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General Γ- Hypermodules: Θ Relation, T- Functor and Fundamental Modules

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Abstract. The main purpose of this paper is to introduce the concept of general Γ-hypermodules as a generalization of Γ-hypermodules, as a generalization of Γ-modules and as a generalization of modules. Then we extended the isomorphism theorems to general Γ-hypermodules. Also, it is observer that if N is a normal Γ-subhypermodule of Γ-hypermodule M, then, $[M : N^*]$ is an abelian group. Finally, we show that there is a covariant functor between the category of general Γ-hypermodules and the category of modules.

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1. Introduction

Hypergroups were introduced in 1934 by a French mathematician Marty [17] at the 8^{th} Congress of Scandinavian Mathematicians. Since then, hundreds of papers and several books have been written set on this topic. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science [18, 20]. One of the first books, dedicated especially to hypergroups, is “Prolegomena of Hypergroup”, written by Corsini in 1993 [3]. A recent book [4] on hyperstructures points out on their applications in cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Another book [7] is devoted especially to the study of hyperring theory; several kinds of hyperrings are introduced and analyzed, and the volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. As an extension of index a subgroup in a group, the theory of index of a subhypergroup in a hypergroup has been recently developed by Zhan *et al.* [24] and generalized orbit-stabilizer theorem by them. Also Jafarabadi *et al.* [13] introduced the notions of simple and completely simple semi hypergroups.

The *fundamental relation* β^* was introduced on hypergroup by Koskas [14] for the first time and studied by many author, for example see [3, 6, 12, 21]. The fundamental relation $\beta^*$is defined on hypergroups are smallest equivalence relation so that the quotient would be

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a group. Let H be a hypergroup and U be the set of all finite products of elements of H and define the relation β on H as follows:

$$x\beta y \Longleftrightarrow \{x,y\} \subseteq u, \text{ for some } u \in U.$$

Freni [11] proved in that for hypergroups we have $\beta^* = \beta$.

Vougiouklis in [20] defined the fundamental relation γ on hyperring R as the smallest equivalence relation on R such that the quotient $[R:\gamma^*]$ is a ring. Let $(R,+,\cdot)$ be a hyperring. Vougiouklis defined the relation γ as follows:

$$a\gamma b \Longleftrightarrow \{a,b\} \subseteq u,$$

where u is a finite sum of finite products elements of R and proved that γ^* is the transitive closure of γ. The fundamental equivalence relation extended to some classes of hyperrings by Spartalis and Vougiouklis [19].

In [23] Zhan *et al.* have considered three isomorphism theorems and the Jordan Holder theorem for hypermodules. Moreover, a fundamental theorem of hypermodules was established by them. Also Davvaz [5] generalized the classical isomorphism theorems of groups to polygroups. By using a certain type of equivalence relations, we can connect *general* Γ-*hypermodule* to Γ-hypermodule. These equivalence relations are called *strong regular* relations. More exactly, starting with a general Γ-hypermodule and using regular relation, we can construct a Γ-*module* structure on the quotient set. Let M be a general Γ-hypermodule and ρ be an equivalence relation on M, M_1 and M_2 be non-empty subsets of M. Then, $M_1\overline{\rho}M_2$ meanies that for every $m_1 \in M_1$ there exists $m_2 \in M_2$ such that $m_1\rho m_2$ and $M_1\overline{\overline{\rho}}M_2$ meanies that for every $m_1 \in M_1$ and $m_2 \in M_2$, $m_1\rho m_2$.

In 1964, Nobusawa introduced Γ-*rings* as a generalization of rings. Barnes [2] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. Barnes [2], Luh [16] and Kyuno [15] studied the structure of Γ-rings and obtained various generalization analogous to corresponding parts in ring theory. Recently, Anvariyeh *et al.*[1] discussed the basic properties of Γ-*hyperideals* in Γ-*semihypergroups*. After that, Dehkordi *et al.* [8, 9, 10] investigated the ideals, rough ideals, homomorphisms and regular relations of Γ-*semihyperrings*.

The plan of this paper is the following: in the next section, we are introduced and analyzed the isomorphism theorems to general Γ-hypermodule. Also, it is observed that if N is a normal Γ-subhypermodules, then $([M:N^*],\oplus)$ is an abelian group. In Sec.3, we introduce a relation Θ and fundamental modules. Finally, we show that there is a covariant functor between the category of Γ-hypermodules and the category of modules.

2. Preliminaries

The purpose of this section is to introduce the basic concepts of algebraic hypergroups and Γ-*hyperrings*. First we shall present the fundamental definitions. Let$H$ be a non-empty set and let $\mathscr{P}^*(H)$ be the set of all non-empty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \mathscr{P}^*(H)$ and the couple $(H,\circ)$ is called a *hypergroupoid*. If A and B are non-empty subsets in H, then we denote

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b, \ \{x\} \circ A = x \circ A \text{ and } B \circ \{x\} = B \circ x.$$

A hypergroupoid $(H,\circ)$ is called a *semihypergroup* if for all a,b,c in H we have $(a\circ b)\circ c = a\circ(b\circ c)$. In addition, if for every $a\in H$, $a\circ H = H = H\circ a$, then $(H,\circ)$ is called a *hypergroup*. A non-empty subset K of a semihypergroup $(H,\circ)$ is called a *subsemihypergroup* if it is a semihypergroup. In other words, a non-empty subset K of a semihypergroup $(H,\circ)$ is a subsemihypergroup if $K\circ K\subseteq K$.

Definition 2.1. *A Krasner hyperring is an algebraic structure $(R,+,\cdot)$ which satisfies the following axioms:*

(1) *$(R,+)$ is a canonical hypergroup, i.e.,*
  - (i) *for every $x,y,z\in R$, $x+(y+z)=(x+y)+z$,*
  - (ii) *for every, $x,y\in R$, $x+y=y+x$,*
  - (iii) *there exists $0\in R$ such that $0+x=x$, for every $x\in R$,*
  - (iv) *for every $x\in R$ there exists a unique element $x^{'}\in R$ such that $0\in x+x^{'}$,*
  - (v) *$z\in x+y$, implies that $x\in z-y$ and $y\in -x+z$.*

(2) *Relating to the multiplication, $(R,\cdot)$ is a semigroup having zero as a bilaterally absorbing element.*

(3) *The multiplication is distributive with respect to the hyperoperation $+$.*

Definition 2.2. [9] *Let R be a commutative semihypergroup and Γ be a commutative group. Then, R is called a Γ-semihyperring if there exists a map $R\times\Gamma\times R\longrightarrow\mathscr{P}^*(R)$ (image to be denoted by $a\alpha b$ for $a,b\in R$ and $\alpha\in\Gamma$) satisfying the following conditions:*

(1) $a\alpha(b+c)=a\alpha b+a\alpha c,$
(2) $(a+b)\alpha c=a\alpha c+b\alpha c,$
(3) $a(\alpha+\beta)c=a\alpha c+a\beta c,$
(4) $a\alpha(b\beta c)=(a\alpha b)\beta c,$

for all $a,b,c\in R$ and $\alpha,\beta\in\Gamma$

In this definition if $(R,+)$ and $(\Gamma,+)$ are canonical hypergroups, then R is called Γ-*hyperring*.

Definition 2.3. [22] *Let $(R,\oplus,\Gamma)$ be a Γ-hyperring and $(M,\oplus)$ be a canonical hypergroup. M is called Γ-hypermodule over R if there exists a mapping $f: R\times\Gamma\times M\longrightarrow M$(the image of (r,α,m) being denoted by $r\alpha m$) such that, for all $r_1,r_2\in R$, $m_1,m_2\in M$ and $\alpha,\beta\in\Gamma$, we have*

(1) $r_1\alpha(m_1+m_2)=r_1\alpha m_1+r_1\alpha m_2,$
(2) $(r_1+r_2)\alpha m_1=r_1\alpha m_1+r_2\alpha m_1,$
(3) $r_1(\alpha+\beta)m_1=r_1\alpha m_1+r_1\beta m_1,$
(4) $r_1\alpha(r_2\beta m_1)=(r_1\alpha r_2)\beta m_1.$

A subset M_1 in M is said to be a Γ-*subhypermodule* of M if it satisfies the following conditions:

(1) M_1 is a subhypergroup of M;
(2) $r\alpha m_1\in M_1$, for all $r\in R,\alpha\in\Gamma,m_1\in M_1$.

Let M and R be the additive commutative canonical hypergroup of all $m\times n$ matrices over the canonical hypergroup G and Γ be the additive commutative canonical hypergroup of all $n\times m$ matrices over the same set. Then, M is a Γ-hypermodule if $a\alpha b$ denotes the usual matrix product of $a;\alpha;b$, where $a\in R$, $\alpha\in\Gamma$ and $b\in M$. In this example, $M_1=\{(x_{ij}): x_{1j}=0, j=1,2,\dots,m\}$ and $M_2=\{(x_{ij}): x_{i1}=0, i=1,2,\dots,m\}$ are Γ-subhypermodules of M.

A Γ-subhypermodule M_1 of M is called *normal* if $m+M_1-m\subseteq M_1$, for all $m\in M$.

3. General Γ-hypermodules

In this section we introduce a new type of hypermodule in which both addition and multiplication are hyperoperations, that satisfy a set of conditions and we extended the isomorphism theorems to general Γ-hypermodules. Also, it is observer that if N is a normal Γ-subhypermodule, then the quotient $[M:N^*]$ considered in the isomorphism theorems by Zhan *et al.* [22] are groups.

Let $(R,\oplus,\Gamma)$ be a Γ-hyperring, $(M,+)$ be a canonical hypergroup and $\widehat{\Gamma}=\{\widehat{\alpha}:\alpha\in\Gamma\}$ be a canonical hypergroup with respect the hyperoperation $\widehat{\alpha}\oplus\widehat{\beta}=\alpha+\beta$ where $\alpha,\beta\in\Gamma$. Then, M is called *general left* $\widehat{\Gamma}$ *hypermodule* if there exists a hyperoperation $R\times\widehat{\Gamma}\times M\longrightarrow\mathscr{P}^*(M)$(the image $(r,\widehat{\alpha},m)$ being denoted by $r\widehat{\alpha}m$) such that for every $r_1,r_2\in R$, $m_1,m_2\in M$ satisfies the following conditions:

(i) $r_1\widehat{\alpha}(m_1+m_2)=r_1\widehat{\alpha}m_1+r_1\widehat{\alpha}m_2$,
(ii) $(r_1+r_2)\widehat{\alpha}m_1=r_1\widehat{\alpha}m_1+r_2\widehat{\alpha}m_1$,
(iii) $r_1(\widehat{\alpha}\oplus\widehat{\beta})m_1=r_1\widehat{\alpha}m_1+r_1\widehat{\beta}m_1$,
(iv) $(r_1\alpha r_2)\widehat{\beta}m_1=r_1\widehat{\alpha}(r_2\widehat{\beta}m_1)$.
(v) $0\in 0\widehat{\alpha}m\cap r\widehat{\alpha}0$, for every $r\in R$ and $m\in M$.

Throughout this paper, by a $\widehat{\Gamma}$-hypermodule we mean a general $\widehat{\Gamma}$-hypermodule.

Example 3.1. Let $(M,+)$ be a Γ-module and M_1 be a Γ-submodule of M. We define the hyperoperation on M as follows:

$$m_1\widehat{\alpha}m_2=m_1\alpha m_2+M_1.$$

Then, M is a $\widehat{\Gamma}$-hypermodule.

Example 3.2. Let R be a Γ-hyperring and I be an ideal of R. Then, I is a Γ-hypermodule under the hyperoperation $\odot:R\times\Gamma\times I\longrightarrow\mathscr{P}^*(I)$, such that $(r,\gamma,a)\longrightarrow r\gamma a$.

Example 3.3. Let R be a Γ-hyperring and $R[x]$ be the set of all polynomial over R. Then, $R[x]$ is a Γ-hypermodule under the following hyperoperation:

$$\begin{aligned} R\times\Gamma\times R[x] &\longrightarrow R[x]\\ \left(r,\alpha,\sum_{i=1}^{n}a_ix^i\right) &\longrightarrow\left\{\sum_{i=1}^{n}c_ix^i:c_i\in r\alpha a_i\right\}.\end{aligned}$$

Example 3.4. Let M be a Γ-hypermodule and $N\subseteq M$. Then, $N\Gamma^{\Sigma}M$ is a Γ- hypermodule with respect the following hyperoperations:

$$\begin{aligned} R\times\Gamma\times N\Gamma^{\Sigma}M &\longrightarrow\mathscr{P}^*(N\Gamma^{\Sigma}M)\\ \left(r,\alpha,\sum_{i=1}^{n}n_i\alpha_im_i\right) &\longrightarrow\left\{\sum_{i=1}^{n}n_i\alpha_ic_i:c_i\in r\alpha m_i\right\}.\end{aligned}$$

Example 3.5. Let R_1 and R_2 be a Γ-hyperrings and M be a left Γ-hypermodule over R_1 and right Γ-hypermodule over R_2. Then,

$$T=\left\{\begin{bmatrix} r_1 & m\\ 0 & r_2\end{bmatrix}:r_1\in R_1,m\in M,r_2\in R_2\right\},$$

is a Γ-hyperring and Γ-hypermodule under following hyperoperation:

$$\begin{array}{ll} T\times\Gamma\times T & \longrightarrow \mathscr{P}^*(T) \\ \left(\begin{bmatrix} r_1 & m_1 \\ 0 & r_2 \end{bmatrix},\gamma,\begin{bmatrix} s_1 & m_2 \\ 0 & s_2 \end{bmatrix}\right) & \longrightarrow \left\{\begin{bmatrix} t_1 & t_2 \\ 0 & t_3 \end{bmatrix} : t_1\in r_1\gamma s_1, t_2\in r_1\gamma m_1+m_2\gamma s_2, t_3\in r_2\gamma s_2\right\}. \end{array}$$

Definition 3.1. *Let X be a subset of M and $\{M_i : i\in J\}$ be a family of all $\widehat{\Gamma}$-subhypermodule of M which contain X. Then, $\bigcap_{i\in J} M_i$ is called the $\widehat{\Gamma}$-hypermodule generated by X and denoted by $< X >$. If $X=\{m_1,m_2,\dots,m_n\}$, then the $\widehat{\Gamma}$-hypermodule $< X >$ denoted by $< m_1,m_2,\dots,m_n >$.*

Proposition 3.1. *Let X be a non-empty subset of M. Then, $<x>=\left\{x\in M : x\in \mathbb{Z}X+R\widehat{\Gamma}^{\Sigma}X\right\}$, where*

$$\mathbb{Z}X=\left\{x\in M : x\in\sum_{i=1}^{n} n_i x_i,\ n\in\mathbb{N}, n_i\in\mathbb{Z},\ x_i\in X\right\},$$

and

$$R\widehat{\Gamma}^{\Sigma}X=\left\{x\in M : x\in\sum_{i=1}^{n} r_i\widehat{\alpha}_i x_i,\ n\in\mathbb{N},\ \alpha_i\in\Gamma,\ x_i\in X\right\}.$$

Proof. The proof is straightforward. ∎

Definition 3.2. *Let M_1 and M_2 be $\widehat{\Gamma}_1$- and $\widehat{\Gamma}_2$-hypermodules, respectively. A map $\varphi : M_1\longrightarrow M_2$ is called a homomorphism if there exist a homomorphism $f:\widehat{\Gamma}_1\longrightarrow\widehat{\Gamma}_2$ and a homomorphism $g: R_1\longrightarrow R_2$ such that*

(i) $\varphi(m_1+m_2)=\varphi(m_1)+\varphi(m_2)$,
(ii) $\varphi(r\widehat{\alpha}m)=g(r)f(\widehat{\alpha})\varphi(m)$.

Definition 3.3. *Let M be a $\widehat{\Gamma}$-hypermodule and ρ be an equivalence relation on M. Then, ρ is called*

(i) *regular if for all $m\in M$, $\alpha\in\Gamma$ and $r\in R$, $m_1\rho m_2$ implies that $(r\widehat{\alpha}m_1)\overline{\rho}(r\widehat{\alpha}m_2)$ and $(m_1+m)\overline{\rho}(m_2+m)$.*
(ii) *strongly regular if for all $m_1,m_2\in M$, $\alpha\in\Gamma$ and $r\in R$ implies that $(r\widehat{\alpha}m_1)\overline{\overline{\rho}}(r\widehat{\alpha}m_2)$ and $(m_1+m)\overline{\overline{\rho}}(m_2+m)$.*
(iii) *additive strong regular if and only if for all $m\in M$ and $m_1\in M$, $m_1\rho m_2$ implies that $(m+m_1)\rho(m+m_2)$.*

Proposition 3.2. *Let ρ be an equivalence relation on M.*

(i) *If ρ is regular, then $[M:\rho]$ is a $\widehat{\Gamma}$-hypermodule, with respect to the following hyperoperation:*

$$\rho_1(m_1)\oplus\rho_1(m_2)=\{\rho(m) : m\in m_1+m_2\},\ \rho_3(r)\widehat{\rho_2(\widehat{\alpha})}\rho_1(m)=\{\rho(t) : t\in r\widehat{\alpha}m\},$$

where ρ_1 and ρ_2 are regular relations on $\widehat{\Gamma}$ and R, respectively.
(ii) *The equivalence relation ρ is strongly regular if and only if $[R:\rho]$ is a $\widehat{\Gamma}$-module.*

Proof. The proof is straightforward. ∎

Let φ be a homomorphism from $\widehat{\Gamma}_1$- hypermodule M_1 into a $\widehat{\Gamma}_2$-hypermodule M_2. Then, the set $\{x\in M_1 : \varphi(x)=0\}$ is called *Kernel* of φ and is denoted by $Ker\varphi$ and the set $\{\varphi(x) : x\in M_1\}$ is called *image* of φ and denoted by $Im\varphi$.

Suppose that N is a $\widehat{\Gamma}$-subhypermodule of M. In this section, we define quotient $[M : N^*]$ and prove that when N is normal the hyperaddition in $[M : N^*]$ is operation.

Suppose that N is a $\widehat{\Gamma}$-subhypermodule of M. We define the relation N^* by

$$x \equiv y \Longleftrightarrow x \in N + y.$$

Proposition 3.3. *Let N be a $\widehat{\Gamma}$-subhypermodule of M. Then, $\equiv$ is an equivalence relation on M.*

Proof. Suppose that $x \in M$. Since $x = x + 0 \in x + N$, the relation $\equiv$ is reflexive. Let $x, y \in M$ such that $x \equiv y$. Then, $x \in y + n$, for some $n \in N$. Hence, $y \in x - n \subseteq x + N$. So, $\equiv$, is symmetric relation. Let $x, y, z \in M$ such that $x \equiv y$ and $y \equiv z$. Then, $x \in y + n_1$ and $y \in z + n_2$ for some $n_1, n_2 \in N$. So, $x \in y + n_1 \subseteq z + n_1 + n_2 \subseteq z + N$. Therefore, the relation $\equiv$ is transitive. This completes the proof. ∎

Proposition 3.4. *Let N be a normal $\widehat{\Gamma}$-subhypermodule. Then, for every $x, y \in M$, following statements are equivalent:*

(1) $y \in x + N$,
(2) $x - y \subseteq N$,
(3) $(x - y) \cap N \neq \emptyset$.

Proof. $(1) \Longrightarrow (2)$. Since $y \in x + N$ and N is normal, we have $y - x \subseteq x + N - x \subseteq N$. Hence, $y - x \subseteq N$ and $x - y = -(y - x) \subseteq N$.
$(2) \Longrightarrow (3)$. Obvious.
$(3) \Longrightarrow (1)$. Since $(x - y) \cap N \neq \emptyset$, there exists $n \in N$ such that $n \in x - y$. Thus, $-y + x \subseteq -y + n + y \subseteq N$. Let $z \in -y + x$. Then, $z \in N$. This implies that $-y \in z - x$ and $y \in x - z \subseteq x + N$. ∎

Let N be a Γ-subhypermodule of M. Then, equivalence relation defined by J. Zhan *et al.* [22], coincide with the following equivalence relation:

$$xN^*y(modN) \Longleftrightarrow x - y \subseteq N.$$

Definition 3.4. *Let M be a Γ-hypermodule defined by Zhan* [22] *and N be a Γ -subhypermodule of M . We define $\Omega(N) = \{x \in M : x - x \subseteq N\}$.*

Proposition 3.5. *Let N be a Γ-subhypermodule of M. Then, $\Omega(N)$ is a Γ-subhypermodule of M and for every $x, y \in \Omega(< 0 >)$, $x + y$ is singleton.*

Proof. Suppose that $x, y \in \Omega(N)$ and $r \in R$. Since, N is a Γ-subhypermodule, $\Omega(N)$ is non-empty. Let $s \in x - y$ and $s_1 \in r\widehat{\alpha}x$. Then, $s - s \subseteq (x - y) - (x - y) = (x - x) + (y - y) \subseteq N$. Therefore, $\Omega(N)$ is a Γ-subhypermodule. Let $x, y \in \Omega(\{0\})$. If $a, b \in x + y$, then $a - b \subseteq (x + y) - (x + y) = (x - x) + (y - y) = 0$. Hence, $a = b$. This means that $x + y$ is singleton. ∎

Corollary 3.1. *Let M be a Γ-hypermodule. Then, $(M, +)$ is an abelian group if and only if $\Omega(\{0\}) = M$.*

Corollary 3.2. *Let N be a Γ-subhypermodule. Then, N is normal if and only if $\Omega(N) = M$.*

Proposition 3.6. *Let M be a Γ-hypermodule and there exist $r \in R$ and $m \in \Omega(\{0\})$ such that $\mid r\alpha m \mid= 1$. Then, $\mid 0\alpha 0 \mid= 1$.*

Proof. we have $r\widehat{\alpha}0 = r\widehat{\alpha}(m - m) = r\widehat{\alpha}m - r\widehat{\alpha}m$. On the other hand $0\widehat{\alpha}0 \subseteq (r - r)\widehat{\alpha}0 = r\widehat{\alpha}0 - r\widehat{\alpha}0$. This implies that $\mid 0\widehat{\alpha}0 \mid= 1$. ∎

Remark 3.1. Let M be a general Γ-hypermodule and N be a normal Γ-subhypermodule of M. Then, the zero element $\{N\}$ is normal and by Proposition 3.6, the hyperoperation $R\widehat{\Gamma}M$ is a operation.

Proposition 3.7. *Let N be a Γ-subhypermodule of M and*

$$F(M) = \left\{ x \in M : x \in \sum_{i=1}^{n} (x_i - x_i),\ \textit{for some},\ n \in \mathbb{N},\ x_i \in M \right\}.$$

Then, $F(M)$ is a subcanonical hypergroup of M and N is normal if and only if $F(M) \subseteq N$. Moreover, $F(M)$ is a $\widehat{\Gamma}$-subhypermodule.

Proof. Suppose that $x, y \in F(M)$. Hence, $x \in \sum_{i=1}^{n}(x_i - x_i)$ and $y \in \sum_{j=1}^{m}(y_j - y_j)$. Then

$$x - y \subseteq \sum_{i=1}^{n}(x_i - x_i) + \sum_{j=1}^{m}(y_j - y_j) \subseteq F(M).$$

This implies that $F(M)$ is a subcanonical hypergroup of M. Let M be a $\widehat{\Gamma}$-hypermodule and $r \in R$. Then,

$$r\widehat{\alpha}x = r\widehat{\alpha}\sum_{i=1}^{n}(x_i - x_i) = \sum_{i=1}^{n}(r\widehat{\alpha}x_i - r\widehat{\alpha}x_i) \subseteq F(M),$$

Hence $F(M)$ is a $\widehat{\Gamma}$-subhypermodule.

Now, for any $x \in F(M)$ and $m \in M$, there exist $n \in \mathbb{N}$, $x_i \in M$ such that $x \in \sum_{i=1}^{n}(x_i - x_i)$. Hence, $m + x - m \subseteq m - m + \sum_{i=1}^{n}(x_i - x_i)$. Thus, $F(M)$ is a normal subcanonical hypergroup of M. Let N be a normal $\widehat{\Gamma}$ of M. Then, for all $m \in M$, $n \in N$, we have $m + n - m \subseteq N$. This implies that $m - m \subseteq N$. Since, N is a $\widehat{\Gamma}$-subhypermodule of M, for all $m_i \in M$, $\sum_{i=1}^{n}(m_i - m_i) \subseteq N$, for every $n \in \mathbb{N}$. This implies that $F(M) \subseteq N$.

Conversely, assume that $F(M) \subseteq N$. Then for every $m \in M$ and $n \in N$, $m + n - m = m - m + n \subseteq F(M) + n \subseteq F(M) + N \subseteq N$. Therefore, N is a normal $\widehat{\Gamma}$-hypermodule. ∎

Proposition 3.8. *Let N_1 and N_2 be two $\widehat{\Gamma}$-subhypermodule of M such that $N_1 \subseteq N_2$ and N_1 is normal. Then, N_2 is also normal.*

Proof. Suppose that N_1 is normal. Then, by Corollary 3.2, $\Omega(N_1) = M$. This implies that $\Omega(N_2) = M$ and by Corollary 3.2, N_2 is normal. ∎

Corollary 3.3. *Let $< 0 >$ is a normal $\widehat{\Gamma}$-hypermodule. Then, all $\widehat{\Gamma}$-subhypermodule of M are normal.*

Theorem 3.1. *The Γ-subhypermodule generated by $\{0\}$ of M is normal if and only if M is an abelian group and $F(M) =< 0 >$ if and only if $(M, +)$ is a abelian group.*

Proof. By Corollary 3.1, $(M, +)$ is abelian group if and only if $\Omega(< 0 >) = M$ and by Corollary 3.2, $\Omega(< 0 >) = M$ if and only if $< 0 >$ is a normal Γ-subhypermodule of M. Hence, $(M, +)$ is an abelian group if and only if $< 0 >$ is normal.

By Proposition 3.7, $F(M)$, is smallest normal subcanonical hypergroup of $(M, +)$. Therefore, $< 0 >$ is normal if and only if $F(M) =< 0 >$. ∎

Remark 3.2. Let N be a $\widehat{\Gamma}$-subhypermodule of M. By Theorem 3.1, all $\widehat{\Gamma}$-subhypermodule of M are normal if and only if $(M, +)$ is an abelian group. For the quotient $\widehat{\Gamma}$-hypermodule

$[M : N^*]$, the zero element $\{N\}$ is normal. On the other hand if N is a normal $\widehat{\Gamma}$ -subhypermodule, then the relation defined by Zhan *et al.* [22], is additive strong regular.

Theorem 3.2. (First Isomorphism Theorem).
Let ψ be a homomorphism from a $\widehat{\Gamma}_1$-hypermodule M_1 into a $\widehat{\Gamma}_2$-hypermodule M_2. Then, $[M_1 : Ker\psi^*]$ is isomorphic to $Im\psi$.

Proof. Suppose that $\varphi : [M_1 : Ker\psi^*] \longrightarrow Im\psi$ by $\varphi(K^*(m)) = \psi(m)$, for all $m \in M$, where $K = Ker\psi$. Let $K^*(m_1) = K^*(m_2)$, where $m_1, m_2 \in M_1$. This implies that $m_1 \in m_2 + k$, for some $k \in K$. Hence,

$$\varphi(m_1) \in \varphi(m_2 + k) = \varphi(m_1) + \varphi(k) = \varphi(m_2) + 0 = \varphi(m_2).$$

So $\varphi(m_1) = \varphi(m_2)$. Thus, φ is well defined. If $m_1, m_2 \in M_1$, then

$$\begin{aligned}\varphi(K^*(m_1) + K^*(m_2)) = \varphi(\{K^*(m) : m \in m_1 + m_2\}) &= \{\varphi(K^*(m)) : m \in m_1 + m_2\} \\ &= \{\varphi(m) : m \in m_1 + m_2\} \\ &= \varphi(K^*(m_1)) + \varphi(K^*(m_2)).\end{aligned}$$

$$\varphi(r\widehat{\alpha}K^*(m)) = \varphi(\{K^*(m_1) : m_1 \in r\widehat{\alpha}m\}) = \{\psi(m_1) : m_1 \in r\widehat{\alpha}m\} = r\widehat{\alpha}\varphi(K^*(m)).$$

Hence, ψ is homomorphism. One can see that φ is one to one and onto. Therefore, φ is a isomorphism. ∎

Theorem 3.3. (Second Isomorphism Theorem)
Let M_1 and M_2 be $\widehat{\Gamma}_1-$ and $\widehat{\Gamma}_2-$ hypermodules. Then,

$$[M_1 : (M_1 \cap M_2)^*] \cong [M_1 + M_2 : M_2^*].$$

Proof. By the First isomorphism, proof is straightforward. ∎

Theorem 3.4. (Third Isomorphism Theorem)
If M_1 and M_2 be $\widehat{\Gamma}$-subhypermodule of M, such that $M_1 \subseteq M_2$, then

$$[M : M_2^*] \cong [[M : M_1^*] : [M_2 : M_1]^*].$$

Proof. By the First isomorphism theorem, proof is straightforward. ∎

4. Fundamental modules and Θ relation

The category $G\Gamma M$ of general Γ- hypermodules in which the objects are Γ-hypermodules. For Γ_1- and Γ_2- hypermodule M_1 and M_2 respectively, $Mor(M_1, M_2)$, are epimorphism from M_1 to M_2 and Mod is the category of all modules. The purpose of this section is to introduce the concept of fundamental modules and T- functor. First we shall present the fundamental definitions.

We define the relation β as follows:

$$a\beta b \text{ if and only if } a, b \subseteq u,$$

where u is a finite sum of finite hyperproduct of R and M. On the other hand, $u \in U$, where

$$U = \left\{ x \in M : x \in \sum_{i=1}^{n} x_i + \sum_{j=1}^{m} \prod_{i=1}^{n_i} r_{ji}\widehat{\alpha}_{ji}y_{ji},\ n, m, n_i \in \mathbb{N},\ r_{ji} \in R, y_{ji} \in M, \alpha_{ji} \in \Gamma \right\}.$$

We denote the transitive closure of β by β^*. The equivalence relation β^* is called *fundamental equivalence relation* on M. We denote the equivalence class of the element $m \in M$ by $\beta^*(m)$. Let γ^* and ν^* be fundamental relations on R and Γ, respectively. We define

$$[M:\Gamma] = \left\{ \Theta_1 \left(\prod_{i=1}^{n} (\beta^*(m_i), \nu^*(\alpha_i)) \right) : m_i \in M, \alpha_i \in \Gamma, n \in \mathbb{N} \right\},$$

where the relation Θ_1 defined as follows:

$$\left(\prod_{i=1}^{n} (\beta^*(m_i), \nu^*(\alpha_i)), \prod_{j=1}^{m} (\beta^*(n_j), \nu^*(\beta_j)) \right) \in \Theta_1,$$

if and only if

$$\bigoplus_{i=1}^{n} \gamma^*(a) \widehat{\nu^*(\alpha_i)} \beta^*(m_i) = \bigoplus_{j=1}^{m} \gamma^*(a) \widehat{\nu^*(\beta_j)} \beta^*(n_j),$$

for every $\gamma^*(a) \in [R : \gamma^*]$. Obviously, this relation is congruence on $[M : \Gamma]$. We define congruence class contain $\prod_{i=1}^{n} (\beta^*(m_i), \nu^*(\alpha_i))$ by $\Theta_1 \left(\prod_{i=1}^{n} (\beta^*(m_i), \nu^*(\alpha_i)) \right)$. Suppose that

$$[R:\Gamma] = \left\{ \prod_{i=1}^{n} (\nu^*(\alpha_i), \gamma^*(x_i)) : \alpha_i \in \Gamma, n \in \mathbb{N}, x_i \in R \right\}.$$

We define a relation Θ_2 on $[R:\Gamma]$ as follows:

$$\left(\prod_{i=1}^{n} (\nu^*(\alpha_i), \gamma^*(x_i)), \prod_{j=1}^{m} (\nu^*(\alpha_j), \gamma^*(y_j)) \right) \in \Theta_2,$$

if and only if

$$\bigoplus_{i=1}^{n} \gamma^*(a) \widehat{\nu^*(\alpha_i)} \gamma^*(x_i) = \bigoplus_{j=1}^{m} \gamma^*(a) \widehat{\nu^*(\beta_j)} \gamma^*(y_j),$$

for every $\gamma^*(a) \in [R : \gamma^*]$. Obviously, this relation is congruence and $[R : \Gamma]$ form a ring with the following multiplication:

$$\Theta_2 \left(\prod_{i=1}^{n} (\nu^*(\alpha_i), \gamma^*(x_i)) \right) \Theta_2 \left(\prod_{j=1}^{m} (\nu^*(\beta_j), \gamma^*(y_j)) \right) = \Theta_2 \left(\prod_{i,j} (\nu^*(\alpha_i), \gamma^*(x_i) \widehat{\nu^*(\beta_j)} \gamma^*(y_j)) \right).$$

Obviously, $[M : \Gamma]$ is a $[R : \Gamma]$-module with respect the following operation:

$$\Theta_2 \left(\prod_{i=1}^{n} (\nu^*(\alpha_i), \gamma^*(x_i)) \right) \Theta_1 \left(\prod_{j=1}^{m} (\nu^*(\beta_j), \beta^*(m_j) \right) = \Theta_1 \left(\prod_{i,j} (\nu^*(\alpha_i), \gamma^*(x_i) \widehat{\nu^*(\beta_j)} \beta^*(m_j)) \right).$$

This operation is well-defined. Indeed, suppose that

$$\Theta_2 \left(\prod_{i=1}^{n_1} (\nu^*(\alpha_i), \gamma^*(x_i)) \right) = \Theta_2 \left(\prod_{j=1}^{n_2} (\nu^*(\alpha_j), \gamma^*(y_j)) \right),$$

and

$$\Theta_1 \left(\prod_{r=1}^{m_1} (\nu^*(\beta_r), \beta^*(m_r)) \right) = \Theta_1 \left(\prod_{s=1}^{m_2} (\nu^*(\beta_s), \beta^*(m_s)) \right).$$

This implies that

$$\sum_{i=1}^{n_1}\gamma^*(x)\widehat{\nu^*(\alpha_i)}\gamma^*(x_i)=\sum_{j=1}^{n_2}\gamma^*(x)\widehat{\nu^*(\alpha_j)}\gamma^*(y_j),$$

and

$$\sum_{r=1}^{m_1}\gamma^*(y)\widehat{\nu^*(\beta_r)}\beta^*(m_r)=\sum_{s=1}^{m_2}\gamma^*(y)\widehat{\nu^*(\beta_s)}\beta^*(m_s),$$

for every $\gamma^*(x),\gamma^*(y)\in[R:\gamma^*]$. Hence,

$$\sum_{i,r}\gamma^*(x)\widehat{\nu^*(\alpha_i)}\gamma^*(x_i)\widehat{\nu^*(\beta_r)}\beta^*(m_r)=\sum_{s,j}\gamma^*(x)\widehat{\nu^*(\alpha_j)}\gamma^*(y_j)\widehat{\nu^*(\beta_s)}\beta^*(m_s),$$

for every $\gamma^*(x)\in[R:\gamma^*]$. This implies that

$$\Theta_1\left(\prod_{i,r}\nu^*(\alpha_i),\gamma^*(x_i)\widehat{\nu^*(\beta_r)}\beta^*(m_r)\right)=\Theta_1\left(\prod_{j,s}(\gamma^*(\alpha_j),\gamma^*(y_j)\widehat{\nu^*(\beta_s)}\beta^*(m_s))\right).$$

Theorem 4.1. *Let M be a $\widehat{\Gamma}$-hypermodule, γ^*, ν^*, β^* be fundamental relations on R, Γ and M, respectively. Then, $[M:\beta^*]$ is a $[R:\gamma^*]$-module.*

Proof. Suppose that $m_1,m_2\in M$, $r\in R$ and $\alpha\in\Gamma$. We define

$$\beta^*(m_1)\oplus\beta^*(m_2)=\{\beta^*(m):m\in\beta^*(m_1)+\beta^*(m_2)\},$$
$$\gamma^*(r)\odot\nu^*(\gamma)\odot\beta^*(m)=\{\beta^*(m):m\in\gamma^*(r)\nu^*(\gamma)\beta^*(m)\}.$$

Let $m_1^{'}\in\beta^*(m_1)$ and $m_2^{'}\in\beta^*(m_2)$. Then, we have $m_1^{'}\beta^*m_1$ if and only if there exist $x_1,x_2,\ldots,x_{n+1}$ with $x_1=m_1^{'},x_{n+1}=m_1$ and $u_1,u_2,\ldots,u_n\in U$ such that $\{x_i,x_{i+1}\}\subseteq u_i$, for $1\le i\le n$ and $m_2^{'}\in\beta^*(m_2)$ if and only if $y_1,y_2,\ldots,y_{t+1}$ with $y_1=m_2^{'}$, $y_{t+1}=m_2$ and $v_1,v_2,\ldots,v_n\in U_M$ such that $\{y_j,y_{j+1}\}\subseteq v_j$. Now, we obtain

$$\{x_i,x_{i+1}\}+y_1\subseteq u_i+v_1,$$
$$x_{n+1}+\{y_j,y_{j+1}\}\subseteq u_n+v_j.$$

The sums $u_i+v_1=t_i$, $1\le i\le m-1$ and $u_m+v_j=t_{m+j-1}$, $1\le j\le n$ are polynomial and so $t_k\in U$ for all $k\in\{1,2,\ldots,n+t-1\}$. Now, pick up the elements $z_1,z_2,\ldots,z_{n+t}$ such that $z_i\in x_i+y_1$, $1\le i\le n$ and $z_{m+j}\in x_{n+1}+y_{j+1}$. Hence we obtain $\{z_k,z_{k+1}\}\subseteq t_k$ for $1\le k\le m+n-1$. Therefore, every element $z_1\in x_1+y_1=m_1^{'}+m_2^{'}$ is β^* equivalent to every element $z_{m+n}\in m_{n+1}+m_{m+1}$. Then, $\beta^*(m_1)\oplus\beta^*(m_2)=\beta^*(m)$, for all $m\in\beta^*(m_1)+\beta^*(m_2)$ $m\in\beta^*(m_1)+\beta^*(m_2)$. In a same way, it prove that $\gamma^*(r)\odot\nu^*(\gamma)\odot\beta^*(m)=\beta^*(d)$, for $d\in\gamma^*(r)\nu^*(\gamma)\beta^*(m)$. For any $\beta^*(m_1),\beta^*(m_2)\in[M:\beta^*]$, $\gamma^*(r)\in[R:\gamma^*]$ and $\nu^*(\gamma)\in[\Gamma:\nu^*]$, we have

$$\begin{aligned}&\beta^*(m)\in\gamma^*(r)\odot\nu^*(\gamma)\odot(\beta^*(m_1)\oplus\beta^*(m_2))\\&\Longleftrightarrow m\in\gamma^*(r)\nu^*(\gamma)(\beta^*(m_1)+\beta^*(m_2))\\&\Longleftrightarrow m\in\gamma^*(r)\nu^*(\gamma)\beta^*(m_1)+\gamma^*(r)\nu^*(\gamma)\beta^*(m_2)\\&\Longleftrightarrow\beta^*(m)\in\gamma^*(r)\nu^*(\gamma)\beta^*(m_1)+\gamma^*(r)\nu^*(\gamma)\beta^*(m_2).\end{aligned}$$

The proof of other property are similar. ∎

Theorem 4.2. *Let M_1 and M_2 be $\widehat{\Gamma}$-hypermodules, respectively, and $\psi:M_1\longrightarrow M_2$ be a epimorphism. Then, there exists a $[R:\gamma^*]$-module homomorphism $\overline{\psi}:[M_1:\Gamma]\longrightarrow[M_2:\Gamma]$. Moreover, if ψ is an isomorphism then, $\overline{\psi}$ is isomorphism.*

Proof. We define

$$\overline{\psi}\left(\Theta_1\left(\prod_{i=1}^{n}(\nu^*(\alpha_i),\beta^*(m_i))\right)\right)=\Theta_1\left(\prod_{i=1}^{n}(\nu^*(\alpha_i),\beta^*(\psi(m_i))\right).$$

First we prove that ψ is well-defined. Let

$$\Theta_1\left(\prod_{i=1}^{n}(\nu^*(\alpha_i),\beta^*(m_i))\right)=\Theta_1\left(\prod_{j=1}^{m}(\nu^*(\delta_j),\beta^*(n_j))\right).$$

We define

$$\begin{aligned}\Lambda:([\Gamma:\nu^*],[M_1:\beta^*]) &\longrightarrow ([\Gamma:\nu^*],[M_2:\beta^*])\\ (\nu^*(x),\beta^*(m_1)) &\longrightarrow (\nu^*(x),\beta^*(\psi(m_1))).\end{aligned}$$

Let $(\gamma^*(y),\beta^*(m_1))=(\gamma^*(y),\beta^*(m_2))$. Then, there exist $x_1,x_2,\ldots,x_n$ such that $x_1=m_1, x_n=m_2$ and $u_1,u_2,\ldots,u_n\in U$ such that $\{x_i,x_{i+1}\}\subseteq u_i$. Since, ψ is a homomorphism $\beta^*(x_i)=\beta^*(x_{i+1})$. This implies that Λ is well-defined. We have Λ is a homomorphism. Because

$$\Lambda(\beta^*(m_1)\oplus\beta^*(m_2))=\Lambda(\beta^*(m_3)),$$

for some $m_3\in\beta^*(m_1)+\beta^*(m_2)$. We know that $\beta^*(m_3)=\beta^*(m)$, for some $m\in m_1+m_2$. Hence,

$$\Lambda(\beta^*(m_1)\oplus\beta^*(m_2))=\beta^*(m)=\Lambda(\beta^*(m_1))\oplus\Lambda(\beta^*(m_2)).$$

In a same way, we can see that

$$\begin{aligned}\Lambda(\gamma^*(x)\widehat{\nu^*(\alpha)}\beta^*(m))=\{\Lambda(t):t\in x\widehat{\alpha}y\} &=\beta^*(\psi(x\widehat{\alpha}m))\\ &=\beta^*(x\widehat{\alpha}\psi(m))\\ &=\gamma^*(x)\widehat{\nu^*(\alpha)}\beta^*(\psi(m)).\end{aligned}$$

Hence,

$$\bigoplus_{i=1}^{n}\gamma^*(x)\widehat{\nu^*(\alpha_i)}\beta^*(\psi(m_i))=\bigoplus_{j=1}^{m}\gamma^*(x)\widehat{\nu^*(\delta_j)}\beta^*(n_j).$$

Since ψ is onto,

$$\overline{\psi}\left(\Theta_1\left(\prod_{i=1}^{n}(\nu^*(\alpha_i),\beta^*(m_i)\right)\right)=\overline{\psi}\left(\Theta_1\left(\prod_{j=1}^{m}(\nu^*(\delta_j),\beta^*(n_j)\right)\right).$$

This implies that ψ is well-defined. ψ is a homomorphism. Indeed,

$$\begin{aligned}&\overline{\psi}\left(\Theta_1\left(\prod_{i=1}^{n}(\nu^*(\alpha_i),\beta^*(m_i))\right)\oplus\Theta_1\left(\prod_{j=1}^{m}(\nu^*(\beta_j),\beta^*(n_j))\right)\right)\\ &=\overline{\psi}\left(\Theta_1\left(\prod_{i=1}^{n}\prod_{j=1}^{m}(\nu^*(\alpha_i),\beta^*(m_i))(\nu^*(\beta_j),\beta^*(n_j))\right)\right)\\ &=\Theta_1\left(\prod_{i=1}^{n}\prod_{j=1}^{m}(\nu^*(\alpha_i),\beta^*(\psi(m_i)))(\nu^*(\beta_j),\beta^*(\psi(n_j)))\right)\\ &=\overline{\psi}\left(\Theta_1\left(\prod_{i=1}^{n}(\nu^*(\alpha_i),\beta^*(m_i))\right)\right)\oplus\overline{\psi}\left(\Theta_1\left(\prod_{j=1}^{m}(\nu^*(\beta_j),\beta^*(n_j))\right)\right).\end{aligned}$$

$$\begin{aligned}&\overline{\psi}\left(\Theta_1\left(\prod_{i=1}^{n}(\nu^*(\alpha_i),\gamma^*(x_i))\right)\Theta_1\left(\prod_{j=1}^{m}(\nu^*(\beta_j),\beta^*(n_j))\right)\right)\\&=\overline{\psi}\left(\Theta_1\left(\prod_{i,j}(\nu^*(\alpha_i),\gamma^*(x_i)\widehat{\nu^*(\beta_j)}\beta^*(n_j))\right)\right).\end{aligned}$$

Hence, for $d_{ij}\in x_i\beta_j n_j$, $\gamma^*(x_i)\widehat{\nu^*(\beta_j)}\beta^*(n_j))=\beta^*(d_{ij})$. This implies that

$$\overline{\psi}\left(\Theta_2\left(\prod_{i=1}^{n}(\nu^*(\alpha_i),\gamma^*(x_i))\right)\Theta_1\left(\prod_{j=1}^{m}(\nu^*(\beta_j),\beta^*(n_j))\right)\right)=\Theta_1\left(\prod_{i,j}(\nu^*(\alpha_i),\beta^*(\psi(d_{ij}))\right).$$

On the other hand,

$$\begin{aligned}&\Theta_2\left(\prod_{i=1}^{n}(\nu^*(\alpha_i),\gamma^*(x_i))\right)\overline{\psi}\left(\Theta_1\left(\prod_{j=1}^{m}(\nu^*(\beta_j),\beta^*(n_j))\right)\right)\\&=\Theta_2\left(\prod_{i=1}^{n}(\nu^*(\alpha_i),\gamma^*(x_i))\right)\left(\Theta_1\left(\prod_{j=1}^{m}(\nu^*(\beta_j),\beta^*(\psi(n_j)))\right)\right)\\&=\Theta_1\left(\prod_{i,j}(\nu^*(\alpha_i),\gamma^*(x_i)\widehat{\nu^*(\beta_j)}\gamma^*(\psi(n_j)))\right)=\Theta_1\left(\prod_{i,j}(\nu^*(\alpha_i),\beta^*(c_{ij})\right)\end{aligned}$$

where $c_{ij}\in x_i\beta_j\psi(n_j)$. Thus, ψ is a $[R:\gamma^*]$-module homomorphism. Let ψ is an isomorphism. We prove that $\overline{\psi}$ is isomorphism. It is enough to prove that ψ is one to one. Firstly, we prove that Λ is one to one. Let

$$\Lambda(\gamma^*(x_1),\beta^*(m_1))=\Lambda(\gamma^*(x_2),\beta^*(m_2)).$$

Then, $\beta^*(\psi(m_1))=\beta^*(\psi(m_2))$. Hence, there exist $y_1,y_2,\dots,y_{m+1}\in M_2$ and $u_i\in U_{M_2}$, for $i\in\{1,2,\dots,m\}$, such that $y_1=\psi(m_1)$ and $y_{m+1}=\psi(m_2)$ and $\{y_i,y_{i+1}\}\subseteq u_i$, for $i\in\{1,2,\dots,m\}$. So, there exist $v_i\in U_{M_1}$ and $x_i\in M_1$ such that $\psi(x_i)=y_i$, for $i\in 1,2,\dots,m$ and $\{x_i,x_{i+1}\}\subseteq v_i$, for $i\in\{1,2,\dots,m\}$. So, $\beta^*(m_1)=\beta^*(m_2)$ and Λ is one to one. Obviously, Λ is onto. Let

$$\overline{\psi}\left(\Theta_1\left(\prod_{i=1}^{n}(\nu^*(x_i),\beta^*(m_i))\right)\right)=\overline{\psi}\left(\Theta_1\left(\prod_{j=1}^{m}(\nu^*(y_j),\beta^*(n_j))\right)\right).$$

Then,

$$\bigoplus_{i=1}^{n}\gamma^*(x)\widehat{\nu^*(x_i)}\beta^*(\psi(m_i))=\bigoplus_{j=1}^{m}\gamma^*(x)\widehat{\nu^*(y_j)}\beta^*(\psi(n_j)).$$

This implies that

$$\Lambda\left(\bigoplus_{i=1}^{n}\gamma^*(x)\widehat{\nu^*(x_i)}\beta^*(m_i)\right)=\Lambda\left(\bigoplus_{j=1}^{m}\gamma^*(x)\widehat{\nu^*(y_j)}\beta^*(n_j)\right),$$

which implies that

$$\bigoplus_{i=1}^{n}\gamma^*(x)\widehat{\nu^*(x_i)}\beta^*(m_i)=\bigoplus_{j=1}^{m}\gamma^*(x)\widehat{\nu^*(y_j)}\beta^*(n_j),$$

for every $\gamma^*(x)\in[R:\gamma^*]$. Therefore, $\overline{\psi}$ is an isomorphism. ∎

Proposition 4.1. *Let M and Γ be canonical hypergroups and σ_1 and σ_2 be regular relations on M and Γ, respectively, such that $[M:\sigma_1]$ is a $[R:\sigma_3]$-modules, where R is a Γ-hyperring and σ_3 is a regular relation on R. Then, M is a general Γ-hypermodule.*

Proof. Suppose that the hyperoperation $*$ defined as follows:

$$*: R\times\Gamma\times M \longrightarrow P^*(M)$$

$$r*\alpha*m \longrightarrow \sigma_3(r)\sigma_2(\alpha)\sigma_1(m),$$

We prove that $r*\alpha*(m_1+m_2)=r*\alpha*m_1+r*\alpha*m_2$. We know that $r*\alpha*(m_1+m_2)=\{r*\alpha*m : m\in m_1+m_2\}$. Let $m\in m_1+m_2$. Then, $\sigma_1(m)=\sigma_1(m_1)+\sigma_1(m_2)$ and we have

$$\sigma_3(r)*\sigma_2(\alpha)*\sigma_1(m)=\sigma_3(r)\sigma_2(\alpha)\sigma_1(m)=\sigma_3(r)\sigma_2(\alpha)(\sigma_1(m_1)+\sigma_1(m_2)).$$

$$\sigma_3(r)\sigma_2(\alpha)\sigma_1(m_1)+\sigma_3(r)\sigma_2(\alpha)\sigma_1(m_2)=r*\alpha*m_1+r*\alpha*m_2$$

Thus, $r*\alpha*(m_1+m_2)\subseteq r*\alpha*m_1+r*\alpha*m_2$. Now, let

$$\sigma_1(m)=\sigma_3(r)\sigma_2(\alpha)\sigma_1(m_1)+\sigma_3(r)\sigma_2(\alpha)\sigma_1(m_2)=\sigma_3(r)\sigma_2(\alpha)(\sigma_1(m_1)+\sigma_1(m_2))$$

Then, there exists $\sigma_1(m_3)\in\sigma_1(m_1)+\sigma_1(m_2)$ such that $\sigma_1(m)=\sigma_3(r)\sigma_2(\alpha)\sigma_1(m_3)=r*\alpha*m_3\subseteq r*\alpha*(m_1+m_2)$. Hence, $r*\alpha*m_1+r*\alpha*m_2\subseteq r*\alpha*(m_1+m_2)$. Therefore, $r*\alpha*m_1+r*\alpha*m_2=r*\alpha*(m_1+m_2)$. In a same way, we can prove another properties. Therefore, M is a Γ-module. ∎

Theorem 4.3. *Let M_1 and M_2 be $\widehat{\Gamma}_1$- and $\widehat{\Gamma}_2$-hypermodules. Then,*

$$[M_1\times M_2:\Gamma_1\times\Gamma_2]\cong[M_1:\Gamma_1]\times[M_2:\Gamma_2].$$

Proof. Suppose that β^*, β_1^* and β_2^* be fundamental relations on $M_1\times M_2$, M_1 and M_2, respectively. It is easy to see that

$$[M_1\times M_2:\beta^*]\cong[M_1:\beta_1^*]\times[M_2:\beta_2^*].$$

We define

$$\begin{aligned}&\psi:[M_1\times M_2:\Gamma_1\times\Gamma_2]\longrightarrow[M_1:\Gamma_1]\times[M_2:\Gamma_2]\\&\Theta\left(\prod_{i=1}^{n}((\nu_1^*(x_i),\nu_2^*(y_i)),(\beta_1^*(m_i),\beta_2^*(n_i))\right)\\&\longrightarrow\left(\Theta_1\left(\prod_{i=1}^{n}(\nu_1^*(x_i),\beta_1^*(m_i))\right),\Theta_2\left(\prod_{i=1}^{n}(\nu_2^*(y_i),\beta_2^*(n_i))\right)\right).\end{aligned}$$

Obviously, this function is well-defined. We proof ψ is a homomorphism.

$$\begin{aligned}&\psi\left(\Theta\left(\prod_{i=1}^{n}((\nu_1^*(x_i),\nu_2^*(y_i)),(\gamma_1^*(t_i),\gamma_2^*(s_i))\right)\Theta\left(\prod_{j=1}^{m}((\nu_1^*(x_j'),\nu_2^*(y_j')),(\beta_1^*(m_j'),\beta_2^*(n_j'))\right)\right)\\&=\psi\left(\Theta\left(\prod_{i,j}(\nu_1^*(x_i),\nu_2^*(y_i)),(\gamma_1^*(t_i)\widehat{\nu_1^*(x_j')}\beta_1^*(m_j'),\gamma_2^*(s_i)\widehat{\nu_2^*(y_j')}\beta_2^*(n_j'))\right)\right)\\&=\psi\left(\Theta\left(\prod_{i,j}(\nu_1^*(x_i),\nu_2^*(y_i)),(\gamma_1^*(c_{ij}),\gamma^*(d_{ij})\right)\right)\\&=\psi\left(\Theta\left(\prod_{i=1}^{n}((\nu_1^*(x_i),\nu_2^*(y_i)),(\gamma_1^*(t_i),\gamma_2^*(s_i))\right)\right)\psi\left(\Theta\left(\prod_{j=1}^{m}((\nu_1^*(x_j'),\nu_2^*(y_j')),(\beta_1^*(m_j'),\beta_2^*(n_j'))\right)\right)\end{aligned}$$

where $c_{ij}\in\gamma_1^*(t_i)\widehat{\nu_1^*(x_j')}\beta_1^*(m_j')$ and $d_{ij}\in\gamma_2^*(s_i)\widehat{\nu_2^*(y_j')}\beta_2^*(n_j')$. This completes the proof. ∎

Theorem 4.4. *There exists a covariant functor between the category of $\widehat{\Gamma}$-hypermodules and the category of modules.*

Proof. Suppose that M_1, M_2 and M_3 be $\widehat{\Gamma}$-hypermodules. We define $T(M_1) = [M_1 : \Gamma]$, $T(M_2) = [M_2 : \Gamma]$ and $T(M_3) = [M_3 : \Gamma]$. Let $\psi_1 : M_1 \longrightarrow M_2$ and $\psi_2 : M_2 \longrightarrow M_3$ be homomorphisms. We define $T(\psi_1) = \overline{\psi}_1$, $T(\psi_2) = \overline{\psi}_2$ and $T(\psi_3) = \overline{\psi}_3$. Then, for $\Theta_1\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(m_i))\right) \in [M_1 : \Gamma]$, we have

$$\begin{aligned}\overline{\psi}_2 \circ \overline{\psi}_1\left(\Theta_1\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(m_i))\right)\right) &= \Theta_3\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\psi_2 \circ \psi_1(m_i))\right)\\ &= \Theta_3\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\psi_2(\psi_1(m_i)))\right).\end{aligned}$$

Thus, $T(\psi_2 \circ \psi_1) = T(\psi_2) \circ T(\psi_1)$. On the other hand, if Id_M is an identity homomorphism, then $T(Id)$ is an identity homomorphism. Therefore, T is a covariant functor. ∎

5. Conclusion

In this paper, we have considered the general Γ-hypermodules as a generalization of Γ-hypermodules, as a generalization of Γ-modules and as a generalization of modules. In particular, we have given three isomorphism theorems of general Γ-hypermodules and we have discussed about Θ relations on general Γ-hypermodules. Also we show that there is a covariant functor between the category of general Γ-hypermodules and the category of modules. A possible future study could be devoted to the introduction and analysis of (m,n)- ary Γ-hypermodules.

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