x,\bar{x})\left[\nabla f(\bar{x},y_i) + \nabla^2 f(\bar{x},y_i)\, p\right]\right) < -\rho_i^1 d^2(x,\bar{x}),\ \ i = 1,2,...,s, \\ & \quad - g_j(\bar{x}) + \frac{1}{2} p^T \nabla^2 g_j(\bar{x})\, p \leqslant 0 \\ & \Rightarrow\ F\left(x,\bar{x};\alpha^2(x,\bar{x})\left[\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x})\, p\right]\right)\ \leqslant -\rho_j^2 d^2(x,\bar{x}),\ \ j = 1,2,...,m. \end{aligned}$$

*If the first implication in the above definition is satisfied under the form*

$$F\left(x,\bar{x};\alpha^1(x,\bar{x})\left[\nabla f(\bar{x},y_i) + \nabla^2 f(\bar{x},y_i)\, p\right]\right) \geqslant -\rho_i^1 d^2(x,\bar{x}),$$

$$\Rightarrow f(x,y_i) > f(\bar{x},y_i) - \frac{1}{2} p^T \nabla^2 f(\bar{x},y_i)\, p,\ \ i = 1,2,...,s,$$

*then we say that for each $j \in M$, $(f, g_j)$ is second-order strictly pseudoquasi $(F, \alpha, \rho, d)$-type I at $\bar{x}$.*

The following result will be needed in the sequel in the proof of strong duality theorem.

**Theorem 2.1.** [6] *Let $x^*$ be a solution of problem (P) and let $\nabla g_j(x^*), j \in J(x^*)$, be linearly independent. Then there exist $(s^*, t^*, \bar{y}^*) \in K(x^*), \lambda^* \in R_+$, and $\mu^* \in R_+^m$ such that*

$$\nabla \sum_{i=1}^{s^*} t_i^* \left(f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)\right) + \nabla \sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0$$

$$f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*) = 0, \quad i = 1, 2, ..., s^*,$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0,$$

$$t_i^* \geqslant 0, \ \sum_{i=1}^{s^*} t_i^* = 1, \quad \bar{y}_i^* \in Y(x^*), \quad i = 1, 2, ..., s^*.$$

## 3. Duality

In this section, we consider a general dual to (P) and discuss duality results in which various generalized second-order $(F, \alpha, \rho, d)$-type I hypothesis are imposed on certain combination of the functions $f, h$ and $g$. This is accomplished by employing a certain type of partitioning scheme which was originally proposed by Mond and Weir [18] for the purpose of constructing generalized dual problem for nonlinear programming problems. We state our general dual model and discuss duality results as follows:

**(GMD)** $$\max_{(s,t,\bar{y}) \in K(z)} \ \sup_{(z,\mu,\lambda,p) \in H_1(s,t,\bar{y})} \lambda,$$

where $H_1(s, t, \bar{y})$ denotes the set of all $(z, \mu, \lambda, p) \in R^n \times R_+^m \times R_+ \times R^n$ satisfying

$$\nabla \sum_{i=1}^{s} t_i \left(f(z, \bar{y}_i) - \lambda h(z, \bar{y}_i)\right) + \nabla^2 \sum_{i=1}^{s} t_i \left(f(z, \bar{y}_i) - \lambda h(z, \bar{y}_i)\right) p$$

$$+ \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p = 0, \tag{3.1}$$

$$\sum_{i=1}^{s} t_i \left(f(z, \bar{y}_i) - \lambda h(z, \bar{y}_i)\right) + \sum_{j \in J_0} \mu_j g_j(z)$$

$$-\frac{1}{2} p^T \nabla^2 \left[\sum_{i=1}^{s} t_i \left(f(z, \bar{y}_i) - \lambda h(z, \bar{y}_i)\right) + \sum_{j \in J_0} \mu_j g_j(z)\right] p \geqslant 0, \tag{3.2}$$

$$\sum_{j \in J_\beta} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \sum_{j \in J_\beta} \mu_j g_j(z) p \geqslant 0, \ \ \beta = 1, 2, ..., r, \tag{3.3}$$

where $J_\beta \subseteq M, \ \beta = 0, 1, 2, ..., r$, with $\bigcup_{\beta=0}^{r} J_\beta = M$ and $J_\gamma \cap J_\beta = \phi$ if $\gamma \neq \beta$. If, for a triplet $(s, t, \bar{y}) \in K(z)$, the set $H_1(s, t, \bar{y}) = \phi$, we define the supremum over it to be -∞.

**Theorem 3.1.** *(Weak duality) Let x and* $(z,\mu,\lambda,s,t,\bar{y},p)$ *be feasible solutions to (P) and (GMD), respectively. Assume that*

(i) $\left(\sum_{i=1}^{s} t_i\left(f(\cdot,\bar{y}_i)-\lambda h(\cdot,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(\cdot),\ \sum_{j\in J_\beta}\mu_j g_j(\cdot)\right)$ *is second order* $(F,\alpha,\rho,d)$*-type I at z,*

(ii) $\frac{\rho_1^1}{\alpha^1(x,z)}+\frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2(x,z)}\geqslant 0.$

*Then*

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}\geqslant\lambda.$$

*Proof.* Suppose contrary to the result that

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}<\lambda.$$

Therefore, we have $f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)<0$ for all $\bar{y}_i\in Y(x),\ i=1,2,...,s$. It follows from $t_i\geqslant 0,\ i=1,2,...,s$, that

$$t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right)\leqslant 0,$$

with at least one strict inequality, since $t=(t_1,t_2,...,t_s)\neq 0$.Taking summation over *i,* we have

$$\sum_{i=1}^{s} t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right)<0,$$

which, by the feasibility of $x$ for (P), $\mu\in R_+^m$ and (3.2) gives

$$\sum_{i=1}^{s} t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(x)<0\leqslant\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(z)$$
$$-\frac{1}{2}p^T\nabla^2\left[\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(z)\right]p.$$

That is,

$$\sum_{i=1}^{s} t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(x)-\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)-\sum_{j\in J_0}\mu_j g_j(z)$$

$$+\frac{1}{2}p^T\nabla^2\left[\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(z)\right]p<0. \tag{3.4}$$

Using (3.3), (3.4) and hypothesis *(i)*, we obtain

$$0>\sum_{i=1}^{s} t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(x)-\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)-\sum_{j\in J_0}\mu_j g_j(z)$$
$$+\frac{1}{2}p^T\nabla^2\left[\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(z)\right]p.$$
$$\geqslant F\left(x,z;\alpha^1(x,z)\left(\nabla\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\nabla^2\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p\right.\right.$$

$$+\nabla\sum_{j\in J_0}\mu_j g_j(z)+\nabla^2\sum_{j\in J_0}\mu_j g_j(z)p\Bigg)\Bigg)+\rho_1^1 d^2(x,z),$$

and

$$0\geqslant-\sum_{j\in J_\beta}\mu_j g_j(z)+\frac{1}{2}p^T\nabla^2\sum_{j\in J_\beta}\mu_j g_j(z)p$$

$$\geqslant F\left(x,z;\alpha^2(x,z)\left(\nabla\sum_{j\in J_\beta}\mu_j g_j(z)+\nabla^2\sum_{j\in J_\beta}\mu_j g_j(z)p\right)\right)+\rho_\beta^2 d^2(x,z),$$

$\beta=1,2,...,r.$

Since $\alpha^1(x,z)>0$ and $\alpha^2(x,z)>0$, by using the sublinearity of $F$, the above two inequalities imply

$$F\left(x,z;\left(\nabla\sum_{i=1}^s t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))+\nabla^2\sum_{i=1}^s t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))p\right)\right.$$

$$\left.+\nabla\sum_{j\in J_0}\mu_j g_j(z)+\nabla^2\sum_{j\in J_0}\mu_j g_j(z)p\right)<-\frac{\rho_1^1 d^2(x,z)}{\alpha^1(x,z)}, \tag{3.5}$$

and

$$F\left(x,z;\left(\nabla\sum_{j\in J_\beta}\mu_j g_j(z)+\nabla^2\sum_{j\in J_\beta}\mu_j g_j(z)p\right)\right)\leqslant-\frac{\rho_\beta^2 d^2(x,z)}{\alpha^2(x,z)},\ \beta=1,2,...,r. \tag{3.6}$$

From (3.1), (3.5), (3.6) and the sublinearity of $F$ in the above inequalities, we summarize to get

$$0=F\left(x,z;\nabla\sum_{i=1}^s t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))+\nabla^2\sum_{i=1}^s t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))p\right.$$

$$\left.+\nabla\sum_{j=1}^m\mu_j g_j(z)+\nabla^2\sum_{j=1}^m\mu_j g_j(z)\,p\right)$$

$$\leqslant F\left(x,z;\left(\nabla\sum_{i=1}^s t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))+\nabla^2\sum_{i=1}^s t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))p\right)\right.$$

$$\left.+\nabla\sum_{j\in J_0}\mu_j g_j(z)+\nabla^2\sum_{j\in J_0}\mu_j g_j(z)p\right)+\sum_{\beta=1}^r F\left(x,z;\left(\nabla\sum_{j\in J_\beta}\mu_j g_j(z)+\nabla^2\sum_{j\in J_\beta}\mu_j g_j(z)p\right)\right)$$

$$<-\left(\frac{\rho_1^1}{\alpha^1(x,z)}+\frac{\sum_{\beta=1}^r\rho_\beta^2}{\alpha^2(x,z)}\right)d^2(x,z)\leqslant 0.\ (\text{by(ii)})$$

Thus, we have a contradiction. Hence, the proof is complete. ∎

**Theorem 3.2.** *(Weak duality) Let $x$ and $(z,\mu,\lambda,s,t,\bar{y},p)$ be feasible solutions to (P) and (GMD), respectively. Assume that*

(i) $\left(\sum_{i=1}^{s} t_i\left(f\left(\cdot,\bar{y}_i\right)-\lambda h\left(\cdot,\bar{y}_i\right)\right)+\sum_{j\in J_0}\mu_j g_j\left(\cdot\right), \sum_{j\in J_\beta}\mu_j g_j\left(\cdot\right)\right)$ *is second order pseudoquasi* $(F,\alpha,\rho,d)$*-type I at z,*

(ii) $\frac{\rho_1^1}{\alpha^1(x,z)}+\frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2(x,z)}\geqslant 0.$

*Then*

$$\sup_{y\in Y}\frac{f\left(x,y\right)}{h\left(x,y\right)}\geqslant\lambda.$$

*Proof.* We proceed as in the proof of Theorem 3.1 and obtain

$$\sum_{i=1}^{s} t_i\left(f\left(x,\bar{y}_i\right)-\lambda h\left(x,\bar{y}_i\right)\right)+\sum_{j\in J_0}\mu_j g_j\left(x\right)<\sum_{i=1}^{s} t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)+\sum_{j\in J_0}\mu_j g_j\left(z\right)$$

$$-\frac{1}{2}p^T\nabla^2\left[\sum_{i=1}^{s} t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)+\sum_{j\in J_0}\mu_j g_j\left(z\right)\right]p. \tag{3.7}$$

Using (3.3), (3.7) and hypothesis (i), we obtain

$$F\left(x,z;\alpha^1\left(x,z\right)\left(\nabla\sum_{i=1}^{s} t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)+\nabla^2\sum_{i=1}^{s} t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)p\right.\right.$$
$$\left.\left.+\nabla\sum_{j\in J_0}\mu_j g_j\left(z\right)+\nabla^2\sum_{j\in J_0}\mu_j g_j\left(z\right)p\right)\right)<-\rho_1^1 d^2\left(x,z\right),$$

and

$$F\left(x,z;\alpha^2\left(x,z\right)\left(\nabla\sum_{j\in J_\beta}\mu_j g_j\left(z\right)+\nabla^2\sum_{j\in J_\beta}\mu_j g_j\left(z\right)p\right)\right)\leqslant-\rho_\beta^2 d^2\left(x,z\right).$$

Since $\alpha^1\left(x,z\right)>0$ and $\alpha^2\left(x,z\right)>0$, and the sublinearity of $F$ in the above inequalities, we summarize to get

$$F\left(x,z;\nabla\sum_{i=1}^{s} t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)+\nabla^2\sum_{i=1}^{s} t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)p\right.$$
$$\left.+\nabla\sum_{j=1}^{m}\mu_j g_j\left(z\right)+\nabla^2\sum_{j=1}^{m}\mu_j g_j\left(z\right)p\right)$$
$$<-\left(\frac{\rho_1^1}{\alpha^1\left(x,z\right)}+\frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2\left(x,z\right)}\right)d^2\left(x,z\right). \tag{3.8}$$

Since $\frac{\rho_1^1}{\alpha^1(x,z)}+\frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2(x,z)}\geqslant 0$, inequality (3.8) yields

$$F\left(x,z;\nabla\sum_{i=1}^{s} t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)+\nabla^2\sum_{i=1}^{s} t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)p\right.$$
$$\left.+\nabla\sum_{j=1}^{m}\mu_j g_j\left(z\right)+\nabla^2\sum_{j=1}^{m}\mu_j g_j\left(z\right)p\right)<0,$$

which contradicts (3.1), as $F(x,z;0)=0$. This completes the proof. ∎

**Theorem 3.3.** *(Strong duality) Assume that $x^*$ is an optimal solution to (P) and $\nabla g_j(x^*), j\in J(x^*)$, are linearly independent. Then there exist $(s^*,t^*,\bar{y}^*)\in K(x^*)$ and $(x^*,\mu^*,\lambda^*,p^*=0)\in H_1(s^*,t^*,\bar{y}^*)$ such that $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ is a feasible solution to (GMD) and the two objectives have the same values. Further, if the hypotheses of weak duality Theorems 3.1 or 3.2 hold for all feasible solutions $(z,\mu,\lambda,s,t,\bar{y},p)$ to (GMD), then $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ is an optimal solution to (GMD).*

*Proof.* Since $x^*$ is an optimal solution to (P) and $\nabla g_j(x^*), j\in J(x^*)$, are linearly independent, then by Theorem 2.1, there exist $(s^*,t^*,\bar{y}^*)\in K(x^*)$ and $(x^*,\mu^*,\lambda^*,p^*=0)\in H_1(s^*,t^*,\bar{y}^*)$ such that $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ is a feasible solution to (GMD) and the two objectives have the same values.
Optimality of $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ for (GMD) thus follows from weak duality Theorems 3.1 or 3.2. ∎

**Theorem 3.4.** *(Strict converse duality) Let $x^*$ be an optimal solution to (P) and $(z^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*)$ be optimal solution to (GMD). Assume that the following conditions are satisfied:*

(i) $\nabla g_j(x^*), j\in J(x^*)$, *are linearly independent,*

(ii) $\left(\sum_{i=1}^{s^*} t_i^*(f(\cdot,\bar{y}_i)-\lambda^* h(\cdot,\bar{y}_i))+\sum_{j\in J_0}\mu_j^* g_j(\cdot),\ \sum_{j\in J_\beta}\mu_j^* g_j(\cdot)\right)$ *is second order strictly $(F,\alpha,\rho,d)$ -type-I at $z^*$,*

(iii) $\frac{\rho_1^1}{\alpha^1(x^*,z^*)}+\frac{\Sigma_{\beta=1}^r\rho_\beta^2}{\alpha^2(x^*,z^*)}\geqslant 0.$

*Then $z^*=x^*$, that is, $z^*$ is an optimal solution of (P).*

*Proof.* Suppose to contrary that $z^*\neq x^*$ and exhibit a contradiction. Since $x^*$ and $(z^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*)$ are optimal solutions of (P) and (GMD), respectively, and $\nabla g_j(x^*), j\in J(x^*)$, are linearly independent, therefore, by Theorem 3.3, we obtain

$$\sup_{y^*\in Y}\frac{f(x^*,y^*)}{h(x^*,y^*)}=\lambda^*.$$

Therefore, we have

$f(x^*,\bar{y}_i^*)-\lambda^* h(x^*,\bar{y}_i^*)\leqslant 0$ for all $\bar{y}_i^*\in Y(x^*),\ i=1,2,...,s^*$.

It follows from $t_i^*\geqslant 0,\ i=1,2,...,s^*$, that

$$t_i^*(f(x^*,\bar{y}_i^*)-\lambda^* h(x^*,\bar{y}_i^*))\leqslant 0,$$

with at least one strict inequality, since $t^*=(t_1^*,t_2^*,...,t_s^*)\neq 0$. Taking summation over *i,* we have

$$\sum_{i=1}^{s^*}t_i^*(f(x^*,\bar{y}_i^*)-\lambda^* h(x^*,\bar{y}_i^*))<0,$$

which by the feasibility of $x$ for (P), $\mu^*\in R_+^m$ and (3.2) gives

$$\sum_{i=1}^{s^*}t_i^*(f(x^*,\bar{y}_i^*)-\lambda^* h(x^*,\bar{y}_i^*))+\sum_{j\in J_0}\mu_j g_j(x^*)<0\leqslant\sum_{i=1}^{s^*}t_i^*(f(z^*,\bar{y}_i^*)-\lambda^* h(z^*,\bar{y}_i^*))$$
$$+\sum_{j\in J_0}\mu_j g_j(z^*)-\frac{1}{2}p^{*T}\nabla^2[\sum_{i=1}^{s}t_i(f(z^*,\bar{y}_i^*)-\lambda h(z^*,\bar{y}_i^*))+\sum_{j\in J_0}\mu_j g_j(z^*)]p^*.$$

That is,

$$\sum_{i=1}^{s^*} t_i^*(f(x^*,\bar{y}_i^*) - \lambda^* h(x^*,\bar{y}_i^*)) + \sum_{j\in J_0} \mu_j g_j(x^*) - \sum_{i=1}^{s^*} t_i^*(f(z^*,\bar{y}_i^*) - \lambda^* h(z^*,\bar{y}_i^*)) - \sum_{j\in J_0} \mu_j g_j(z^*)$$

$$+\frac{1}{2}p^{*T}\nabla^2\left[\sum_{i=1}^{s^*} t_i^*\left(f\left(z^*,\bar{y}_i^*\right) - \lambda^* h\left(z^*,\bar{y}_i^*\right)\right) + \sum_{j\in J_0} \mu_j g_j\left(z^*\right)\right] p^* < 0. \tag{3.9}$$

Using (3.3), (3.9) and hypothesis (ii), we obtain

$$0 > \sum_{i=1}^{s^*} t_i^*\left(f\left(x^*,\bar{y}_i^*\right) - \lambda^* h\left(x^*,\bar{y}_i^*\right)\right) + \sum_{j\in J_0} \mu_j^* g_j\left(x^*\right) - \sum_{i=1}^{s^*} t_i^*\left(f\left(z^*,\bar{y}_i^*\right) - \lambda^* h\left(z^*,\bar{y}_i^*\right)\right)$$
$$- \sum_{j\in J_0} \mu_j^* g_j\left(z^*\right) + \frac{1}{2}p^{*T}\nabla^2\left[\sum_{i=1}^{s^*} t_i^*\left(f\left(z^*,\bar{y}_i^*\right) - \lambda^* h\left(z^*,\bar{y}_i^*\right)\right) + \sum_{j\in J_0} \mu_j^* g_j\left(z^*\right)\right] p^*$$
$$> F\left(x^*,z^*;\alpha^1\left(x^*,z^*\right)\left(\nabla \sum_{i=1}^{s^*} t_i^*\left(f\left(z^*,\bar{y}_i^*\right) - \lambda^* h\left(z^*,\bar{y}_i^*\right)\right) + \nabla^2 \sum_{i=1}^{s^*} t_i^*\left(f\left(z^*,\bar{y}_i^*\right)\right.\right.\right.$$
$$\left.\left.\left. -\lambda^* h\left(z^*,\bar{y}_i^*\right)\right) p^* + \nabla \sum_{j\in J_0} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j\in J_0} \mu_j^* g_j\left(z^*\right) p^*\right)\right) + \rho_1^1 d^2\left(x^*,z^*\right),$$

and

$$0 \geqslant - \sum_{j\in J_\beta} \mu_j^* g_j\left(z^*\right) + \frac{1}{2}p^{*T}\nabla^2 \sum_{j\in J_\beta} \mu_j^* g_j\left(z^*\right)p^*,$$
$$\geqslant F\left(x^*,z^*;\alpha^2\left(x^*,z^*\right)\left(\nabla \sum_{j\in J_\beta} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j\in J_\beta} \mu_j^* g_j\left(z^*\right)p^*\right)\right)$$
$$+ \rho_\beta^2 d^2\left(x^*,z^*\right),\ \beta = 1,2,...,r.$$

Since $\alpha^1\left(x^*,z^*\right) > 0$ and $\alpha^2\left(x^*,z^*\right) > 0$, by using the sublinearity of $F$, the above two inequalities imply

$$F\left(x^*,z^*;\left(\nabla \sum_{i=1}^{s^*} t_i^*\left(f\left(z^*,\bar{y}_i^*\right) - \lambda^* h\left(z^*,\bar{y}_i^*\right)\right) + \nabla^2 \sum_{i=1}^{s^*} t_i^*\left(f\left(z^*,\bar{y}_i^*\right) - \lambda^* h\left(z^*,\bar{y}_i^*\right)\right)p^*\right)\right.$$
$$\left. +\nabla \sum_{j\in J_0} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j\in J_0} \mu_j^* g_j\left(z^*\right)p^*\right) < -\frac{\rho_1^1 d^2\left(x^*,z^*\right)}{\alpha^1\left(x^*,z^*\right)}, \tag{3.10}$$

and

$$F\left(x^*,z^*;\ \left(\nabla \sum_{j\in J_\beta} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j\in J_\beta} \mu_j^* g_j\left(z^*\right)p^*\right)\right) \leqslant -\frac{\rho_\beta^2 d^2\left(x^*,z^*\right)}{\alpha^2\left(x^*,z^*\right)} \tag{3.11}$$

$\beta = 1,2,...,r.$

From (3.1), (3.10), (3.11) and the sublinearity of $F$, we get

$$0 = F\left(x^*,z^*;\nabla \sum_{i=1}^{s^*} t_i^*\left(f\left(z^*,\bar{y}_i^*\right) - \lambda^* h\left(z^*,\bar{y}_i^*\right)\right) + \nabla^2 \sum_{i=1}^{s^*} t_i^*\left(f\left(z^*,\bar{y}_i^*\right) - \lambda^* h\left(z^*,\bar{y}_i^*\right)\right)p^*\right.$$

$$+\nabla\sum_{j=1}^{m}\mu_j^*g_j(z^*)+\nabla^2\sum_{j=1}^{m}\mu_j^*g_j(z^*)\,p^*\Bigg)$$

$$\leq F\left(x^*,z^*;\left(\nabla\sum_{i=1}^{s^*}t_i^*\left(f(z^*,\bar{y}_i^*)-\lambda^*h(z^*,\bar{y}_i^*)\right)+\nabla^2\sum_{i=1}^{s^*}t_i^*\left(f(z^*,\bar{y}_i^*)-\lambda^*h(z^*,\bar{y}_i^*)\right)p^*\right.\right.$$

$$\left.\left.+\nabla\sum_{j\in J_0}\mu_j^*g_j(z^*)+\nabla^2\sum_{j\in J_0}\mu_j^*g_j(z^*)p^*\right)\right)$$

$$+\sum_{\beta=1}^{r}F\left(x^*,z^*;\left(\nabla\sum_{j\in J_\beta}\mu_j^*g_j(z^*)+\nabla^2\sum_{j\in J_\beta}\mu_j^*g_j(z^*)p^*\right)\right)$$

$$<-\left(\frac{\rho^1}{\alpha^1(x^*,z^*)}+\frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2(x^*,z^*)}\right)d^2(x^*,z^*)\leq 0.\ (\text{by(iii)})$$

Thus, we have a contradiction. Hence $z^*=x^*$. ∎

**Theorem 3.5.** *(Strict converse duality) Let $x^*$ be an optimal solution to (P) and $(z^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*)$ be optimal solution to (GMD). Assume that the following conditions are satisfied:*

(i) $\nabla g_j(x^*), j\in J(x^*)$, *are linearly independent,*

(ii) $\left(\sum_{i=1}^{s^*}t_i^*\left(f(\cdot,\bar{y}_i)-\lambda^*h(\cdot,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j^*g_j(\cdot),\sum_{j\in J_\beta}\mu_j^*g_j(\cdot)\right)$ *is second order strictly pseudoquasi* $(F,\alpha,\rho,d)$ *-type I at* $z^*$,

(iii) $\frac{\rho_1^1}{\alpha^1(x^*,z^*)}+\frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2(x^*,z^*)}\geqslant 0.$

*Then, $z^*=x^*$; that is, $z^*$ is an optimal solution of (P).*

*Proof.* We proceed as in the proof of Theorem 3.4 and obtain

$$\sum_{i=1}^{s^*}t_i^*\left(f(x^*,\bar{y}_i^*)-\lambda^*h(x^*,\bar{y}_i^*)\right)+\sum_{j\in J_0}\mu_jg_j(x^*)<\sum_{i=1}^{s^*}t_i^*\left(f(z^*,\bar{y}_i^*)-\lambda^*h(z^*,\bar{y}_i^*)\right)+\sum_{j\in J_0}\mu_jg_j(z^*)$$

$$-\frac{1}{2}p^{*T}\nabla^2\left[\sum_{i=1}^{s^*}t_i^*\left(f(z^*,\bar{y}_i^*)-\lambda^*h(z^*,\bar{y}_i^*)\right)+\sum_{j\in J_0}\mu_jg_j(z^*)\right]p^*. \tag{3.12}$$

From (3.3), and by the second part of the hypothesis on $\left(\sum_{i=1}^{s^*}t_i^*\left(f(\cdot,\bar{y}_i)-\lambda^*h(\cdot,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j^*g_j(\cdot),\sum_{j\in J_\beta}\mu_j^*g_j(\cdot)\right)$ at $z^*$, we have

$$F\left(x^*,z^*;\alpha^2(x^*,z^*)\left(\nabla\sum_{j\in J_\beta}\mu_j^*g_j(z^*)+\nabla^2\sum_{j\in J_\beta}\mu_j^*g_j(z^*)p^*\right)\right)\leqslant-\rho_\beta^2d^2(x^*,z^*),$$

$\beta=1,2,...,r$. As $\alpha^2(x^*,z^*)>0$ and as $F$ is sublinear, it follows that

$$F\left(x^*,z^*;\left(\nabla\sum_{j\in J_\beta}\mu_j^*g_j(z^*)+\nabla^2\sum_{j\in J_\beta}\mu_j^*g_j(z^*)p^*\right)\right)\leqslant-\frac{\rho_\beta^2d^2(x^*,z^*)}{\alpha^2(x^*,z^*)}, \tag{3.13}$$

$\beta = 1, 2, ..., r$. On the other hand from relation (3.1) and the sublinearity of $F$, we obtain

$$0 = F\left(x^*, z^*; \nabla \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) + \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) p^* \right.$$
$$\left. + \nabla \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) p^* \right)$$
$$\leqslant F\left(x^*, z^*; \nabla \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) + \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) p^* \right.$$
$$\left. + \nabla \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) p^* \right)$$
$$+ \sum_{\beta=1}^{r} F\left(x^*, z^*; \nabla \sum_{j \in J_\beta} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j \in J_\beta} \mu_j^* g_j\left(z^*\right) p^* \right).$$

That is,

$$\sum_{\beta=1}^{r} F\left(x^*, z^*; \nabla \sum_{j \in J_\beta} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j \in J_\beta} \mu_j^* g_j\left(z^*\right) p^* \right)$$
$$\geqslant -F\left(x^*, z^*; \nabla \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) \right.$$

(3.14)

$$\left. + \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) p^* + \nabla \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) p^* \right).$$

From relation (3.13), (3.14), we obtain

$$F\left(x^*, z^*; \left(\nabla \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) + \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) p^* \right.\right.$$
$$\left.\left. + \nabla \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) p^* \right)\right) \geqslant \frac{\sum_{\beta=1}^{r} \rho_\beta^2 d^2\left(x^*, z^*\right)}{\alpha^2\left(x^*, z^*\right)}.$$

In view of $\frac{\rho_1^1}{\alpha^1(x^*,z^*)} + \frac{\sum_{\beta=1}^{r} \rho_\beta^2}{\alpha^2(x^*,z^*)} \geqslant 0$, $\alpha^1\left(x^*, z^*\right) > 0$ and the sublinearity of $F$, the above inequality becomes

$$F\left(x^*, z^*; \alpha^1\left(x^*, z^*\right) \left(\nabla \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) + \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) p^* \right.\right.$$
$$\left.\left. + \nabla \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) p^* \right)\right) \geqslant -\rho_1^1 d^2\left(x^*, z^*\right).$$

Using the first part of the hypothesis on $\left(\sum_{i=1}^{s^*} t_i^* \left(f(\cdot,\bar{y}_i)-\lambda^* h(\cdot,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j^* g_j(\cdot),\ \sum_{j\in J_\beta}\mu_j^* g_j(\cdot)\right)$ at $z^*$, it follows that

$$\sum_{i=1}^{s^*} t_i^* \left(f(x^*,\bar{y}_i^*)-\lambda^* h(x^*,\bar{y}_i^*)\right)+\sum_{j\in J_0}^{\mu_j g_j}(x^*) > \sum_{i=1}^{s^*} t_i^* \left(f(z^*,\bar{y}_i^*)-\lambda^* h(z^*,\bar{y}_i^*)\right)+\sum_{j\in J_0}\mu_j g_j(z^*)$$

$$-\frac{1}{2}p^{*T}\nabla^2\left[\sum_{i=1}^{s^*} t_i^* \left(f(z^*,\bar{y}_i^*)-\lambda^* h(z^*,\bar{y}_i^*)\right)+\sum_{j\in J_0}\mu_j g_j(z^*)\right]p^* \quad ,$$

which is a contradiction to (3.12). Hence $z^*=x^*$. ∎

## 4. Conclusions

In this paper, we have discussed the second order duality to minmax fractional programming problems under the assumptions of generalized $(F,\alpha,\rho,d)$-type I convexity. It will be interesting to see whether or not the second order duality results developed in this paper still hold for the following nondifferentiable minmax fractional programming problems:

**(P2)** Min $\sup_{y\in Y} \frac{\phi(x,y)+\left(x^T Bx\right)^{1/2}}{\psi(x,y)-\left(x^T Dx\right)^{1/2}}$

subject to $g(x) \leqslant 0,\ x\in R^n$,

where $Y$ is a compact subset of $R^m$, $\phi(.,.)$, $\psi(.,.) : R^n\times R^m \to R$ and $g(.,.) : R^n \to R$ are continuously differentiable function, and $B$ and $D$ are two positive semidefinite $n\times n$ symmetric matrices.

**(P3)** Min $\sup_{v\in W} \frac{Re\left[\phi(\xi,v)+\left(z^T Bz\right)^{1/2}\right]}{Re\left[\psi(\xi,v)-\left(z^T Dz\right)^{1/2}\right]}$,

subject to $-g(\xi)\in S^0,\ \xi\in C^{2n}$,

where $\xi=(z,\bar{z})$, $v=(w,\bar{w})$ for $z\in C^n$, $w\in C^l$, $\phi(\cdot,\cdot): C^{2n}\times C^{2l}\to C$ and $\psi(\cdot,\cdot): C^{2n}\times C^{2l}\to C$ are analytic with respect to $\xi$, W is a specified compact subset in $C^{2l}$, $S^0$ is a polyhedral cone in $C^m$ and $g: C^{2n}\to C^m$ is analytic. Also $B,D\in C^{n\times n}$ are positive semidefinite Hermitian matrices.

This would be task of some of our forthcoming work.

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