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Fundamental PDE's of the Canonical Almost Geodesic Mappings of Type $\tilde{\pi}_1$

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Abstract. For modelling of various physical processes, geodesic lines and almost geodesic curves serve as a useful tool. Trasformations or mappings between spaces (endowed with the metric or connection) which preserve such curves play an important role in physics, particularly in mechanics, and in geometry as well. Our aim is to continue investigations concerning existence of almost geodesic mappings of manifolds with linear (affine) connection, particularly of the so-called $\tilde{\pi}_1$ mappings, i.e. canonical almost geodesic mappings of type π_1 according to Sinyukov. First we give necessary and sufficient conditions for existence of $\tilde{\pi}_1$ mappings of a manifold endowed with a linear connection onto pseudo-Riemannian manifolds. The conditions take the form of a closed system of PDE's of first order of Cauchy type. Further we deduce necessary and sufficient conditions for existence of $\tilde{\pi}_1$ mappings onto generalized Ricci-symmetric spaces. Our results are generalizations of some previous theorems obtained by N.S. Sinyukov.

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1. Introduction

Geodesic and almost geodesic lines serve as a useful tool for modelling of various physical processes, and mappings between spaces (endowed with the metric or connection) and transformations which preserve such curves, play an important role in geometry as well as in physics, particularly in mechanics, optics and the theory of relativity, [10–12].

Many geometric problems connected with the topic of differential geometry are solved by means of differential equations, particularly, the problems are often answered by solving systems of partial differential equations (PDE's) for components of some geometric objects (e.g. tensors), [2, 4–10, 15]. We intend to study here the existence problem of canonical almost geodesic mappings, and as we shall see, our main tool will be to construct and solve a suitable system of PDE's of Cauchy type that controls the situation. One of characteristic

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properties of a system of PDE's of Cauchy type is that the solution of such a system depends on a finite number of real (or complex) parameters. Moreover, solutions of such systems can be effectively enumerated, eventually some approximation can be found.

Unless otherwise specified, all objects under consideration are supposed to be differentiable of a sufficiently high class (mostly, differentiability of the class C^3 is sufficient).

Let $A_n = (M, \nabla)$ be an n-dimensional (C^k, C^∞ or C^ω) manifold endowed with a linear connection ∇. Let $c : I \to M, t \mapsto c(t)$ defined on an open interval $I \subset \mathbb{R}$ be a (C^k, or smooth) curve on M satisfying the regularity condition

$$c'(t) = dc(t)/dt \neq 0 \quad \text{for all } t \in I.$$

Denote by ξ the corresponding (C^{k-1}, or smooth) tangent vector field along c ("velocity field"), $\xi(t) = (c(t), c'(t))$, $t \in I$, and let

$$\xi_1 = \nabla(\xi;\xi) = \nabla_\xi \xi, \quad \xi_2 = \nabla^2(\xi;\xi,\xi) = \nabla_\xi \xi_1.$$

Geodesics $c(s)$, parametrized by canonical affine parameter (given up to the affine transformations $s \mapsto as + b$), are characterized by $\nabla_\xi \xi = 0$ while unparametrized geodesic curves (i.e. arbitrarily parametrized, called also *pregeodesics* in the literature) can be characterized by the formula $\nabla_\xi \xi = \lambda \xi$ where $\lambda(t) : I \to \mathbb{R}$ is a real function.

Let $D = \text{span}\,(X_1, X_2)$ (i.e. the vector fields X_1, X_2 along c form a basis of D). Recall that D is parallel (along c) if and only if the covariant derivatives along c of basis vector fields belong to the distribution (the property is independent of reparametrization of the curve) [14–16].

As a generalization of (an unparametrized) geodesic, let us introduce an *almost geodesic curve* as a curve c satisfying: there exists a two-dimensional (differentiable) distribution D parallel along c (relative to ∇) such that for any tangent vector of c, its parallel translation along c (to any other point) belongs to the distribution D. Equivalently, c is almost geodesic if and only if there exist vector fields X_1, X_2 parallel along c (i.e. satisfying $\nabla_\xi X_i = a^j X_j$ for some differentiable functions $a_i^j(t) : I \to \mathbb{R}$) and differentiable real functions $b^i(t)$, $t \in I$ along c, such that $\xi = b^1 X_1 + b^2 X_2$ holds. For almost geodesic curves, the vector fields ξ_1 and ξ_2 belong to the corresponding distribution D. If the vector fields ξ and ξ_1 are independent at any point (and hence the (local) curve c is not a geodesic one), we can write $D = \text{span}\,(\xi, \xi_1)$. So we get another equivalent characterization: *a curve is almost geodesic if and only if* $\xi_2 \in \text{span}\,(\xi, \xi_1)$.

2. Almost geodesic mappings

Geodesic mappings of manifolds with linear connection are (C^k)-diffeomorphisms characterized by the property that all geodesics are send onto (unparametrized in general) geodesic curves. The classification of geodesic mappings is more or less known. Recall that even for Riemannian spaces, there is a lack of a nice simple criterion for decision when a given Riemannian space admits non-trivial geodesic mappings.

Let $A_n = (M, \nabla)$, $\bar{A}_n = (\bar{M}, \bar{\nabla})$ be n-dimensional manifolds ($n > 2$) each endowed with a torsion-free linear connection.

We may ask which (C^k-)diffeomorphisms of manifolds send almost geodesic curves onto almost geodesic again. The answer is: such mappings reduce to geodesic ones, since there are "too many" almost geodesic curves. It appears that the following definition is more acceptable.

We say that a (C^k-)diffeomorphism $f\colon M \to \bar{M}$ is *almost geodesic* if any geodesic curve of (M,∇) is mapped under f onto an almost geodesic curve in $(\bar{M},\bar{\nabla})$.

This concept of an almost geodesic mapping was introduced by Sinyukov [14], and before by Chernyshenko [3], from a rather different point of view. The theory of almost geodesic mappings was treated in [14–16].

Due to the fact that f is a diffeomorphism we can accept the useful convention that both linear connections ∇ and $\bar{\nabla}$ are in fact defined on the same underlying manifold M, so that we can consider their difference tensor field of type $(1,2)$, $P = \bar{\nabla} - \nabla$, called sometimes a *deformation tensor* of the given connections under f [15], given by $\bar{\nabla}(X,Y) = \nabla(X,Y) + P(X,Y)$ for $X,Y \in \mathscr{X}(M)$ (componentwise, $\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + P^h_{ij}$). Since the connections are symmetric, P is also symmetric in X,Y. Of course, we identify objects on M with their corresponding objects on $\bar{M}$: a curve c on M identifies with its image $\bar{c} = f \circ c$, its tangent vector field $\xi(t)$ with the corresponding vector field $\bar{\xi}(t) = Tf(\xi(t))$ etc.

Besides the deformation tensor, we will use the tensor field of type $(1,3)$, denoted by the same symbol P, introduced by

$$P(X,Y,Z) = \sum_{CS(X,Y,Z)} \nabla_Z P(X,Y) + P(P(X,Y),Z), \quad X,Y,Z \in \mathscr{X}(M),$$

where $\sum_{CS(,,)}$ means the cyclic sum on arguments in brackets (i.e. symmetrization without coefficients).

Almost geodesic diffeomorphisms $f\colon (M,\nabla) \to (M,\bar{\nabla})$ are characterized by the following condition on the type $(1,3)$ tensor P:

$$P(X_1,X_2,X_3) \wedge P(X_4,X_5) \wedge X_6 = 0, \quad X_i \in \mathscr{X}(M),\ i = 1,\dots,6;$$

$X \wedge Y$ means the decomposable bivector, the exterior product of X and Y.

Sinyukov [14–16] distinguished three kinds of almost geodesic mappings, namely π_1, π_2, and π_3, characterized, respectively, by the conditions for the deformation tensor:

$$\pi_1\colon \nabla_X P(X,X) + P(P(X,X),X) = a(X,X)\cdot X + b(X)\cdot P(X,X), \quad X \in \mathscr{X}(M),$$

where $a \in S^2(M)$ is a symmetric tensor field of type $(0,2)$ and b is a 1-form;

$$\pi_2\colon P(X,X) = \psi(X)\cdot X + \varphi(X)\cdot F(X), \quad X \in \mathscr{X}(M),$$

where ψ and φ are 1-forms, and F is a type $(1,1)$ tensor field satisfying

$$\nabla_X F(X) + \varphi(X)\cdot F(F(X)) = \mu(X)\cdot X + \rho(X)\cdot F(X), \quad X \in \mathscr{X}(M)$$

for some 1-forms μ, ρ;

$$\pi_3\colon P(X,X) = \psi(X)\cdot X + a(X,X)\cdot Z, \quad X \in \mathscr{X}(M)$$

where ψ is a 1-form, $a \in S^2(M)$ is a symmetric bilinear form and $Z \in \mathscr{X}(M)$ is a vector field satisfying

$$\nabla_X Z = h\cdot X + \theta(X)\cdot Z$$

for some scalar function $h\colon M \to \mathbb{R}$ and some 1-form θ. Remark that the above classes are not disjoint.

3. Canonical almost geodesic mappings $\tilde{\pi}_1$

We are interested here in a particular subclass of π_1-mappings, the so-called $\tilde{\pi}_1$*-mappings*, or *canonical* almost geodesic mappings, distinguished by the condition $b = 0$. That is, $\tilde{\pi}_1$-mappings are just morphisms satisfying

$$\nabla_X P(X,X) + P(P(X,X),X) = a(X,X)\cdot X, \quad a \in S^2(M),\ X \in \mathscr{X}(M).$$

In local coordinates, the condition reads

$$P^h_{(ij,k)} = a_{(ij}\delta^h_{k)} - P^h_{\alpha(i}P^\alpha_{jk)}. \tag{3.1}$$

Here and in what follows, the comma "," denotes covariant derivative with respect to ∇, the round bracket denote the cyclic sum on indices involved, δ^h_i is the Kronecker delta, and P^h_{ij} are component of deformations tensor P.

Recall that the equation $\bar{\nabla}(X,Y) = \nabla(X,Y) + P(X,Y)$ is in local coordinates expressed as

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + P^h_{ij},$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of connections ∇ and $\bar{\nabla}$, respectively.

Any geodesic mapping is a π_1-mapping (the characterizing condition can be checked), and any π_1-mapping can be written as a composition of a geodesic mapping followed by a $\tilde{\pi}_1$-mapping. So we can consider geodesic mappings as trivial almost geodesic mappings, and we will omit them in further considerations; they were analysed in [1].

Recall ([7,13]) that a manifold (M,∇) is called a *Ricci-symmetric space*[1] when the Ricci tensor is parallel with respect to the connection ∇, i.e.

$$\nabla \mathrm{Ric} = 0.$$

It was proven by Sinyukov [15], that the basic partial differential equations (PDE's) of $\tilde{\pi}_1$-mappings of a manifold (M,∇) onto Ricci-symmetric pseudo-Riemannian manifolds $(\bar{M},\bar{g})$ (of arbitrary signature) can be transformed into (an equivalent) closed system of PDE's of first order of the Cauchy type. Hence the solution (if it exists) depends on a finite set of parameters. Consequently, for a manifold with a symmetric connection admitting $\tilde{\pi}_1$-mappings onto Ricci-symmetric spaces, the set of all Ricci-symmetric spaces $(\bar{M},\bar{g})$ which can serve as images of the given manifold (M,∇) under $\tilde{\pi}_1$-mappings is finite. The cardinality r of such a set is bounded by the number of free parameters.

On the other hand, geodesic mappings form a subclass among $\tilde{\pi}_1$-mappings (they obey the definition). Basic equations describing geodesic mappings of manifolds with linear connection do not form a closed system of Cauchy type (the general solution depends on n arbitrary functions; if the given manifold admits geodesic mappings, the cardinality of the set of possible images is big). It follows that the conditions (3.1) describing $\tilde{\pi}_1$-mappings of manifolds, in general, cannot be transformed into a closed system of Cauchy type. But if we choose a suitable subclass of images and restrict ourselves (for the given manifold) only onto mappings with co-domain in the apropriate subclass we might succeed to get an equivalent closed system of Cauchy type. If this is the case then the given manifold admits either non (if the system is non-integrable) or a finite number of $\tilde{\pi}_1$-images in the given class.

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[1] In analogy to symmetric spaces that are characterized by parallel Riemannian curvature tensor: $\nabla R = 0$.

Our aim is to analyse $\tilde{\pi}_1$-mappings of manifolds onto manifolds with linear connection in general, and to use the reached results for examining $\tilde{\pi}_1$-mappings of manifolds onto (pseudo-)Riemannian spaces (in general, without any restrictive conditions onto the Ricci tensor), which will generalize the above result by Sinyukov. In the rest, we will omit "pseudo".

All $\tilde{\pi}_1$-mappings $f\colon M \to M$ can be described by the following system of differential equations [15, 16]:

$$3(\nabla_Z P(X,Y) + P(Z, P(X,Y))) = \sum_{CS(X,Y)} (R(Y,Z)X - \bar{R}(Y,Z)X) + \sum_{CS(X,Y,Z)} a(X,Y)Z. \tag{3.2}$$

In what follows, we prefer to express our equalities in local coordinates (with respect to a map (U,φ) on M) since the invariant formulas are rather complicated. The above formula has the local expression

$$3(P^h_{ij,k} + P^h_{k\alpha}P^\alpha_{ij}) = R^h_{(ij)k} - \bar{R}^h_{(ij)k} + a_{(ij}\delta^h_{k)}, \tag{3.3}$$

where P^h_{ij}, a_{ij}, R^h_{ijk}, $\bar{R}^h_{ijk}$ are local components of tensors P, a, R, and $\bar{R}$.

4. Properties of the fundamental equations of the canonical almost geodesic mappings $\tilde{\pi}_1$

Assuming (3.3) as a system of PDE's for functions P^h_{ij} on M, the corresponding integrability conditions read

$$\begin{aligned}\bar{R}^h_{(ij)[k,\ell]} &= R^h_{(ij)[k,\ell]} + \delta^h_{(i}a_{jk),\ell} - \delta^h_{(i}a_{j\ell),k} + 3\left(P^\alpha_{ij}\bar{R}^h_{\alpha k\ell} - P^h_{\alpha(j}R^\alpha_{i)k\ell}\right)\\ &\quad - P^h_{\alpha k}\left(R^\alpha_{(ij)\ell} - \bar{R}^\alpha_{(ij)\ell} + \delta^\alpha_{(i}a_{j\ell)}\right) + P^h_{\alpha\ell}\left(R^\alpha_{(ij)k} - \bar{R}^\alpha_{(ij)k} + \delta^\alpha_{(i}a_{jk)}\right).\end{aligned}$$

Passing from $\nabla\bar{R}$ to $\bar{\nabla}\bar{R}$ on the left hand side we get integrability conditions of the system (3.3) in the form

$$\bar{R}^h_{(ij)[k;\ell]} = \delta^h_{(i}a_{jk),\ell} - \delta^h_{(i}a_{j\ell),k} + \Theta^h_{ijk\ell}; \tag{4.1}$$

here we denoted

$$\begin{aligned}\Theta^h_{ijk\ell} &= R^h_{(ij)[k,\ell]} + 3\left(P^\alpha_{ij}\bar{R}^h_{\alpha k\ell} - P^h_{\alpha(j}R^\alpha_{i)k\ell}\right) - P^h_{\alpha k}\left(R^\alpha_{(ij)\ell} + \delta^\alpha_{(i}a_{j\ell)}\right)\\ &\quad + P^h_{\alpha\ell}\left(R^\alpha_{(ij)k} + \delta^\alpha_{(i}a_{jk)}\right) - P^\alpha_{\ell(i}\bar{R}^h_{|\alpha|j)k} - P^\alpha_{\ell(i}\bar{R}^h_{j)\alpha k} + P^\alpha_{k(i}\bar{R}^h_{|\alpha|j)\ell} + P^\alpha_{k(i}\bar{R}^h_{j)\alpha\ell}\end{aligned}$$

where ";" denotes covariant derivative with respect to $\bar{\nabla}$.

If we apply covariant differentiation with respect to $\bar{\nabla}$ to the integrability conditions (4.1) of the system (3.3), and then pass from covariant derivation $\bar{\nabla}$ to ∇, we get

$$\bar{R}^h_{(ij)k;\ell m} - \bar{R}^h_{(ij)\ell;mk} = \delta^h_{(i}a_{jk),\ell m} - \delta^h_{(i}a_{j\ell),km} + T^h_{ijk\ell m}, \tag{4.2}$$

where we denoted

$$\begin{aligned}T^h_{ijk\ell m} &= \bar{R}^h_{\alpha mk}\bar{R}^\alpha_{(ij)\ell} - \bar{R}^\alpha_{\ell mk}\bar{R}^h_{(ij)\alpha} - \bar{R}^\alpha_{jmk}\bar{R}^h_{(i\alpha)\ell} - \bar{R}^\alpha_{imk}\bar{R}^h_{(j\alpha)\ell} - P^h_{m\alpha}\delta^\alpha_{(i}a_{jk),\ell}\\ &\quad - P^\alpha_{mj}\delta^h_{(i}a_{\alpha k),\ell} - P^\alpha_{mi}\delta^h_{(\alpha}a_{jk),\ell} - P^\alpha_{mk}\delta^h_{(\alpha}a_{ij),\ell} - P^\alpha_{ml}\delta^h_{(i}a_{jk),\alpha} - P^h_{m\alpha}\delta^\alpha_{(i}a_{j\ell),k}\\ &\quad + P^\alpha_{mi}\delta^h_{(\alpha}a_{j\ell),k} + P^\alpha_{mj}\delta^h_{(i}a_{\alpha\ell),k} + P^\alpha_{mk}\delta^h_{(i}a_{j\ell),\alpha} - P^\alpha_{ml}\delta^h_{(i}a_{j\alpha),k} - \Theta^h_{ijk\ell,m}\\ &\quad + P^h_{\alpha m}\Theta^\alpha_{ijk\ell} - P^\alpha_{mi}\Theta^h_{\alpha jk\ell} - P^\alpha_{mj}\Theta^h_{i\alpha k\ell} - P^\alpha_{mk}\Theta^h_{ij\alpha\ell} - P^\alpha_{m\ell}\Theta^h_{ijk\alpha}.\end{aligned} \tag{4.3}$$

Alternating (4.2) in ℓ, m we get

$$\begin{aligned}\bar{R}^h_{(ij)m;\ell k} - \bar{R}^h_{(ij)\ell;mk} = \delta^h_{(i} a_{jm),k\ell} - \delta^h_{(i} a_{j\ell),km} + T^h_{ijk[lm]} + \bar{R}^h_{(i|\alpha k|} \bar{R}^\alpha_{j)m\ell} \\ + \bar{R}^h_{(ij)\alpha} \bar{R}^\alpha_{km\ell} - \bar{R}^\alpha_{(ij)k} \bar{R}^h_{\alpha m\ell} + \bar{R}^h_{\alpha(i|k|} \bar{R}^\alpha_{j)m\ell} + \delta^h_{(\alpha} a_{jk)} R^\alpha_{i\ell m} \\ + \delta^h_{(\alpha} a_{ik)} R^\alpha_{j\ell m} + \delta^h_{(i} a_{j\alpha)} R^\alpha_{k\ell m} - \delta^h_{(i} a_{jk)} R^\alpha_{\alpha\ell m}.\end{aligned} \tag{4.4}$$

Using properties of the Riemannian tensor, we rewrite (4.4) as

$$\bar{R}^h_{im\ell;jk} + \bar{R}^h_{jm\ell;ik} = \delta^h_{(i} a_{j\ell),km} - \delta^h_{(i} a_{jm),k\ell} - N^h_{ijk\ell m}, \tag{4.5}$$

where the last term is

$$\begin{aligned}N^h_{ijk\ell m} = T^h_{ijk[\ell m]} + \bar{R}^\alpha_{im\ell} \bar{R}^h_{(\alpha j)k} + \bar{R}^\alpha_{jm\ell} \bar{R}^h_{(\alpha i)k} + \bar{R}^\alpha_{km\ell} \bar{R}^h_{(ij)\alpha} - \bar{R}^h_{\alpha m\ell} \bar{R}^\alpha_{(ij)k} \\ + \delta^h_{(\alpha} a_{jk)} R^\alpha_{i\ell m} + \delta^h_{(\alpha} a_{ik)} R^\alpha_{j\ell m} + \delta^h_{(\alpha} a_{ij)} R^\alpha_{k\ell m} - a_{(ij} R^h_{k)\ell m}.\end{aligned}$$

Alternating (4.5) over j, k we get

$$\begin{aligned}\bar{R}^h_{jm\ell;ik} - \bar{R}^h_{km\ell;ij} = \delta^h_{(i} a_{j\ell),km} - \delta^h_{(i} a_{jm),k\ell} - \delta^h_{(i} a_{k\ell),jm} + \delta^h_{(i} a_{km),j\ell} - N^h_{i[jk]\ell m} \\ + \bar{R}^h_{\alpha m\ell} \bar{R}^\alpha_{ikj} + \bar{R}^h_{i\alpha\ell} \bar{R}^\alpha_{mkj} + \bar{R}^h_{im\alpha} \bar{R}^\alpha_{\ell kj} - \bar{R}^\alpha_{im\ell} \bar{R}^h_{\alpha kj}.\end{aligned} \tag{4.6}$$

Let us change mutually i and k in (4.5), and then use (4.6). We evaluate

$$\begin{aligned}2\bar{R}^h_{jm\ell;ik} = \delta^h_{(i} a_{j\ell),km} - \delta^h_{(i} a_{jm),k\ell} - \delta^h_{(k} a_{jm),i\ell} + \delta^h_{(i} a_{km),j\ell} \\ - \delta^h_{(i} a_{k\ell),jm} + \delta^h_{(j\ell} a_{k),im} + \Omega^h_{ijk\ell m},\end{aligned} \tag{4.7}$$

where we used the notation

$$\begin{aligned}\Omega^h_{ijk\ell m} = -N^h_{ijk\ell m} + N^h_{k[ij]k\ell m} - \bar{R}^h_{\alpha m\ell} \bar{R}^\alpha_{(kj)i} + \bar{R}^h_{j\alpha\ell} \bar{R}^\alpha_{mik} + \bar{R}^h_{jm\alpha} \bar{R}^\alpha_{\ell ik} - \bar{R}^h_{\alpha i(j} \bar{R}^\alpha_{k)m\ell} \\ + \bar{R}^h_{j\alpha\ell} \bar{R}^\alpha_{mik} + \bar{R}^h_{jm\alpha} \bar{R}^\alpha_{\ell ik} - \bar{R}^h_{\alpha m\ell} \bar{R}^\alpha_{ikj} - \bar{R}^h_{i\alpha\ell} \bar{R}^\alpha_{mkj} + \bar{R}^\alpha_{im[\ell} \bar{R}^h_{\alpha]kj}.\end{aligned}$$

On the left side of (4.7), let us pass from the covariant derivation $\bar{\nabla}$ to ∇:

$$\begin{aligned}2\bar{R}^h_{jm\ell,ik} = \delta^h_{(i} a_{j\ell),km} - \delta^h_{(i} a_{jm),k\ell} - \delta^h_{(k} a_{jm),i\ell} + \delta^h_{(i} a_{km),j\ell} \\ - \delta^h_{(i} a_{k\ell),jm} - \delta^h_{(k} a_{j\ell),im} + S^h_{ijk\ell m},\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}S^h_{ijk\ell m} = \Omega^h_{ijk\ell m} - 2\Big[\bar{R}^\alpha_{jm\ell,i} P^h_{\ell k} - \bar{R}^h_{\alpha m\ell,i} P^\alpha_{jk} - \bar{R}^h_{j\alpha\ell,i} P^\alpha_{mk} - \bar{R}^h_{jm\alpha,i} P^\alpha_{\ell k} - \bar{R}^h_{jm\ell,\alpha} P^\alpha_{ik} \\ + \Big(\bar{R}^\alpha_{jm\ell} P^\beta_{\alpha i} - \bar{R}^h_{\alpha m\ell} P^\alpha_{ij} - \bar{R}^h_{j\alpha\ell} P^\alpha_{im} - \bar{R}^h_{jm\alpha} P^\alpha_{i\ell}\Big) P^h_{\beta k} - \Big(\bar{R}^\alpha_{jm\ell} P^h_{\alpha\beta} \\ - \bar{R}^h_{\alpha m\ell} P^\alpha_{\beta j} - \bar{R}^h_{j\alpha\ell} P^\alpha_{\beta m} - \bar{R}^h_{jm\alpha} P^\alpha_{\beta\ell}\Big) P^\beta_{ik} - \Big(\bar{R}^\alpha_{\beta m\ell} P^h_{\alpha i} - \bar{R}^h_{\alpha m\ell} P^\alpha_{\beta i} \\ - \bar{R}^h_{\beta\alpha\ell} P^\alpha_{im} - \bar{R}^h_{\beta m\alpha} P^\alpha_{i\ell}\Big) P^\beta_{jk} - \Big(\bar{R}^\alpha_{j\beta\ell} P^h_{\alpha i} - \bar{R}^h_{\alpha\beta\ell} P^\alpha_{ji} - \bar{R}^h_{j\alpha\ell} P^\alpha_{\beta i} \\ - \bar{R}^h_{j\beta\alpha} P^\alpha_{i\ell}\Big) P^\beta_{km} - \Big(\bar{R}^\alpha_{jm\beta} P^h_{\alpha i} - \bar{R}^h_{\alpha m\beta} P^\alpha_{ji} - \bar{R}^h_{j\alpha\beta} P^\alpha_{mi} - \bar{R}^h_{jm\alpha} P^\alpha_{\beta i}\Big) P^\beta_{k\ell}\Big].\end{aligned} \tag{4.9}$$

5. Canonical almost geodesic mappings $\tilde{\pi}_1$ onto Riemannian spaces

Let there exist a $\tilde{\pi}_1$-mapping of a manifold $A_n = (M,\nabla)$ onto a Riemannian manifold $\bar{V}_n = (M,\bar{g})$ where $\bar{g} \in T_2^0 M$ is a metric tensor with components $\bar{g}_{ij}$. Recall that the Riemannian tensor $\bar{R}_{hijk} = \bar{R}^{\alpha}_{ijk}\bar{g}_{\alpha h}$ of type $(0,4)$ satisfies

$$\bar{R}_{hijk} + \bar{R}_{ihjk} = 0. \tag{5.1}$$

In (4.7), let us lower the index h by means of the metric tensor $\bar{g}$ and then use symmetrization with respect to h and j. According to (5.1) we get

$$\begin{aligned}&\bar{g}_{ih}\big(a_{m[k,j]l} + a_{l[j,k]m}\big) + \bar{g}_{ij}\big(a_{m[k,h]l} + a_{l[h,k]m}\big) + \bar{g}_{kh}\big(a_{m[i,j]l} + a_{l[j,i]}\big) + \bar{g}_{kj}\big(a_{m[i,h]l}\\ &+ a_{l[h,i]m}\big) + \bar{g}_{mh}\big(a_{k[i,j]l} - a_{ij,kl}\big) + \bar{g}_{mj}\big(a_{k[i,h]l} - a_{ih,kl}\big) + \bar{g}_{lj}\big(a_{kh,il} - a_{i(h,k)m}\big)\\ &+ 2\bar{g}_{jh}\big(a_{k(l,i)m} - a_{m(i,k)l}\big) + \bar{g}_{lh}\big(a_{k[j,i]m} - a_{ij,km}\big) = -\Omega^{\alpha}_{i(j|klm}\bar{g}_{\alpha|h)}.\end{aligned} \tag{5.2}$$

Contraction of the last formula with the dual tensor $\bar{g}^{jh}$ ($\|\bar{g}^{ij}\| = \|\bar{g}_{ij}\|^{-1}$) gives

$$a_{kl,im} - a_{im,kl} - a_{km,il} + a_{il,km} = -\frac{2}{n+1}\Omega^{\alpha}_{i\alpha klm}. \tag{5.3}$$

Let us symmetrize the above formula over k and l. From (5.3) we get

$$\begin{aligned}2a_{kl,im} - 2a_{im,kl} = {}& 2a_{\alpha m}R^{\alpha}_{lik} + a_{\alpha i}R^{\alpha}_{mlk} + a_{\alpha k}R^{\alpha}_{mil} + a_{\alpha l}R^{\alpha}_{mik}\\ &+ \frac{2}{n+1}\Big(\Omega^{\alpha}_{l\alpha kim} - \Omega^{\alpha}_{i\alpha(kl)m}\Big).\end{aligned} \tag{5.4}$$

Using (5.3) and (5.4) the Equation (5.2) reads

$$\begin{aligned}&2\bar{g}_{ih}(a_{km,jl} - a_{jm,kl}) + 2\bar{g}_{ij}(a_{km,hl} - a_{hm,kl}) + 2\bar{g}_{kh}(a_{im,jl} - a_{jm,il})\\ &+ 2\bar{g}_{kj}(a_{im,hl} - a_{hm,il}) + \bar{g}_{mk}(a_{ki,jl} - a_{kj,il} - a_{ij,kl}) + \bar{g}_{mj}(a_{ki,hl}\\ &- a_{kh,il} - a_{ih,kl}) + \bar{g}_{lj}(a_{kh,im} - a_{i(h,k)m}) + \bar{g}_{lh}(a_{kj,im} - a_{i(k,j)m}) = C_{ijkhl},\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}C_{ijkhl} = {}& -\Omega^{\alpha}_{i(j|klm}\bar{g}_{\alpha|h)} + \frac{2}{n+1}\Omega^{\alpha}_{i\alpha klm}\bar{g}_{jh} - \bar{g}_{kh}a_{\alpha l}R^{\alpha}_{mij}\\ &+ \bar{g}_{ih}\left(\frac{2}{n+1}\Omega^{\alpha}_{m\alpha ljk} - a_{\alpha k}R^{\alpha}_{(ml)j} - a_{\alpha j}R^{\alpha}_{(l|k|m)} - a_{\alpha m}R^{\alpha}_{lkj} - a_{\alpha l}R^{\alpha}_{mkj}\right)\\ &+ \bar{g}_{ij}\left(\frac{2}{n+1}\Omega^{\alpha}_{m\alpha lhk} - a_{\alpha k}R^{\alpha}_{(ml)h} - a_{\alpha h}R^{\alpha}_{(l|k|m)} - a_{\alpha m}R^{\alpha}_{lkh} - a_{\alpha l}R^{\alpha}_{mkh}\right)\\ &+ \bar{g}_{kh}\left(\frac{2}{n+1}\Omega^{\alpha}_{m\alpha lji} - a_{\alpha i}R^{\alpha}_{(ml)j} - a_{\alpha j}R^{\alpha}_{(l|i|m)} - a_{\alpha m}R^{\alpha}_{lij} + a_{\alpha l}R^{\alpha}_{mij}\right)\\ &+ \bar{g}_{kj}\left(\frac{2}{n+1}\Omega^{\alpha}_{m\alpha lhi} - a_{\alpha i}R^{\alpha}_{(ml)h} - a_{\alpha h}R^{\alpha}_{(l|i|m)} - a_{\alpha m}R^{\alpha}_{lih} + a_{\alpha l}R^{\alpha}_{mih}\right).\end{aligned}$$

If we contract (5.5) with the dual $\bar{g}^{ij}$ of the metric tensor, use (5.4) and the Ricci identity we get

$$a_{km,hl} - a_{kl,hm} = \frac{1}{2(n+3)}(\bar{g}_{hm}\mu_{kl} - \bar{g}_{hl}\mu_{km}) + B_{kmhl}, \tag{5.6}$$

where $\mu_{km} = a_{\alpha\beta,km}\bar{g}^{\alpha\beta}$, and

$$B_{kmhl} = C_{\alpha\beta kmhl}\bar{g}^{\alpha\beta} + 3a_{m\alpha}R^{\alpha}_{lhk} + \frac{3}{2}\Big(a_{h\alpha}R^{\alpha}_{mkl} + a_{k\alpha}R^{\alpha}_{mhl} + a_{l\alpha}R^{\alpha}_{mhk}\Big)$$
$$+ \frac{3}{n+1}\Big(\Omega^{\alpha}_{l\alpha khm} - \Omega^{\alpha}_{h\alpha(kl)m}\Big) - \frac{1}{2}\Big(a_{m\alpha}R^{\alpha}_{lkm} + a_{k\alpha}R^{\alpha}_{mhl} + a_{h\alpha}R^{\alpha}_{mkl} + a_{l\alpha}R^{\alpha}_{mkh}\Big)$$
$$- \frac{1}{n+1}\Big(\Omega^{\alpha}_{l\alpha hkm} - \Omega^{\alpha}_{k\alpha(hl)m}\Big) - a_{\alpha(h}R^{\alpha}_{k)lm} + \frac{1}{2}\Big(a_{k\alpha}R^{\alpha}_{lmh} + a_{h\alpha}R^{\alpha}_{lkm} + a_{m\alpha}R^{\alpha}_{lkh}\Big).$$

Now contract (5.5) with $\bar{g}^{ih}$. According to (5.6) we get

$$\bar{g}_{kl}\mu_{jm} - \bar{g}_{jl}\mu_{km} + \bar{g}_{km}\mu_{jl} - \bar{g}_{jkm}\mu_{kl} = \frac{n+3}{n+1}C_{kljm}, \tag{5.7}$$

where

$$C_{kljm} = C_{\alpha jkl(m|\beta|l)}\bar{g}^{\alpha\beta} - 2(n+1)\Big(B_{k(ml)j} - a_{\alpha(l}R^{\alpha}_{m)jk} + a_{j\alpha}R^{\alpha}_{(m|k|l)} + a_{k\alpha}R^{\alpha}_{(lm)j}\Big).$$

Contracting (5.7) with $\bar{g}^{k\ell}$ and using the notation $K = \mu_{\alpha\beta}\bar{g}^{\alpha\beta}$ we obtain components of the tensor μ:

$$\mu_{jm} = \frac{1}{n}K\bar{g}_{jm} + \frac{n+3}{n(n+1)}C_{\alpha\beta jm}\bar{g}^{\alpha\beta}. \tag{5.8}$$

Using (5.8) we can rewrite (5.6) in the form

$$a_{km,hl} - a_{hm,kl} = \frac{K}{2n(n+3)}\left(\bar{g}_{mh}\bar{g}_{kl} - \bar{g}_{lh}\bar{g}_{km}\right) + A_{kmhl}, \tag{5.9}$$

where

$$A_{kmhl} = B_{kmhl} + \frac{1}{2n(n+1)}\Big(\bar{g}_{hm}C_{\alpha\beta kl}\bar{g}^{\alpha\beta} - \bar{g}_{hl}C_{\alpha\beta km}\bar{g}^{\alpha\beta}\Big).$$

Combining (5.5) and (5.9) we get

$$\begin{aligned} &\bar{g}_{jl}a_{ih,km} + \bar{g}_{hl}a_{ij,km} - \bar{g}_{jm}a_{ih,kl} - \bar{g}_{hm}a_{ij,kl} \\ &= -\frac{K}{n(n+3)}\big(\bar{g}_{ih}\bar{g}_{kl}\bar{g}_{jm} - \bar{g}_{ih}\bar{g}_{km}\bar{g}_{jl} + \bar{g}_{ij}\bar{g}_{kl}\bar{g}_{hm} - \bar{g}_{ij}\bar{g}_{km}\bar{g}_{hl} \\ &\quad + 3\bar{g}_{kh}\bar{g}_{il}\bar{g}_{jm} - 3\bar{g}_{kh}\bar{g}_{jl}\bar{g}_{im} + 3\bar{g}_{kj}\bar{g}_{il}\bar{g}_{hm} - 3\bar{g}_{lh}\bar{g}_{jk}\bar{g}_{im}\big) + A_{ijkmhl}, \end{aligned} \tag{5.10}$$

where we have denoted

$$A_{ijkmhl} = C_{ijkmhl} - 2\big(\bar{g}_{i(h}A_{|km|j)l} + \bar{g}_{k(h}A_{|im|jl)} - \bar{g}_{m(h}A_{|ki|j)l} - \bar{g}_{l(h}A_{|k|j)im)}\big).$$

Finally, symetrization of (5.10) over the indices i, j, followed by contraction with $\bar{g}^{\ell h}$ enables us to express second covariant derivatives of the tensor a,

$$a_{ij,km} = \frac{K}{n(n+3)}\left(\bar{g}_{ij}\bar{g}_{km} + 3\bar{g}_{k(j}\bar{g}_{i)m}\right) + A_{(ij)km\alpha\beta}\bar{g}^{\alpha\beta}. \tag{5.11}$$

Now we can consider (5.11) as the first order system of PDE's of Cauchy type relative to the tensor ∇a (i.e. in $a_{ij,k}$), find the integrability conditions and contract them with $\bar{g}^{ij}$ and $\bar{g}^{km}$, respectively. We calculate ∇K,

$$K_{,\beta} = \frac{n(n+3)}{n^2+5n-6}A_{\beta}, \tag{5.12}$$

where we denoted

$$A_\rho = \Big[a_{\alpha(j,|k} R^{\alpha}_{i)m\rho} + a_{ij,\alpha} R^{\alpha}_{km\rho} - \frac{K}{n(n+3)} \big(\bar{g}_{ij,[\rho} \bar{g}_{m]k} + \bar{g}_{ij} \bar{g}_{k[m,\rho]} + 3\bar{g}_{kj,[\rho} \bar{g}_{m]i} + 3\bar{g}_{kj} \bar{g}_{i[m,\rho]}$$
$$+ 3\bar{g}_{ki,[\rho} \bar{g}_{m]j} + 3\bar{g}_{ki} \bar{g}_{j[m,\rho]} \big) + A_{(ij)k[m|\alpha\beta|,\rho]} \bar{g}^{\alpha\beta} + A_{(ij)k[m|\alpha\beta|} \bar{g}^{\alpha\beta}_{\ \ ,\rho]} \Big] \bar{g}^{ij} \bar{g}^{km}.$$

We use $\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + P^h_{ij}$ and get

$$\bar{g}_{ij,k} = P^{\alpha}_{ik} \bar{g}_{\alpha j} + P^{\alpha}_{jk} \bar{g}_{\alpha i}. \tag{5.13}$$

Assume the tensors ∇a and $\nabla \bar{R}$, and denote their components by $a_{ijk} := a_{ij,k}$ and $\bar{R}^h_{ijk\ell} := \bar{R}^h_{ijk,\ell}$, respectively. Then (4.8) and (5.11) take the form

$$\begin{aligned} 2R^h_{jmli,k} = {} & \delta^h_{(i} a_{jl)k,m} - \delta^h_{(i} a_{jm)k,l} + \delta^h_{(k} a_{jl)i,m} - \delta^h_{(k} a_{jm)i,l} \\ & + \delta^h_{(i} a_{km)j,l} - \delta^h_{(i} a_{kl)j,m} + S^h_{ijklm}, \end{aligned} \tag{5.14}$$

$$a_{ijk,m} = \frac{K}{n(n+3)} (\bar{g}_{ij} \bar{g}_{km} + 3\bar{g}_{k(j} \bar{g}_{i)m}) + A_{(ij)km\alpha\beta} \bar{g}^{\alpha\beta}, \tag{5.15}$$

where covariant derivatives of the tensor a_{ijk} in (5.14) are supposed to be expressed according to (5.15), the tensor S was introduced componentwise in (4.9).

The formulas (3.3), (5.12)–(5.15) represent a closed system of Cauchy type for unknown functions

$$\bar{g}_{ij}(x),\ P^h_{ij}(x),\ a_{ij}(x),\ a_{ijk}(x),\ K(x),\ \bar{R}^h_{ijk}(x),\ R^h_{ijkl}(x), \tag{5.16}$$

which, moreover, must satisfy a finite set of algebraic conditions

$$\bar{g}_{[ij]} = P^h_{[ij]} = a_{[ij]} = a_{[ij]k} = \bar{R}^h_{i(jk)} = R^h_{i(jk)l} = 0, \quad \det \|\bar{g}_{ij}(x)\| \neq 0. \tag{5.17}$$

So we have proven:

Theorem 5.1. *The given manifold $A_n = (M, \nabla)$ admits $\tilde{\pi}_1$-mappings (i.e. canonical almost geodesic mappings of type π_1) onto Riemannian spaces $\bar{V}_n = (M, \bar{g})$ if and only if there exists solution of the mixed system of Cauchy type* (3.3), *(5.12)-(5.15), (5.17) for the functions (5.16).*

As a consequence of the additional algebraic conditions, we get an upper boundary for the number r of possible solutions:

Corollary 5.1. *The family of all Riemannian manifolds $\bar{V}_n$ which can serve as images of the given manifold $A_n = (M, \nabla)$, depends on at most*

$$\frac{1}{2} n^2 (n^2 - 1) + n(n+1)^2 + 1$$

parameters.

The above Theorem generalizes the result of Sinyukov [16] already mentioned as well as his results on geodesic mappings of Riemannian spaces.

6. Ricci-symmetric and generalized Ricci-symmetric spaces

It was proven in [15] that the family of all $\tilde{\pi}_1$-mappings of a manifold (M,∇) onto Ricci-symmetric (pseudo-)Riemannian spaces $(\bar{M},\bar{g})$ is given by the integrable system (of Cauchy type) of partial differentiable equations (in covariant derivatives). Consequently, given a manifold with a symmetric connection, the family of all Ricci-symmetric Riemannian spaces $(\bar{M},\bar{g})$ which can serve as images of the given manifold (M,∇) under some $\tilde{\pi}_1$-mapping, depends on a finite set of parameters.

On the other hand, the geodesic mappings form a subset in the set of $\tilde{\pi}_1$-mappings; they obey the definition. But the basic equations describing geodesic mappings of a manifold with the linear connection do not form an integrable system of Cauchy type, since the general solution depends on n arbitrary functions. It follows that the conditions (3.1) describing $\tilde{\pi}_1$-mappings (i.e. canonical almost geodesic mappings) of manifolds do not, in general, induce an integrable system.

In the following, we consider a particular case when (3.1) can be transformed into an integrable system, generalizing the results of Sinyukov. Namely, we will investigate $\tilde{\pi}_1$-mappings of a manifold (M,∇) onto the so-called generalized Ricci-symmetric manifolds.

Definition 6.1. *A manifold* (M,∇) *will be called a* generalized Ricci-symmetric manifold *if its Ricci tensor satisfies*

$$\nabla\mathrm{Ric}\,(Y,Z;X)+\nabla\mathrm{Ric}\,(X,Z;Y)=0, \tag{6.1}$$

that is, $\nabla_X\mathrm{Ric}\,(Y,Z)+\nabla_Y\mathrm{Ric}\,(X,Z)=0.$

We do not a priori suppose the Ricci tensor be symmetric. If Ric is symmetric and (6.1) holds then Ric is parallel, $\nabla\mathrm{Ric}=0$, and (M,∇) is a Ricci-symmetric manifold. Einstein spaces (Riemannian spaces characterized by the property Ric = $\frac{R}{n}\,g$, see [10]) satisfy (6.1) since they satisfy $\nabla\mathrm{Ric}=0$, hence are generalized Ricci-symmetric. In this sense, the generalized Ricci-symmetric spaces can be considered as a certain generalization of Einstein spaces.

7. Almost geodesic mappings $\tilde{\pi}_1$ onto generalized Ricci-symmetric manifolds

Given the n-dimensional manifolds $\mathbb{A}=(M,\nabla)$ and $\bar{\mathbb{A}}=(\bar{M},\bar{\nabla})$ with the corresponding curvature tensors R and $\bar{R}$, respectively, all connection-preserving mappings $f\colon M\to\bar{M}$ can be described by the system of differential equations (3.2), [15–17]. These formulas have the local expression (3.3). As we have already proved, from (3.2) it follows (4.1). Using the Bianchi identity we can write (4.1) in local coordinates as

$$\bar{R}^h_{i\ell k;j}+\bar{R}^h_{j\ell k;i}=\delta^h_{(i}a_{jk),\ell}-\delta^h_{(i}a_{j\ell),k}+\Theta^h_{ijk\ell}\,,$$

where ";" denotes the covariant derivative with respect to $\bar{\nabla}$. Contraction in h and k gives the following equality for covariant derivatives of components of the Ricci tensor $\bar{\mathrm{Ric}}$ of $\bar{\nabla}$:

$$\bar{R}_{i\ell;j}+\bar{R}_{j\ell;i}=(n+1)a_{ij,\ell}-a_{\ell(i,j)}+\Theta^\alpha_{ij\alpha\ell}\,.$$

In the following let us suppose that the manifold $(\bar{M},\bar{\nabla})$ is a generalized Ricci-symmetric space, that is, (6.1) holds. In local coordinates, (6.1) reads

$$\bar{R}_{ij;k}+\bar{R}_{kj;i}=0.$$

Under this assumption, (5.6) reads

$$(n+1)a_{ij,\ell} - a_{\ell i,j} - a_{\ell j,i} = -\Theta^{\alpha}_{ij\alpha\ell}\,. \tag{7.1}$$

Using symmetrization in ℓ, i gives

$$a_{\ell i,j} + a_{\ell j,i} = -\frac{1}{n}\Theta^{\alpha}_{(i|\ell\alpha|j)} + \frac{2}{n}a_{ij,\ell}\,.$$

Now (7.1) reads

$$\frac{n^2+n-2}{n}\,a_{ij,\ell} = -\,\Theta^{\alpha}_{ij\alpha\ell} - \frac{1}{n}\Theta^{\alpha}_{(i|\ell\alpha|j)}\,. \tag{7.2}$$

Applying the covariant differentiation with respect to $\bar{\nabla}$ to the integrability conditions (5.4), followed by passing from the covariant derivative $\bar{\nabla}$ to ∇ on the right hand side, we get

$$\bar{R}^h_{(ij)k;\ell m} - \bar{R}^h_{(ij)\ell;mk} = \delta^h_{(i}a_{jk),\ell m} - \delta^h_{(i}a_{j\ell),km} + T^h_{ijk\ell m}\,, \tag{7.3}$$

where

$$\begin{aligned} T^h_{ijk\ell m} = {} & \bar{R}^h_{\alpha mk}\bar{R}^{\alpha}_{(ij)\ell} - \bar{R}^{\alpha}_{\ell mk}\bar{R}^h_{(ij)\alpha} - \bar{R}^{\alpha}_{jmk}\bar{R}^h_{(i\alpha)\ell} - \bar{R}^{\alpha}_{imk}\bar{R}^h_{(j\alpha)\ell} - P^h_{m\alpha}\delta^{\alpha}_{(i}a_{jk),\ell} \\ & - P^{\alpha}_{mj}\delta^h_{(i}a_{\alpha k),\ell} - P^{\alpha}_{mi}\delta^h_{(\alpha}a_{jk),\ell} - P^{\alpha}_{mk}\delta^h_{(\alpha}a_{ij),\ell} - P^{\alpha}_{ml}\delta^h_{(i}a_{jk),\alpha} - P^h_{m\alpha}\delta^{\alpha}_{(i}a_{j\ell),k} \\ & + P^{\alpha}_{mi}\delta^h_{(\alpha}a_{j\ell),k} + P^{\alpha}_{mj}\delta^h_{(i}a_{\alpha\ell),k} + P^{\alpha}_{mk}\delta^h_{(i}a_{j\ell),\alpha} - P^{\alpha}_{ml}\delta^h_{(i}a_{j\alpha),k} - \theta^h_{ijk\ell,m} \\ & + P^h_{\alpha m}\theta^{\alpha}_{ijk\ell} - P^{\alpha}_{mi}\theta^h_{\alpha jk\ell} - P^{\alpha}_{mj}\theta^h_{i\alpha k\ell} - P^{\alpha}_{mk}\theta^h_{ij\alpha\ell} - P^{\alpha}_{m\ell}\theta^h_{ijk\alpha}\,. \end{aligned}$$

Alternating (7.3) over ℓ, m we obtain

$$\begin{aligned} \bar{R}^h_{(ij)m;\ell k} - \bar{R}^h_{(ij)\ell;mk} = {} & \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(i}a_{j\ell),km} + T^h_{ijk[lm]} + \bar{R}^h_{(i|\alpha k|}\bar{R}^{\alpha}_{j)m\ell} \\ & + \bar{R}^h_{(ij)\alpha}\bar{R}^{\alpha}_{km\ell} - \bar{R}^{\alpha}_{(ij)k}\bar{R}^h_{\alpha m\ell} + \bar{R}^h_{\alpha(i|k|}\bar{R}^{\alpha}_{j)m\ell} + \delta^h_{(\alpha}a_{jk)}R^{\alpha}_{i\ell m} \\ & + \delta^h_{(\alpha}a_{ik)}R^{\alpha}_{j\ell m} + \delta^h_{(i}a_{j\alpha)}R^{\alpha}_{k\ell m} - \delta^h_{(i}a_{jk)}R^{\alpha}_{\alpha\ell m}\,. \end{aligned} \tag{7.4}$$

Due to the properties of the Riemannian tensor, (7.4) can be written as

$$\bar{R}^h_{im\ell;jk} + \bar{R}^h_{jm\ell;ik} = \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - N^h_{ijk\ell m}\,, \tag{7.5}$$

where

$$\begin{aligned} N^h_{ijk\ell m} = {} & T^h_{ijk[\ell m]} + \bar{R}^{\alpha}_{im\ell}\bar{R}^h_{(\alpha j)k} + \bar{R}^{\alpha}_{jm\ell}\bar{R}^h_{(\alpha i)k} + \bar{R}^{\alpha}_{km\ell}\bar{R}^h_{(ij)\alpha} - \bar{R}^h_{\alpha m\ell}\bar{R}^{\alpha}_{(ij)k} \\ & + \delta^h_{(\alpha}a_{jk)}R^{\alpha}_{i\ell m} + \delta^h_{(\alpha}a_{ik)}R^{\alpha}_{j\ell m} + \delta^h_{(\alpha}a_{ij)}R^{\alpha}_{k\ell m} - a_{(ij}R^h_{k)\ell m}\,. \end{aligned}$$

Let us alternate (7.5) over j, k. We get

$$\begin{aligned} \bar{R}^h_{jm\ell;ik} - \bar{R}^h_{km\ell;ij} = {} & \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(i}a_{k\ell),jm} + \delta^h_{(i}a_{km),j\ell} - N^h_{i[jk]\ell m} \\ & + \bar{R}^h_{\alpha m\ell}\bar{R}^{\alpha}_{ikj} + \bar{R}^h_{i\alpha\ell}\bar{R}^{\alpha}_{mkj} + \bar{R}^h_{im\alpha}\bar{R}^{\alpha}_{\ell kj} - \bar{R}^{\alpha}_{im\ell}\bar{R}^h_{\alpha kj}\,. \end{aligned} \tag{7.6}$$

Let us change mutually i and k in (7.5), and then use (7.6). We evaluate

$$\begin{aligned} 2\bar{R}^h_{jm\ell;ik} = {} & \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(k}a_{jm),i\ell} + \delta^h_{(i}a_{km),j\ell} \\ & - \delta^h_{(i}a_{k\ell),jm} + \delta^h_{(j\ell}a_{k),im} + \Omega^h_{ijk\ell m}\,, \end{aligned} \tag{7.7}$$

where

$$\begin{aligned} \Omega^h_{ijk\ell m} = {} & -N^h_{ijk\ell m} + N^h_{k[ij]k\ell m} - \bar{R}^h_{\alpha m\ell}\bar{R}^{\alpha}_{(kj)i} + \bar{R}^h_{j\alpha\ell}\bar{R}^{\alpha}_{mik} + \bar{R}^h_{jm\alpha}\bar{R}^{\alpha}_{\ell ik} - \bar{R}^h_{\alpha i(j}\bar{R}^{\alpha}_{k)m\ell} \\ & + \bar{R}^h_{j\alpha\ell}\bar{R}^{\alpha}_{mik} + \bar{R}^h_{jm\alpha}\bar{R}^{\alpha}_{\ell ik} - \bar{R}^h_{\alpha m\ell}\bar{R}^{\alpha}_{ikj} - \bar{R}^h_{i\alpha\ell}\bar{R}^{\alpha}_{mkj} + \bar{R}^{\alpha}_{im[\ell}\bar{R}^h_{\alpha]kj}\,. \end{aligned}$$

On the left hand side of (7.7), let us pass from the covariant derivative with respect to $\bar{\nabla}$ to the covariant derivative with respect to ∇:

$$\begin{aligned}2\bar{R}^h_{jm\ell,ik} = \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(k}a_{jm),i\ell} + \delta^h_{(i}a_{km),j\ell}\\ - \delta^h_{(i}a_{k\ell),jm} - \delta^h_{(k}a_{j\ell),im} + S^h_{ijk\ell m}\,,\end{aligned} \tag{7.8}$$

where

$$\begin{aligned}S^h_{ijk\ell m} = \Omega^h_{ijk\ell m} - 2\Big[&\bar{R}^\alpha_{jm\ell,i}P^h_{\ell k} - \bar{R}^h_{\alpha m\ell,i}P^\alpha_{jk} - \bar{R}^h_{j\alpha\ell,i}P^\alpha_{mk} - \bar{R}^h_{jm\alpha,i}P^\alpha_{\ell k} - \bar{R}^h_{jm\ell,\alpha}P^\alpha_{ik}\\ &+ \Big(\bar{R}^\alpha_{jm\ell}P^\beta_{\alpha i} - \bar{R}^h_{\alpha m\ell}P^\alpha_{ij} - \bar{R}^h_{j\alpha\ell}P^\alpha_{im} - \bar{R}^h_{jm\alpha}P^\alpha_{i\ell}\Big)P^h_{\beta k} - \Big(\bar{R}^\alpha_{jm\ell}P^h_{\alpha\beta} - \bar{R}^h_{\alpha m\ell}P^\alpha_{\beta j}\\ &- \bar{R}^h_{j\alpha\ell}P^\alpha_{\beta m} - \bar{R}^h_{jm\alpha}P^\alpha_{\beta\ell}\Big)P^\beta_{ik} - \Big(\bar{R}^\alpha_{\beta m\ell}P^h_{\alpha i} - \bar{R}^h_{\alpha m\ell}P^\alpha_{\beta i} - \bar{R}^h_{\beta\alpha\ell}P^\alpha_{im}\\ &- \bar{R}^h_{\beta m\alpha}P^\alpha_{i\ell}\Big)P^\beta_{jk} - \Big(\bar{R}^\alpha_{j\beta\ell}P^h_{\alpha i} - \bar{R}^h_{\alpha\beta\ell}P^\alpha_{ji} - \bar{R}^h_{j\alpha\ell}P^\alpha_{\beta i} - \bar{R}^h_{j\beta\alpha}P^\alpha_{i\ell}\Big)P^\beta_{km}\\ &- \Big(\bar{R}^\alpha_{jm\beta}P^h_{\alpha i} - \bar{R}^h_{\alpha m\beta}P^\alpha_{ji} - \bar{R}^h_{j\alpha\beta}P^\alpha_{mi} - \bar{R}^h_{jm\alpha}P^\alpha_{\beta i}\Big)P^\beta_{k\ell}\Big]\,.\end{aligned}$$

Let us introduce a $(1,4)$-tensor field $R^h_{jm\ell i} = \bar{R}^h_{jm\ell,i}$. Then we get

$$\bar{R}^h_{jm\ell,i} = R^h_{jm\ell i}\,. \tag{7.9}$$

From (7.8), the covariant derivative of the tensor (7.9) satisfies

$$\begin{aligned}2R^h_{jm\ell i,k} = \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(k}a_{jm),i\ell} + \delta^h_{(i}a_{km),j\ell}\\ - \delta^h_{(i}a_{k\ell),jm} + \delta^h_{(k}a_{j\ell),im} + S^h_{ijk\ell m}\,,\end{aligned} \tag{7.10}$$

where we used (7.2).

It can be verified that the equations (5.2), (7.2), (7.9) and (7.10) for the functions $P^h_{ij}(x)$, $a_{ij}(x)$, $\bar{R}^h_{ijk}(x)$ and $R^h_{ijkm}(x)$ on (M,∇) form an integrable system; the above functions must satisfy also additional algebraic conditions

$$\begin{aligned}P^h_{ij}(x) = P^h_{ji}(x), \quad a_{ij}(x) = a_{ji}(x),\\ \bar{R}^h_{i(jk)}(x) = \bar{R}^h_{(ijk)}(x) = 0, \quad R^h_{i(jk)\ell}(x) = R^h_{(ijk)\ell}(x) = 0.\end{aligned} \tag{7.11}$$

So we have succeeded to prove the following generalization of the result of Sinyukov [16, 17] (we use the above notation).

Theorem 7.1. *Let (M,∇) be a manifold with linear connection and $(\bar{M},\bar{\nabla})$ a generalized Ricci-symmetric manifold. There is a $\tilde{\pi}_1$ mapping $f\colon M \to \bar{M}$ (i.e. a canonical almost geodesic mapping of type π_1) if and only if there exist functions $P^h_{ij}(x)$, $a_{ij}(x)$, $\bar{R}^h_{ijk}(x)$ and $R^h_{ijkm}(x)$ which satisfy the equations (5.2), (7.2), (7.9), (7.10), and (7.11). The system of equations (5.2), (7.2), (7.9) and (7.10) forms a Cauchy type system of PDE's in covariant derivatives.*

As a consequence we obtain

Corollary 7.1. *The family of all generalized Ricci-symmetric manifolds, which can serve as an image of the given manifold (M,∇) under some $\tilde{\pi}_1$-mapping, depends on at most*

$$\frac{1}{6}\,n(n+1)(2n^3 - 4n^2 + 5n + 3)$$

parameters.

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