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Some Sharp Poincaré Inequalities with Weights on Carnot Groups

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Abstract. We prove some weighted Poincaré inequalities on Carnot groups and obtain the sharp constants. These improve the corresponding results of Fausto Ferrari and Enrico Valdinoci.

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1. Introduction

Let Ω be a domain in $\mathbb{R}^n$ and A be a smooth symmetric matrix function on Ω. Assume that there exist $\Lambda > 0$ such that for every $\xi \in \mathbb{R}^n$ and all $x \in \Omega$, there hiolds

$$0 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2.$$

F. Ferrari and E. Valdinocihad [4] have showed that if u is a C^2 stable week solution of the following PDE

$$\operatorname{div}(A(x)\nabla u(x)) = f(u(x)),$$

then the following general Poincaré inequality holds:

$$\int_\Omega \langle A(x)\nabla\phi, \nabla\phi\rangle|\nabla u|^2 \geq \int_{\Omega\bigcap\{\nabla u\neq 0\}} W(x)\phi^2, \tag{1.1}$$

where

$$W(x) = \sum_{i=1}^n \langle A(x)\nabla u_i, \nabla u_i\rangle - \langle A(x)(D^2u\cdot N), (D^2u\cdot N)\rangle - \sum_{i=1}^n \operatorname{div}(A_i(x)\nabla u)u_i$$

and $N = \nabla u(x)/|\nabla u(x)|$. By the choice of suitable $A(x)$ and stable solutions, F. Ferrari and E. Valdinocihad [4] obtained some weighted Poincaré inequalities for the Laplace operator in the Euclidean space $\mathbb{R}^n$, Kohn's sublaplace operator in the Heisenberg group $\mathbb{H}^n$, the sublaplace operator in the Engel group, the Franchi-Grushin-Lanconelli operators and the p-laplacian in the Euclidean space $\mathbb{R}^n$.

The aim of this note is to prove analogous Poincaré inequalities with weights on Carnot groups. We refer to [7, 11] for Hardy inequalities on this groups. Our proof shows if one considers the weighted Poincaré inequalities on Carnot groups (not the general Poincaré

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inequality (1.1)), one needs not use the stable solutions. In fact, we offer here an alternative elementary proof, in which some calculus is enough.

To state our results, we need some notations. Recall that a Carnot group G is a stratified, simply connected nilpotent Lie group with the Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^{r} V_i$ satisfying $[V_1, V_j] = V_{j+1}$ for all $1 \le j \le r-1$. The integer r is called the step of the group G. Set $n_j = \dim V_j$ $(1 \le j \le r)$. Let $\{X_1, \ldots, X_{n_1}\}$ be a basis of V_1 and denote by $\nabla_G = (X_1, \ldots, X_{n_1})$. Set $\xi^{(1)} = \xi_1 X_1 + \cdots + \xi_{n_1} X_{n_1}$ and $|\xi^{(1)}| = \sqrt{\xi_1^2 + \cdots + \xi_{n_1}^2}$. To this end, we have

Theorem 1.1. *Let $p > 1$ and $\alpha \ge 0$. If G is a Carnot group of step 2, then for all $\phi \in C_0^\infty(G)$, there holds*

$$\left(\frac{n_1+\alpha}{p}\right)^p \int_G |\phi|^p |\xi^{(1)}|^\alpha \le \int_G |\nabla_G \phi|^p |\xi^{(1)}|^{\alpha+p} \tag{1.2}$$

and the constant $((n_1+\alpha)/p)^p$ in (1.2) is sharp.

Let $\mathbb{H}^n$ be the Heisenberg group whose group structure is given by

$$(x,t) \circ (x',t') = \left(x + x', t + t' + 2\sum_{j=1}^{n} (x_{2j} x'_{2j-1} - x_{2j-1} x'_{2j})\right).$$

The vector fields $X_{2j-1} = \partial/\partial x_{2j-1} + 2x_{2j}\partial/\partial t$, $X_{2j} = \partial/\partial x_{2j} - 2x_{2j-1}\partial/\partial t$, $(j = 1, \ldots, n)$ are left invariant and generate the Lie algebra of $\mathbb{H}^n$. It is easy to check that $\mathbb{H}^n$ is a Carnot group of step two. By Theorem 1.1, we have the following corollary which improves [4, Theorem 1.3].

Corollary 1.1. *Let $p > 1$ and $\alpha \ge 0$. There holds, for all $\phi \in C_0^\infty(\mathbb{H}^n)$,*

$$\left(\frac{2n+\alpha}{p}\right)^p \int_{\mathbb{H}^n} |\phi|^p |x|^\alpha \le \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} \phi|^p |x|^{\alpha+p}, \tag{1.3}$$

where $\nabla_{\mathbb{H}} = (X_1, \ldots, X_{2n})$, and the constant $((2n+\alpha)/p)^p$ in (1.3) is sharp.

When G is a Carnot group of step r with $r > 2$, we have the following theorem.

Theorem 1.2. *Let $p > 1$ and $\alpha \ge 0$. If G is a Carnot group of step r with $r > 2$, then for $\alpha \ge 0$ and $\phi \in C_0^\infty(G)$, there holds,*

$$\left(\frac{n_1+\alpha}{p}\right)^p \int_G |\phi|^p |\xi^{(1)}|^\alpha \le \int_G |\nabla_G \phi|^p |\xi^{(1)}|^{\alpha+p}. \tag{1.4}$$

Furthermore, if $p = 2$, the constant $((n_1+\alpha)/2)^2$ in (1.4) is sharp.

Denote by $\mathbb{E}$ the Engel group. Let

$$Z_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} + x_2^2 \frac{\partial}{\partial x_4} \quad \text{and} \quad Z_2 = \frac{\partial}{\partial x_2}.$$

It is known that Z_1 and Z_2 generate the Lie algebra of Engel group $\mathbb{E}$ and $\mathbb{E}$ is a Carnot group of step 3. By Theorem 4, we have the following corollary which generalizes the [4, inequality (1.18)].

Corollary 1.2. *Let $\alpha \ge 0$. There holds, for all $\phi \in C_0^\infty(\mathbb{E})$,*

$$(1+\alpha)^2 \int_{\mathbb{R}^4} |\phi|^2 (x_1^2 + x_2^2)^\alpha \le \int_{\mathbb{R}^4} |\nabla_{\mathbb{E}} \phi|^2 (x_1^2 + x_2^2)^{\alpha+1}, \tag{1.5}$$

where $\nabla_{\mathbb{E}} = (Z_1, Z_2)$*, and the constant* $(1+\alpha)^2$ *in (1.5) is sharp.*

We fail to show the constant $((n_1+\alpha)/p)^p$ in (1.4) is sharp when $p \neq 2$. For the reasons, see Remark 3.1.

2. Proof of Theorem 1.1

Recall that if G is a Carnot group of step two, then there exists $m \times m$ linearly independent skew-symmetric matrices $U^{(1)}, \dots, U^{(n)}$, such that G is isomorphic to $(\mathbb{R}^{m+n}, \circ)$ with the following Lie group law ($x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$) (see A.4, [2])

$$(x,t) \circ (x',t') = \begin{pmatrix} x_i + x_i', \quad i = 1,2,\dots,m \\ t_j + t_j' + \frac{1}{2} < x, U^{(j)} x' >, \quad j = 1,2,\dots,n \end{pmatrix}.$$

The vector fields in the Lie algebra $\mathfrak{g}$ of $G = (\mathbb{R}^{m+n}, \circ)$ that agree at the origin with $\partial/\partial x_j (j = 1,\dots,m)$ are given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^{n} \left(\sum_{i=1}^{m} U_{i,j}^{(k)} x_i \right) \frac{\partial}{\partial t_k},$$

and that $\mathfrak{g}$ is spanned by the left-invariant vector fields

$$X_1, \dots, X_m, \quad T_1 = \frac{\partial}{\partial t_1}, \dots, T_n = \frac{\partial}{\partial t_n}.$$

The Kohn's sub-Laplacian on the group G is given by

$$\Delta_G = \sum_{j=1}^{m} X_j^2 = \sum_{j=1}^{m} \left(\frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^{n} \left(\sum_{i=1}^{m} U_{j,i}^{(k)} x_i \right) \frac{\partial}{\partial t_k} \right)^2$$

and the corresponding horizontal gradient is the m-dimensional vector given by

$$\nabla_G = (X_1, \dots, X_m) = \nabla_x - \frac{1}{2} U^{(1)} x \frac{\partial}{\partial t_1} - \cdots - \frac{1}{2} U^{(n)} x \frac{\partial}{\partial t_n}. \tag{2.1}$$

To finish the proof of Theorem 1.1, it is enough to show that the following inequality is held for all $\phi \in C_0^\infty(\mathbb{R}^{m+n})$

$$\left(\frac{m+\alpha}{p} \right)^p \int_{\mathbb{R}^{m+n}} |\phi|^p |x|^\alpha \le \int_{\mathbb{R}^{m+n}} |\nabla_G \phi|^p |x|^{\alpha+p} \tag{2.2}$$

and the corresponding constant is sharp.

For a function $\phi \in C_0^\infty(G)$, we denote by $\phi_\varepsilon := [(|\phi|^2 + \varepsilon^2)^{p/2} - \varepsilon^p]^{1/p}$ with $\varepsilon > 0$. Then $0 \le \phi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$. In fact, ϕ_ε has the same support as ϕ. Notice that for $|x| \neq 0$,

$$\Delta_G |x|^{\alpha+2} = \sum_{j=1}^{m} \left(\frac{\partial}{\partial x_j} \right)^2 |x|^{\alpha+2} = (\alpha+2)(m+\alpha)|x|^\alpha.$$

Integrating by parts yields

$$\begin{aligned} (\alpha+2)(m+\alpha) \int_{\mathbb{R}^{m+n}} |\phi_\varepsilon|^p |x|^\alpha &= \int_{\mathbb{R}^{m+n}} |\phi_\varepsilon|^p \Delta_G |x|^{\alpha+2} = -\int_{\mathbb{R}^{m+n}} \langle \nabla_G |\phi_\varepsilon|^p, \nabla_G |x|^{\alpha+2} \rangle \\ &= -p \int_{\mathbb{R}^{m+n}} (|\phi|^2 + \varepsilon^2)^{\frac{p-2}{2}} \phi \langle \nabla_G \phi, \nabla_G |x|^{\alpha+2} \rangle \\ &\le p \int_{\mathbb{R}^{m+n}} (|\phi|^2 + \varepsilon^2)^{\frac{p-2}{2}} |\phi| \cdot |\nabla_G \phi| \cdot |\nabla_G |x|^{\alpha+2}| \end{aligned}$$

$$\leq p(\alpha+2)\int_{\mathbb{R}^{m+n}}(|\phi|^2+\varepsilon^2)^{\frac{p-1}{2}}|\nabla_G\phi||x|^{\alpha+1}.$$

Letting $\varepsilon\to 0$, we obtain

$$(m+\alpha)\int_{\mathbb{R}^{m+n}}|\phi|^p|x|^\alpha\leq p\int_{\mathbb{R}^{m+n}}|\phi|^{p-1}|\nabla_G\phi||x|^{\alpha+1}.$$

By Hölder's inequality:

$$(m+\alpha)\int_{\mathbb{R}^{m+n}}|\phi|^p|x|^\alpha\leq p\left(\int_{\mathbb{R}^{m+n}}|\phi|^p|x|^\alpha\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^{m+n}}|\nabla_G\phi|^p|x|^{\alpha+2}\right)^{\frac{1}{p}}.$$

Canceling and raising both sides to the power p, we get (2.2).

To see the constant $((m+\alpha)/p)^p$ in (2.2) is sharp, we follow [9, Lemma 2.5]. Choosing $f(x,t)=u(|x|)g(t)$, where $u(|x|)\in C_0^\infty(\mathbb{R}^m)$ and $g(t)=\prod_{j=1}^n w_j(t_j)$ with $w_j(t_j)\in C_0^\infty(\mathbb{R})$ for all $1\leq j\leq n$. We have, by (2.1),

$$\begin{aligned}|\nabla_G f(x,t)|^2&=\left|g(t)\nabla_x u(|x|)-\frac{1}{2}u(|x|)\sum_{j=1}^n U^{(j)}x\frac{\partial g}{\partial t_j}\right|^2\\&=g^2|\nabla_x u(|x|)|^2+\frac{1}{4}u^2(|x|)\left|\sum_{j=1}^n U^{(j)}x\frac{\partial g}{\partial t_j}\right|^2-u(|x|)g\cdot\sum_{j=1}^n\langle\nabla_x u(|x|),U^{(j)}x\rangle\frac{\partial g}{\partial t_j}.\end{aligned}$$

Since $U^{(j)}$ $((1\leq j\leq n))$ is a skew-symmetric matric,

$$\langle\nabla_x u(|x|),U^{(j)}x\rangle=u'(|x|)\left\langle\frac{x}{|x|},U^{(j)}x\right\rangle=0.$$

Therefore,

$$|\nabla_G f(x,t)|^2=|\nabla_x u|^2g^2+\frac{1}{4}u^2\left|\sum_{j=1}^n U^{(j)}x\frac{\partial g}{\partial t_j}\right|^2. \tag{2.3}$$

Using the inequality (cf. [9, inequality (4)])

$$(s_1^2+s_2^2)^{p/2}\leq(1-\lambda)^{1-p}s_1^p+\lambda^{1-p}s_2^p,\quad s_1,s_2>0,\quad 0<\lambda<1, \tag{2.4}$$

we have, for $0<\lambda<1$,

$$\begin{aligned}&\frac{\int_{\mathbb{R}^{m+n}}|\nabla_G f|^p|x|^{\alpha+p}}{\int_{\mathbb{R}^{m+n}}|f|^p|x|^\alpha}\\&\leq(1-\lambda)^{1-p}\frac{\int_{\mathbb{R}^m}|\nabla u|^p|x|^{p+\alpha}dx}{\int_{\mathbb{R}^m}|u|^p|x|^\alpha dx}+\lambda^{1-p}2^{-p}\frac{\int_{\mathbb{R}^{m+n}}|u|^p\left|\sum_{j=1}^n U^{(j)}x\frac{\partial g}{\partial t_j}\right|^p|x|^{p+\alpha}}{\int_{\mathbb{R}^{m+n}}|u(|x|)|^p|g|^p|x|^\alpha}.\end{aligned} \tag{2.5}$$

Notice that there exists a positive constant $C_{n,p}$, depending only on n and p, such that

$$\left|\sum_{j=1}^n U^{(j)}x\frac{\partial g}{\partial t_j}\right|^p\leq C_{n,p}\sum_{j=1}^n\left|U^{(j)}x\frac{\partial g}{\partial t_j}\right|^p=C_{n,p}\sum_{j=1}^n\left|U^{(j)}x\right|^p\left|\frac{\partial g}{\partial t_j}\right|^p.$$

We have, by using (2.5) and $g(t)=\prod_{j=1}^n w_j(t_j)$,

$$
\begin{aligned}
&\frac{\int_{\mathbb{R}^{m+n}} |\nabla_G f|^p |x|^{\alpha+p}}{\int_{\mathbb{R}^{m+n}} |f|^p |x|^{\alpha}}\\
\leq&(1-\lambda)^{1-p}\frac{\int_{\mathbb{R}^m} |\nabla u|^p |x|^{p+\alpha} dx}{\int_{\mathbb{R}^m} |u|^p |x|^{\alpha} dx} + 2^{-p}\lambda^{1-p} C_{n,p} \frac{\int_{\mathbb{R}^{m+n}} |u|^p \left(\sum_{j=1}^n \left|U^{(j)}x\right|^p \left|\frac{\partial g}{\partial t_j}\right|^p\right) |x|^{p+\alpha}}{\int_{\mathbb{R}^{m+n}} |u|^p |g|^p |x|^{\alpha}}\\
=&(1-\lambda)^{1-p}\frac{\int_{\mathbb{R}^m} |\nabla u|^p |x|^{p+\alpha} dx}{\int_{\mathbb{R}^m} |u|^p |x|^{\alpha} dx} + 2^{-p}\lambda^{1-p} C_{n,p} \sum_{j=1}^n \frac{\int_{\mathbb{R}^m} |u|^p |x|^{p+\alpha} |U^{(j)}x|^p dx}{\int_{\mathbb{R}^m} |u(|x|)|^p |x|^{\alpha}} \cdot \frac{\int_{\mathbb{R}} |w_j'(t_j)|^p dt_j}{\int_{\mathbb{R}} |w_j(t_j)|^p dt_j}.
\end{aligned}
$$

Since

$$\inf_{w(t)\in C_0^\infty(\mathbb{R})\setminus\{0\}} \frac{\int_{\mathbb{R}} |w'(t)|^p dt}{\int_{\mathbb{R}} |w(t)|^p dt} = 0,$$

we obtain, for $0 < \lambda < 1$,

$$
\begin{aligned}
\inf_{\phi\in C_0^\infty(\mathbb{R}^{m+n})\setminus\{0\}} \frac{\int_{\mathbb{R}^{m+n}} |\nabla_G \phi|^p |x|^{\alpha+p}}{\int_{\mathbb{R}^{m+n}} |\phi|^p |x|^{\alpha}} &\leq (1-\lambda)^{1-p} \inf_{u(|x|)\in C_0^\infty(\mathbb{R}^m)\setminus\{0\}} \frac{\int_{\mathbb{R}^m} |\nabla u|^p |x|^{p+\alpha} dx}{\int_{\mathbb{R}^m} |u|^p |x|^{\alpha} dx}\\
&= (1-\lambda)^{1-p} \left(\frac{m+\alpha}{p}\right)^p.
\end{aligned}
$$

Here we use the fact (see e.g. [9, Theorem 2.1])

$$\left(\frac{m+\alpha}{p}\right)^p = \inf_{u(|x|)\in C_0^\infty(\mathbb{R}^m)\setminus\{0\}} \frac{\int_{\mathbb{R}^m} |\nabla u(|x|)|^p |x|^{p+\alpha} dx}{\int_{\mathbb{R}^m} |u(|x|)|^p |x|^{\alpha} dx}.$$

By letting $\lambda \to 0+$, we can see the constant $((m+\alpha)/p)^p$ is sharp.

Remark 2.1. A simple calculation shows, if $|x| \neq 0$, then

$$|\nabla_G |x|^{-\frac{\alpha+m}{p}}|^p = \left(\frac{m+\alpha}{p}\right)^p |x|^{-\alpha-m-p}.$$

It looks like that $|x|^{-(\alpha+m)/p}$ can realize the sharp constant $((m+\alpha)/p)^p$. However, since

$$\int_{\mathbb{R}^{m+n}} ||x|^{-\frac{\alpha+m}{p}}|^p |x|^{\alpha} dxdt = +\infty,$$

inequality (1.2) is strict for every $\phi \neq 0$. It seems that one can anticipate improving this inequality by adding some nonnegative correction term to the left-hand side of the inequality (1.2) when $\mathbb{R}^{m+n}$ is replaced by a bounded domain. We refer to [3] and [10] for more information about this subject. Moreover, in their papers, it also shows that the sharp constant appeared in this type of inequality has some applications in PDE.

We can not directly use the the family of function found in [9, Lemma 2.3]. In fact, if we set

$$u_\varepsilon(x,t) = \begin{cases} 1, & |x| \leq 1; \\ |x|^{-\frac{\alpha+m}{p}-\varepsilon}, & |x| \geq 1, \end{cases}$$

then

$$\int_{\mathbb{R}^{m+n}} |u_\varepsilon(x,t)|^p |x|^{\alpha} dxdt = \int_{\mathbb{R}^n} dt \cdot \int_{\mathbb{R}^m} |u_\varepsilon(x,t)|^p |x|^{\alpha} dx = +\infty.$$

3. Proof of Theorem 1.2

We begin by quoting some preliminary facts which will be needed in the sequel and refer to $[1,5,6,8]$ for more precise information about Carnot group. Let G be a Carnot group with the Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^{r} V_i$ satisfying $[V_1, V_j] = V_{j+1}$ for all $1 \le j \le r-1$. As a simply connected nilpotent group, G is differential with $\mathbb{R}^N$, $N = \sum_{i=1}^{r} \dim V_i = \sum_{i=1}^{r} n_i$, via the exponential map $\exp : \mathfrak{g} \to G$. The Haar measure on G is induced by the exponential mapping from the Lebesgue measure on $\mathfrak{g} = \mathbb{R}^N$ and coincides the Lebesgue measure on $\mathbb{R}^N$.

Consider $\xi = (\xi^{(1)}, \dots, \xi^{(r)}) \in \mathbb{R}^N$ with $\xi^{(i)} = (\xi_1^{(i)}, \dots, \xi_{n_i}^{(i)}) \in \mathbb{R}^{n_i}$. For $j = 1, \dots, n_1$, let X_j be the unique vector field in $\mathfrak{g}$ that coincides with $\partial/\partial \xi_j^{(1)}$ at the origin. The second order differential operator

$$\Delta_G = -\sum_{j=1}^{n_1} X_j^* X_j = \sum_{j=1}^{n_1} X_j^2$$

is called a sub-Laplacian on G. We shall denote by the $\nabla_G = (X_1, \dots, X_{n_1})$ the related subelliptic gradient. By the Campbell-Hausdorff formula (see e.g. [5, page 2–4]), $X_j (1 \le j \le n_1)$ can be expressed as the following

$$X_j = \frac{\partial}{\partial \xi_j^{(1)}} + \sum_{k=2}^{r} \sum_{s=1}^{n_k} p_{k,s}^j (\xi^{(1)}, \dots, \xi^{(k-1)}) \frac{\partial}{\partial \xi_s^{(k)}}, \tag{3.1}$$

where $p_{k,s}^j (\xi^{(1)}, \dots, \xi^{(k-1)})$ is a polynomial of $\xi^{(1)}, \dots, \xi^{(k-1)}$.

Now we prove inequality (1.4). The proof is similar to that given in Section 2. Following the proof of Theorem 1.1, we also denote by $\phi_\varepsilon := [(|\phi|^2 + \varepsilon^2)^{p/2} - \varepsilon^p]^{1/p}$ with $\varepsilon > 0$ if $\phi \in C_0^\infty(G)$. Since, for $|\xi^{(1)}| \neq 0$,

$$\Delta_G |\xi^{(1)}|^{\alpha+2} = \sum_{j=1}^{n_1} \left(\frac{\partial}{\partial \xi_j^{(1)}} \right)^2 |\xi^{(1)}|^{\alpha+2} = (\alpha+2)(n_1+\alpha) |\xi^{(1)}|^{\alpha},$$

we have, through integrating by parts,

$$\begin{aligned}
(\alpha+2)(n_1+\alpha) \int_G |\phi_\varepsilon|^p |\xi^{(1)}|^\alpha &= \int_G |\phi_\varepsilon|^p \Delta_G |\xi^{(1)}|^{\alpha+2} = -\int_G \langle \nabla_G |\phi_\varepsilon|^p, \nabla_G |\xi^{(1)}|^{\alpha+2} \rangle \\
&= -p \int_G (|\phi|^2 + \varepsilon^2)^{\frac{p-2}{2}} \phi \langle \nabla_G \phi, \nabla_G |\xi^{(1)}|^{\alpha+2} \rangle \\
&\le p \int_G (|\phi|^2 + \varepsilon^2)^{\frac{p-2}{2}} |\phi| \cdot |\nabla_G \phi| \cdot |\nabla_G |\xi^{(1)}|^{\alpha+2}| \\
&\le p(\alpha+2) \int_G (|\phi|^2 + \varepsilon^2)^{\frac{p-1}{2}} |\nabla_G \phi| |\xi^{(1)}|^{\alpha+1}.
\end{aligned}$$

Letting $\varepsilon \to 0$ yields

$$(n_1+\alpha) \int_G |\phi|^p |\xi^{(1)}|^\alpha \le p \int_G |\phi|^{p-1} |\nabla_G \phi| |\xi^{(1)}|^{\alpha+1}.$$

By Hölder's inequality:

$$(n_1+\alpha) \int_G |\phi|^p |\xi^{(1)}|^\alpha \le p \left(\int_G |\phi|^p |\xi^{(1)}|^\alpha \right)^{\frac{p-1}{p}} \left(\int_G |\nabla_G \phi|^p |\xi^{(1)}|^{\alpha+2} \right)^{\frac{1}{p}}.$$

Canceling and raising both sides to the power p, we get (1.4).

Next we shall show the constant $((n_1+\alpha)/2)^2$ in (1.4) is sharp when $p=2$. Choosing $f(\xi)=u(\xi^{(1)})g(\xi^{(2)},\dots,\xi^{(r)})$, where $u(\cdot)\in C_0^\infty(\mathbb{R}^{n_1})$ and $g=\prod_{k=2}^{r}\prod_{s=1}^{n_k}w_{k,s}(\xi_s^{(k)})$ with $w_{k,s}(\cdot)\in C_0^\infty(\mathbb{R})$ for all $2\le k\le r$ and $1\le s\le n_l$. For convenience, we set $w_{k,s}\equiv 1$ for all $1\le s\le n_k$ if $k=1$. By (3.1),

$$\begin{aligned}&\int_G|\nabla_G f|^2|\xi^{(1)}|^{\alpha+2}d\xi\\ &=\sum_{j=1}^{n_1}\left\{\int_{\mathbb{R}^N}\left(\frac{\partial u}{\partial\xi_j^{(1)}}\right)^2 g^2|\xi^{(1)}|^{\alpha+2}d\xi+\int_{\mathbb{R}^N}u^2\left|\sum_{k=2}^{r}\sum_{s=1}^{n_k}p_{k,s}^j\frac{\partial g}{\partial\xi_s^{(k)}}\right|^2|\xi^{(1)}|^{\alpha+2}d\xi+(*)\right\},\end{aligned}$$

where

$$\begin{aligned}(*)&=2\int_{\mathbb{R}^N}u^2g\sum_{k=2}^{r}\sum_{s=1}^{n_k}p_{k,s}^j\frac{\partial g}{\partial\xi_s^{(k)}}|\xi^{(1)}|^{\alpha+2}d\xi\\ &=\sum_{k=2}^{r}\sum_{s=1}^{n_k}\int_{\mathbb{R}^N}u^2p_{k,s}^j(\xi^{(1)},\dots,\xi^{(k-1)})\frac{\partial g^2}{\partial\xi_s^{(k)}}|\xi^{(1)}|^{\alpha+2}d\xi.\end{aligned}$$

Since $w_{k,s}(\cdot)\in C_0^\infty(\mathbb{R})$ for all $2\le k\le r$ and $1\le s\le n_k$,

$$\int_{\mathbb{R}}\frac{\partial w_{k,s}^2(\xi_s^{(k)})}{\partial\xi_s^{(k)}}d\xi_s^{(k)}=0$$

and hence

$$\int_{\mathbb{R}}\frac{\partial g^2}{\partial\xi_s^{(k)}}d\xi_s^{(k)}=0.$$

Thus $(*)=0$ and

$$\begin{aligned}&\int_G|\nabla_G f|^2|\xi^{(1)}|^{\alpha+2}d\xi\\ &=\sum_{j=1}^{n_1}\left\{\int_{\mathbb{R}^N}\left(\frac{\partial u}{\partial\xi_j^{(1)}}\right)^2 g^2|\xi^{(1)}|^{\alpha+2}d\xi+\int_{\mathbb{R}^N}u^2\left|\sum_{k=2}^{r}\sum_{s=1}^{n_k}p_{k,s}^j\frac{\partial g}{\partial\xi_s^{(k)}}\right|^2|\xi^{(1)}|^{\alpha+2}d\xi\right\}.\end{aligned}\tag{3.2}$$

Choose a positive constant C such that

$$\left|\sum_{k=2}^{r}\sum_{s=1}^{n_k}p_{k,s}^j\frac{\partial g}{\partial\xi_s^{(k)}}\right|^2\le C\sum_{k=2}^{r}\sum_{s=1}^{n_k}\left|p_{k,s}^j\frac{\partial g}{\partial\xi_s^{(k)}}\right|^2.$$

We have, by (3.2),

$$\begin{aligned}&\frac{\int_G|\nabla_G f|^2|\xi^{(1)}|^{\alpha+2}d\xi}{\int_G f^2|\xi^{(1)}|^{\alpha}d\xi}\\ &\le\frac{\int_{\mathbb{R}^{n_1}}\sum_{j=1}^{n_1}\left(\frac{\partial u}{\partial\xi_j^{(1)}}\right)^2|\xi^{(1)}|^{\alpha+2}d\xi^{(1)}}{\int_{\mathbb{R}^{n_1}}u^2|\xi^{(1)}|^{\alpha}d\xi^{(1)}}+C\sum_{k=2}^{r}\sum_{s=1}^{n_k}\frac{\int_G u^2\left|p_{k,s}^j\frac{\partial g}{\partial\xi_s^{(k)}}\right|^2|\xi^{(1)}|^{\alpha+2}d\xi}{\int_G f^2|\xi^{(1)}|^{\alpha}d\xi}.\end{aligned}$$

Since for all $2 \le k \le r$ and $1 \le s \le n_l$,

$$\inf_{w_{k,s}\in C_0^\infty(\mathbb{R})\setminus\{0\}} \frac{\int_{\mathbb{R}} |w'_{k,s}|^2 d\xi_s^{(k)}}{\int_{\mathbb{R}} |w_{k,s}|^2 d\xi_s^{(k)}} = 0,$$

we have

$$\begin{aligned}
&\inf_{w_{k,s}\in C_0^\infty(\mathbb{R})\setminus\{0\}} \frac{\int_G u^2 \left| p_{k,s}^j \frac{\partial g}{\partial \xi_s^{(k)}} \right|^2 |\xi^{(1)}|^{\alpha+2} d\xi}{\int_G f^2 |\xi^{(1)}|^\alpha d\xi} \\
&= \frac{\int_{\mathbb{R}^{n_1+\cdots+n_{k-1}}} u^2 |p_{k,s}^j|^2 \prod_{l=1}^{k-1}\prod_{i=1}^{n_l} w_{l,i}^2 |\xi^{(1)}|^{\alpha+2}}{\int_{\mathbb{R}^{n_1+\cdots+n_{k-1}}} u^2 \prod_{l=1}^{k-1}\prod_{i=1}^{n_l} w_{l,i}^2 |\xi^{(1)}|^{\alpha}} \cdot \inf_{w_{k,s}\in C_0^\infty(\mathbb{R})\setminus\{0\}} \frac{\int_{\mathbb{R}} |w'_{k,s}|^2 d\xi_s^{(k)}}{\int_{\mathbb{R}} |w_{k,s}|^2 d\xi_s^{(k)}} = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\inf_{\phi\in C_0^\infty(G)\setminus\{0\}} \frac{\int_G |\nabla_G \phi|^p |\xi^{(1)}|^{\alpha+2}}{\int_G |\phi|^p |\xi^{(1)}|^\alpha} &\le \inf_{u\in C_0^\infty(\mathbb{R}^{n_1})\setminus\{0\}} \frac{\int_{\mathbb{R}^{n_1}} \sum_{j=1}^{n_1} \left(\frac{\partial u}{\partial \xi_j^{(1)}}\right)^2 |\xi^{(1)}|^{\alpha+2} d\xi^{(1)}}{\int_{\mathbb{R}^{n_1}} u^2 |\xi^{(1)}|^\alpha d\xi^{(1)}} \\
&= \left(\frac{n_1+\alpha}{2}\right)^2.
\end{aligned}$$

To get the last inequality above, we use the fact (see e.g. [9, Theorem 2.1])

$$\left(\frac{n_1+\alpha}{2}\right)^2 = \inf_{u(x)\in C_0^\infty(\mathbb{R}^{n_1})\setminus\{0\}} \frac{\int_{\mathbb{R}^{n_1}} |\nabla u(x)|^2 |x|^{2+\alpha} dx}{\int_{\mathbb{R}^{n_1}} |u(|x|)|^2 |x|^\alpha dx}.$$

The desired result follows.

Remark 3.1. We note that the equality (2.3) play an important role in the proof of sharpness of constant $((n_1+\alpha)/p)^p$ in (1.2) when G is a Carnot group of step 2. However, if G is a Carnot group of step r with $r > 2$, we can only obtain similar equality for $p = 2$ (see (3.2)). This is the reason that we can not prove the sharpness of $((n_1+\alpha)/p)^p$ in (1.4) when $p \neq 2$.

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