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Time-Dependent Problems for a Wedge via Kontorovich-Lebedev's Transform

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Abstract. The aim of this paper is to solve a certain type of time-dependent problems for a wedge in terms of the Kontorovich-Lebedev integral. In order to do that we will consider a certain testing-function space for distributions associated with the two-dimensional Kontorovich-Lebedev transformation.

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1. Introduction

Time-dependent problems are a challenge for the comprehension of many phenomena of mathematical-physics. Solutions of this type of problems is of great importance for a large variety of practical applications. They can be very useful in evaluating effects of the exposition of a complex structure to impulsive electromagnetic excitations, as for example, in the cases of the lightning flash or the nuclear pulse [3], or in the analysis of acoustic waves [7]. In fact, it is expected that a solution to the time-dependent problems contain many of the elementary patterns from the stationary solution. The main difference is that the patterns will be slowly varying in time instead of time-independent and, consequently, the elementary wave patterns may bifurcate. An example of this occurs when a shock is incident on a wedge [3] and it is connected, for example, to the study of the first passage behavior of fractional Brownian motion in two-dimensional wedge domains [2] or to the scattering of an electromagnetic time-dependent plane wave by the edge of an impedance wedge [4].

A possible approach in the analysis of time evolution equations involves the use of integral transforms and special functions. In [4] the authors use Sommerfeld's integrals to present the solution of the time-dependent wedge problem. Another type of integral transforms that we can use to study this type of problems is the index transforms, in particular, the

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Kontorovich-Lebedev transform [8]. This special type of integral transform has been used in many applications including, for instance, fluid mechanics, quantum and nano-optics, plasmonics and in an analysis of the harmonic equation [10]. This transform, where the integration is with respect to the index of a function rather than by the argument, proves to be useful in solving differential equations. It arises naturally when one uses the method of separation of variables to solve boundary-value problems in terms of cylindrical coordinate system [6]. As far as the authors aware, this type of integral transforms was only used to study the regularized Schrödinger equation [8]. The aim of this paper is to show that the Kontorovich-Lebedev transform can be used to solve time-dependent problems on a wedge under some boundary conditions.

The paper is structured as follows: initially we recall some basic notions about special functions and the Kontorovich-Lebedev transform. In Section 3 we will study the two dimensional Kontorovich-Lebedev transform for distributions, namely we prove its existence, uniqueness and inversion properties on a manner to be found in [12]. In the last section we solve a certain class of time-dependent problems for a wedge.

2. Preliminary results

The Macdonald function $K_s(x)$, with $s=\mu+i\tau$, is the eigenfunction of the differential operator [1]

$$A_x = x^2 - x\frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}\right], \tag{2.1}$$

and satisfies the property

$$A_x K_s(x) = -s^2 K_s(x). \tag{2.2}$$

It has the following asymptotic behavior (cf. [1, Vol. I], [11])

$$K_s(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left[1+O\left(\frac{1}{z}\right)\right], \quad z\to\infty, \tag{2.3}$$

$$K_s(z) = O\left(z^{-|\mathrm{Re}(s)|}\right), \quad z\to 0, \tag{2.4}$$

$$K_0(z) = -\log(z) + O(1), \quad z\to 0. \tag{2.5}$$

This function has the following integral representation [1, 11]

$$K_s(z) = \int_{\mathbb{R}_+} e^{-z\cosh(u)}\cosh(su)\,du = \frac{1}{2}\int_{i\delta-\infty}^{i\delta+\infty} e^{-z\cosh(u)+su}\,du, \tag{2.6}$$

where $\mathrm{Re}(z)>0$, $\delta\in[0,\pi/2[$. Hence one can show that $K_s(z)$ is an even entire function with respect to the index s and it is analytic in a right half-plane with respect to the argument z. Moreover, the uniform inequality holds [10]

$$\begin{aligned} |K_s(z)| \le{}& e^{-\delta|\tau|}\left(\frac{\mathrm{Re}(z)+\mathrm{Im}(z)\tan(\delta)}{\mathrm{Re}(z)-\mathrm{Im}(z)\tan(\delta)}\right)^{\frac{\mu}{2}} \\ &\times K_\mu\left(\sqrt{(\mathrm{Re}(z)\cos(\delta))^2-(\mathrm{Im}(z)\sin(\delta))^2}\right), \end{aligned} \tag{2.7}$$

in the sector $|\arg(z)|<\pi/2-\delta$, $\delta\in[0,\pi/2[$. In particular, $\delta=0$ leads to the inequality $|K_s(z)|\le K_\mu(\mathrm{Re}(z))$, $\mathrm{Re}(z)>0$, $s=\mu+i\tau$.

Taking into account ideas presented in [10], for $2 \leq p < +\infty$ and $0 < \nu < 1$, we will consider the class $\mathscr{A}_{\nu,p}$ of complex-valued smooth functions $\varphi(x,t)$ on $\mathbb{R}^2_+$ for which the following quantity

$$\alpha_{k,\nu,p}(\varphi) = \alpha_{0,\nu,p}(A_x^k \varphi) = \left(\int_{\mathbb{R}^2_+} \left| (A_x^k \varphi)(x,t) \right|^p (xt)^{\nu p - 1} dxdt \right)^{\frac{1}{p}} \tag{2.8}$$

is finite, for each $k \in \mathbb{N}_0$ and A_x defined in (2.1). In analogue to [10] we have that $\mathscr{A}_{\nu,p}$ is a complete testing function space and therefore a Frechet space. Moreover, correspondingly , for the two-dimensional case, an analog of [10, Lemma 2] is

Lemma 2.1. (cf. [10]). *Let $\varphi \in \mathscr{D}(\mathbb{R}^2_+)$. Then φ can be represented by the Lebedev integral*

$$\varphi(x,t) = \lim_{\varepsilon \to 0} \frac{4}{\pi^4} \int_{\mathbb{R}^2_+} \tau_1 \tau_2 \sinh((\pi - \varepsilon)\tau_1) \sinh((\pi - \varepsilon)\tau_2) K_{i\tau_1}(x) K_{i\tau_2}(t)$$
$$\times \int_{\mathbb{R}^2_+} K_{i\tau_1}(y_1) K_{i\tau_2}(y_2) \varphi(y_1, y_2) \frac{dy_1 dy_2}{y_1 y_2} d\tau_1 d\tau_2,$$

where the limit is understood as a convergence in $\mathscr{A}_{\nu,p}$, with $0 < \nu < 1$.

We consider $\mathscr{D}(\mathbb{R}^2_+)$, $\mathscr{E}(\mathbb{R}^2_+)$ the usual spaces of testing functions used in distribution theory [12] and it is clear that $\mathscr{D}(\mathbb{R}^2_+) \subset \mathscr{A}_{\nu,p} \subset \mathscr{E}(\mathbb{R}^2_+)$ and that $\mathscr{A}_{\nu,p}$ is dense in $\mathscr{E}(\mathbb{R}^2_+)$ (see [10]). We will denote by $\mathscr{A}'_{\nu,p}$ the dual of $\mathscr{A}_{\nu,p}$. It's equipped with the weak topology and represents a Hausdorff locally convex space of distributions. From the imbedding above we obtain that $\mathscr{E}'(\mathbb{R}^2_+) \subset \mathscr{A}'_{\nu,p}$. Since $\mathscr{A}'_{\nu,p} \subset L_{\nu,p}(\mathbb{R}^2_+)$ we imbed the dual space $L_{1-\nu,q}(\mathbb{R}^2_+)$, $q = p/(p-1)$, into $\mathscr{A}'_{\nu,p}$ as a subspace of regular distributions. They act upon elements φ from $\mathscr{A}_{\nu,p}$ according to

$$\langle f, \varphi \rangle := \int_{\mathbb{R}^2_+} f(x,t) \varphi(x,t) dxdt, \tag{2.9}$$

where f is a locally integrable function.

Taking into account the multidimensional Kontorovich-Lebedev transform presented in [9], we have for $n = 2$

$$\mathscr{K}L[f](\tau_1, \tau_2) = \int_{\mathbb{R}^2_+} f(x,t) K_{i\tau_1}(x) K_{i\tau_2}(t) dxdt. \tag{2.10}$$

Considering the weighted space $L_{\nu,p}(\mathbb{R}^2_+)$ with $0 < \nu < 1$ and the norm

$$||f||_{\nu,p} = \left(\int_{\mathbb{R}^2_+} |f(x,t)|^p (xt)^{\nu p - 1} dxdt \right)^{\frac{1}{p}}, \tag{2.11}$$

integral (2.10) exists as a Lebesgue integral and $\mathscr{K}L[f]$ is bounded from $L_{\nu,p}(\mathbb{R}^2_+)$ into $L_r(\mathbb{R}^2_+)$, where $p, r \in [1, +\infty]$ have no dependence.

3. Two-dimensional Kontorovich-Lebedev transform in distributional spaces

We introduce the two-dimensional Kontorovich-Lebedev transformation on distributions $f \in \mathscr{A}'_{\nu,p}$ in a similar way as in [10, 13] and taking into account the multidimensional Kontorovich-Lebedev transform introduced in [9] for $n = 2$. Namely, it is defined by

$$\mathscr{K}L[f](\tau_1, \tau_2) := \langle f(\cdot,\cdot), K_{i\tau_1}(\cdot) K_{i\tau_2}(\cdot) \rangle, \quad \tau_1, \tau_2 \in \mathbb{R}_+. \tag{3.1}$$

From (2.3), (2.4), (2.7) and (2.9) we conclude that the product $K_{i\tau_1}(x)K_{i\tau_2}(t)$ is in $L_{\nu,p}(\mathbb{R}^2_+)$ when $0 < \nu < 1$. Moreover, it belongs to $\mathscr{A}_{\nu,p}$ via (2.2). Hence for regular distributions $f \in L_{1-\nu,p}(\mathbb{R}^2_+)$ the Kontorovich-Lebedev transform $\mathscr{K}L[f]$ can be written in the form (2.10) and we immediately conclude that $\mathscr{K}L[f]$ represents an analytic function (cf. [11, Theorem 2.5]).

Furthermore, from (2.7) we have the following estimate

$$
\begin{aligned}
|\mathscr{K}L[f](\tau_1,\tau_2)| &\leq C \max_{0\leq k\leq r} \alpha_{k,\nu,p}(K_{i\tau_1}(x)K_{i\tau_2}(t)) \\
&\leq C e^{-\left(\frac{\pi}{2}-\delta_1\right)\tau_1-\left(\frac{\pi}{2}-\delta_2\right)\tau_2} \max_{0\leq k\leq r} \tau_1^{2k} \int_{\mathbb{R}^2_+} K_0(x\sin(\delta_1))K_0(t\sin(\delta_2))dxdt \\
&\leq C_{f,\delta_1,\delta_2} e^{-\left(\frac{\pi}{2}-\delta_1\right)\tau_1-\left(\frac{\pi}{2}-\delta_2\right)\tau_2} \max\{1,\tau_1^{2r}\}, \qquad (3.2)
\end{aligned}
$$

where $\delta_1, \delta_2 \in]0,\pi/2]$.

We are ready to prove now an inversion theorem for the transformation (3.1). Indeed we have

Theorem 3.1. *Let $f \in \mathscr{A}'_{\nu,p}$. Then*

$$
\begin{aligned}
f(x,t) = \lim_{\varepsilon\to 0^+} \frac{4}{xt\pi^4} \int_{\mathbb{R}^2_+} &\tau_1\tau_2 \sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2) \\
&\times K_{i\tau_1}(x)K_{i\tau_2}(t)\mathscr{K}L[f](\tau_1,\tau_2)d\tau_1 d\tau_2, \qquad (3.3)
\end{aligned}
$$

where the convergence is understood in $\mathscr{D}'(\mathbb{R}^2_+)$.

Proof. We observe that formula (3.3) means the following equality

$$
\begin{aligned}
\langle f,\varphi\rangle = \lim_{\varepsilon\to 0^+} \Bigg\langle \frac{4}{\cdot\cdot\pi^4} \int_{\mathbb{R}^2_+} &\tau_1\tau_2 \sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2) \\
&\times K_{i\tau_1}(\cdot)K_{i\tau_2}(\cdot)\mathscr{K}L[f](\tau_1,\tau_2)d\tau_1 d\tau_2, \varphi \Bigg\rangle, \qquad (3.4)
\end{aligned}
$$

for every $\varphi \in \mathscr{D}(\mathbb{R}^2_+)$ having a compact support, let say, in the closed rectangular $[a,b]\times[c,d] \subset \mathbb{R}^2_+$. The integrals with respect to τ_1 and τ_2 in (3.4) are convergent for each $\varepsilon > 0$ and can be treated as Riemann improper integrals. Furthermore, with inequality (2.7) we show that the expression under the limit sign is a regular distribution. Therefore it is equal to

$$
\begin{aligned}
\frac{4}{\pi^4}\int_a^b\int_c^d y_1^{-1}y_2^{-1}\varphi(y_1,y_2)\int_{\mathbb{R}^2_+} &\tau_1\tau_2\sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2) \\
&\times K_{i\tau_1}(y_1)K_{i\tau_2}(y_2)\mathscr{K}L[f](\tau_1,\tau_2)d\tau_1 d\tau_2 dy_1 dy_2. \qquad (3.5)
\end{aligned}
$$

Appealing to the Fubini theorem we change the order of integration in (3.5) and we write in the form

$$\begin{aligned}&\frac{4}{\pi^4}\int_{\mathbb{R}^2_+}\tau_1\tau_2\sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2)\mathscr{K}L[f](\tau_1,\tau_2)\\ &\qquad\times\int_a^b\int_c^d y_1^{-1}y_2^{-1}\varphi(y_1,y_2)K_{i\tau_1}(y_1)K_{i\tau_2}(y_2)dy_1dy_2d\tau_1d\tau_2\\ &=\frac{4}{\pi^4}\lim_{T\to+\infty}\int_0^T\int_0^T\tau_1\tau_2\sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2)\mathscr{K}L[f](\tau_1,\tau_2)\\ &\qquad\times\int_a^b\int_c^d y_1^{-1}y_2^{-1}\varphi(y_1,y_2)K_{i\tau_1}(y_1)K_{i\tau_2}(y_2)dy_1dy_2d\tau_1d\tau_2.\end{aligned}\tag{3.6}$$

Invoking (3.1) and the Riemann sums technique [12, 13] we prove that

$$\begin{aligned}&\frac{4}{\pi^4}\int_0^T\int_0^T\tau_1\tau_2\sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2)\mathscr{K}L[f](\tau_1,\tau_2)\\ &\qquad\times\int_a^b\int_c^d y_1^{-1}y_2^{-1}\varphi(y_1,y_2)K_{i\tau_1}(y_1)K_{i\tau_2}(y_2)dy_1dy_2d\tau_1d\tau_2=\langle f,\Theta_{T,\varepsilon}\rangle,\end{aligned}$$

where

$$\begin{aligned}\Theta_{T,\varepsilon}(x,t)=&\frac{4}{\pi^4}\int_0^T\int_0^T\tau_1\tau_2\sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2)K_{i\tau_1}(x)K_{i\tau_2}(t)\\ &\times\int_a^b\int_c^d y_1^{-1}y_2^{-1}\varphi(y_1,y_2)K_{i\tau_1}(y_1)K_{i\tau_2}(y_2)dy_1dy_2d\tau_1d\tau_2\end{aligned}$$

is an element of $\mathscr{A}_{\nu,p}$. Meanwhile, we will show that $\Theta_{T,\varepsilon}(x,t)\to\varphi_\varepsilon(x,t)$ in $\mathscr{D}(\mathbb{R}^2_+)$ as $T\to+\infty$, where $\varphi_\varepsilon(x,t)$ is defined by

$$\begin{aligned}\varphi_\varepsilon(x,t)=&\frac{4}{\pi^4}\int_{\mathbb{R}^2_+}\tau_1\tau_2\sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2)K_{i\tau_1}(x)K_{i\tau_2}(t)\\ &\times\int_{\mathbb{R}^2_+}\varphi(y_1,y_2)K_{i\tau_1}(y_1)K_{i\tau_2}(y_2)\frac{dy_1dy_2}{y_1y_2}d\tau_1d\tau_2.\end{aligned}$$

Indeed, choosing $0<\delta_1,\delta_2<\varepsilon/2$ we employ (2.2), (2.7) and the generalized Minkowski inequality to obtain

$$\begin{aligned}&\alpha_{k,\nu,p}(\Theta_{T,\varepsilon}-\varphi_\varepsilon)\\ =&\frac{4}{\pi^4}\left(\int_{\mathbb{R}^2_+}(xt)^{\nu p-1}\left|\int\int_{\mathbb{R}^2_+\setminus[0,T]^2}\tau_1\tau_2\sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2)\right.\right.\\ &\left.\left.\times A_x^kK_{i\tau_1}(x)K_{i\tau_2}(t)\int_a^b\int_c^d y_1^{-1}y_2^{-1}\varphi(y_1,y_2)K_{i\tau_1}(y_1)K_{i\tau_2}(y_2)dy_1dy_2dxdt\right|^p\right)^{\frac{1}{p}}dxdt\\ \le&\frac{4}{\pi^4}\left(\int_{\mathbb{R}^2_+}K_0(x\sin(\delta_1))K_0(t\sin(\delta_2))(xt)^{\nu p-1}dxdt\right)^{\frac{1}{p}}\\ &\times\int_a^b\int_c^d y_1^{-1}y_2^{-1}|\varphi(y_1,y_2)|K_0(y_1\sin(\delta_1))K_0(t\sin(\delta_2))dy_2dy_1\\ &\times\int\int_{\mathbb{R}^2_+\setminus[0,T]^2}\tau_1^{2k+1}\tau_2\sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2)e^{(2\delta_1-\pi)\tau_1+(2\delta_2-\pi)\tau_2}d\tau_1d\tau_2\end{aligned}$$

$$\leq C_{\nu,p,\delta_1,\delta_2} \int\int_{\mathbb{R}^2_+\setminus[0,T]^2} \tau_1^{2k+1}\tau_2 e^{(2\delta_1-\varepsilon)\tau_1+(2\delta_2-\varepsilon)\tau_2} d\tau_1 d\tau_2, \tag{3.7}$$

where the last integral converges to zero when $T \to +\infty$ and $C_{\nu,p,\delta_1,\delta_2}$ is a constant. Thus combining with (3.6) we arrive to the equality

$$\begin{aligned} &\frac{4}{\pi^4}\int_0^T\int_0^T \tau_1\tau_2 \sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2)\mathscr{K}L[f](\tau_1,\tau_2) \\ &\times \int_a^b\int_c^d y_1^{-1}y_2^{-1}\varphi(y_1,y_2)K_{i\tau_1}(y_1)K_{i\tau_2}(y_2)dy_1dy_2d\tau_1d\tau_2 = \langle f,\varphi_\varepsilon\rangle. \end{aligned} \tag{3.8}$$

To end the proof we pass to the limit through (3.8) when $\varepsilon \to 0^+$. Hence taking into account Lemma 2.1, we get (3.4) and we establish the inversion formula (3.3). ∎

4. Time-dependent problems for a wedge

Let us consider a generic space-time problem for a time-dependent domain with the form of a wedge (r,θ_1,t,θ_2), where $\theta_1 \in [0,\alpha_1]$ and $\theta_2 \in [0,\alpha_2]$, with $0<\alpha_1\leq\pi$ and $0<\alpha_2\leq\pi/2$. The problem for the interior of this wedge is to find a function $u(r,\theta_1,t,\theta_2)$ that satisfies the following equation

$$(\partial^2_{\theta_1,\theta_1}+a\partial_{\theta_2})u = A_r^* u, \quad 0<r,t<+\infty,\ 0<\theta_i<\alpha_i (i=1,2), \tag{4.1}$$

where $a \in \mathbb{R}^0_+$ and A_r^* is the adjoint of operator to (2.1) defined as

$$A_r^* = r^2 - 1 - 3r\frac{\partial}{\partial r} - r^2\frac{\partial^2}{\partial r^2}.$$

When $a = 0,1$ we obtain, respectively, the harmonic and forward heat equations. The first particular case was already studied in [10].

We assume that $u(r,\theta_1,t,\theta_2)$ is twice differentiable with respect to θ_1 and differentiable with respect to θ_2 in a sense of a conventional derivative [12]. We also impose the following boundary conditions:

- As $\theta_1 \to 0^+$, with $\theta_2=\alpha_2$, $u(r,\theta_1,t,\theta_2) \to v(r,t) \in \mathscr{A}'_{\nu,p}$, $0<\nu<1$, $p\geq 2$ for any φ such that $\varphi(r,t), rt\varphi(r,t) \in \mathscr{A}'_{\nu,p}$.
- As $\theta_1 \to \alpha_1^-$, with $\theta_2=\alpha_2$, $u(r,\theta_1,t,\theta_2)$ converges to zero under the same conditions.

This problem can be solved through an operational technique by the two-dimensional Kontorovich-Lebedev transform on distributions. Indeed, applying (3.1) to both sides of the equation (4.1) and appealing to the definition of adjoint operator and a conventional derivative we arrive

$$\left(\partial^2_{\theta_1,\theta_1}+a\partial_{\theta_2}\right)\langle u(\cdot,\theta_1,\cdot,\theta_2), K_{i\tau_1}(\cdot)K_{i\tau_2}(\cdot)\rangle = \langle u(\cdot,\theta_1,\cdot,\theta_2), A.K_{i\tau_1}(\cdot)K_{i\tau_2}(\cdot)\rangle.$$

Hence via (2.2) we obtain

$$(\partial^2_{\theta_1,\theta_1}+a\partial_{\theta_2})\mathscr{K}L[u(\cdot,\theta_1,\cdot,\theta_2)](\tau_1,\tau_2) = \tau_1^2\mathscr{K}L[u(\cdot,\theta_1,\cdot,\theta_2)](\tau_1,\tau_2).$$

Solving this differential equation via [5] we find

$$\mathscr{K}L[u(\cdot,\theta_1,\cdot,\theta_2)](\tau_1,\tau_2) = A(\tau_1,\tau_2)\frac{i\sqrt{a}}{2\sqrt{\pi\theta_2}}e^{\frac{\theta_1^2 a}{4\theta_2}+\frac{\tau_1^2\theta_2}{a}} + B(\tau_1,\tau_2)e^{\frac{-\tau_2^2\theta_2}{a}-\sqrt{\tau_1^2+\tau_2^2}\theta_1}, \tag{4.2}$$

where the functions $A(\tau_1,\tau_2)$ and $B(\tau_1,\tau_2)$ do not depend on θ_1 and θ_2. To determinate them we will use the boundary conditions. Indeed, taking into account [10] we know that the product $K_{i\tau_1}(r)K_{i\tau_2}(t) \in \mathscr{A}_{\nu,p}$ and $rtK_{i\tau_1}(r)K_{i\tau_2}(t) \in \mathscr{A}_{\nu,p}$. Invoking (4.2) and (3.1) we have

$$\begin{aligned}\lim_{\theta_1\to 0^+,\theta_2=\alpha_2} \mathscr{K}L[u(\cdot,\theta_1,\cdot,\theta_2)](\tau_1,\tau_2) &= A(\tau_1,\tau_2)\frac{i\sqrt{a}}{2\sqrt{\pi\alpha_2}}e^{\frac{\tau_1^2\alpha_2}{a}} + B(\tau_1,\tau_2)e^{\frac{-\tau_2^2\alpha_2}{a}} \\ &= \mathscr{K}L[v(r,t)](\tau_1,\tau_2),\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\lim_{\theta_1\to \alpha_1^-,\theta_2=\alpha_2} \mathscr{K}L[u(\cdot,\theta_1,\cdot,\theta_2)](\tau_1,\tau_2) \\ &= A(\tau_1,\tau_2)\frac{i\sqrt{a}}{2\sqrt{\pi\alpha_2}}e^{\frac{\alpha_1^2 a}{4\alpha_2}+\frac{\tau_1^2\alpha_2}{a}} + B(\tau_1,\tau_2)e^{\frac{-\tau_2^2\alpha_2}{a}-\sqrt{\tau_1^2+\tau_2^2}\alpha_1} = 0.\end{aligned} \tag{4.4}$$

Solving the previous system of equations we get

$$A(\tau_1,\tau_2) = \mathscr{K}L[v(r,t)](\tau_1,\tau_2)\frac{2i\sqrt{\pi\alpha_2}}{\sqrt{a}e^{\frac{\tau_1^2\alpha_2}{a}}\left(e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1\right)},$$

$$B(\tau_1,\tau_2) = \mathscr{K}L[v(r,t)](\tau_1,\tau_2)\frac{e^{\frac{\alpha_1^2 a}{4\alpha_2}-\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}}{e^{\frac{\tau_1^2\alpha_2}{a}}\left(e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1\right)}.$$

Substituting functions A and B into (4.2) and making some calculations we obtain

$$\begin{aligned}&\mathscr{K}L[u(\cdot,\theta_1,\cdot,\theta_2)](\tau_1,\tau_2) \\ &= \mathscr{K}L[v(r,t)](\tau_1,\tau_2)\frac{\sqrt{\theta_2}e^{\frac{\alpha_1^2 a}{4\alpha_2}-\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}+\sqrt{\tau_1^2+\tau_2^2}(\alpha_1-\theta_1)} - \sqrt{\alpha_2}e^{\frac{\theta_1^2 a}{4\theta_2}-\frac{(\tau_1^2+\tau_2^2)\theta_2}{a}}}{\sqrt{\theta_2}e^{\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}}\left(e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1\right)}.\end{aligned}$$

Consequently, by (3.3) we obtain as our possible solution

$$\begin{aligned}&u(r,\theta_1,t,\theta_2) \\ &= \lim_{\varepsilon\to 0^+}\frac{4}{rt\pi^4}\int_{\mathbb{R}_+^2}\tau_1\tau_2\sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2)K_{i\tau_1}(r)K_{i\tau_2}(t)\mathscr{K}L[v(r,t)](\tau_1,\tau_2) \\ &\times\frac{\sqrt{\theta_2}e^{\frac{\alpha_1^2 a}{4\alpha_2}-\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}+\sqrt{\tau_1^2+\tau_2^2}(\alpha_1-\theta_1)} - \sqrt{\alpha_2}e^{\frac{\theta_1^2 a}{4\theta_2}-\frac{(\tau_1^2+\tau_2^2)\theta_2}{a}}}{\sqrt{\theta_2}e^{\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}}\left(e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1\right)}d\tau_1 d\tau_2.\end{aligned} \tag{4.5}$$

Appealing to (2.7) and (3.2) we get the uniform estimate on $\varepsilon \in [0,\pi]$

$$\begin{aligned}&|\tau_1\tau_2\sinh((\pi-\varepsilon)\tau_1)\sinh((\pi-\varepsilon)\tau_2)K_{i\tau_1}(r)K_{i\tau_2}(t)\mathscr{K}L[v(r,t)](\tau_1,\tau_2)| \\ &\le C\tau_1\tau_2 e^{(\delta_1+\delta_3)\tau_1}e^{(\delta_2+\delta_4)\tau_2}K_0(r_0\sin(\delta_1))K_0(t_0\sin(\delta_2)),\end{aligned} \tag{4.6}$$

where $\delta_i \in]0,\pi/2]$, $i=1,2,3,4$, $0<r_0<r<+\infty$ and $0<t_0<t<+\infty$. For every $\tau_1,\tau_2\geq 0$, we have

$$\left| \frac{\sqrt{\theta_2}e^{\frac{\alpha_1^2 a}{4\alpha_2}-\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}+\sqrt{\tau_1^2+\tau_2^2}(\alpha_1-\theta_1)}-\sqrt{\alpha_2}e^{\frac{\theta_1^2 a}{4\theta_2}-\frac{(\tau_1^2+\tau_2^2)\theta_2}{a}}}{\sqrt{\theta_2}e^{\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}}\left(e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1\right)} \right| \leq e^{-\frac{2(\tau_1^2+\tau_2^2)\alpha_2}{a}}\sqrt{\frac{\alpha_2}{\theta_2}}\frac{e^{\frac{\alpha_1^2 a}{4\alpha_2}+(\tau_1+\tau_2)\alpha_1}+e^{\frac{\theta_1^2 a}{4\theta_2}+\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}}}{e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1}. \tag{4.7}$$

From (4.6) and (4.7) we conclude that one can pass the limit under the integral sign in (4.5) when $\varepsilon\to 0^+$, via Lebesgue's dominated convergence theorem. Hence our solution has the form

$$\begin{aligned} u(r,\theta_1,t,\theta_2) &= \frac{4}{rt\pi^4}\int_{\mathbb{R}^2_+}\tau_1\tau_2\sinh(\pi\tau_1)\sinh(\pi\tau_2)K_{i\tau_1}(r)K_{i\tau_2}(t)\mathscr{K}L[v(r,t)](\tau_1,\tau_2) \\ &\times \frac{\sqrt{\theta_2}e^{\frac{\alpha_1^2 a}{4\alpha_2}-\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}+\sqrt{\tau_1^2+\tau_2^2}(\alpha_1-\theta_1)}-\sqrt{\alpha_2}e^{\frac{\theta_1^2 a}{4\theta_2}-\frac{(\tau_1^2+\tau_2^2)\theta_2}{a}}}{\sqrt{\theta_2}e^{\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}}\left(e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1\right)}d\tau_1 d\tau_2. \end{aligned} \tag{4.8}$$

Now we need to prove that (4.8) is indeed a solution, which satisfies the differential equation (4.1) and corresponding boundary conditions. In order to verify that (4.8) is a solution of (4.1) we use (4.6) combined with (4.7) together with the fact that the integrand in (4.8) is analytic. Consequently the differentiation may be interchanged with the integration. Moreover, after some calculations we see that the function

$$K_{i\tau_1}(r)K_{i\tau_2}(t)\frac{\sqrt{\theta_2}e^{\frac{\alpha_1^2 a}{4\alpha_2}-\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}+\sqrt{\tau_1^2+\tau_2^2}(\alpha_1-\theta_1)}-\sqrt{\alpha_2}e^{\frac{\theta_1^2 a}{4\theta_2}-\frac{(\tau_1^2+\tau_2^2)\theta_2}{a}}}{\sqrt{\theta_2}e^{\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}}\left(e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1\right)}$$

satisfies (4.1). Thus $u(r,\theta_1,t,\theta_2)$ is a solution of (4.1).

We turn to the boundary conditions. First we show that $rtu(r,\theta_1,t,\theta_2)\in\mathscr{A}_{\nu,p}\cap L_{1-\nu,p}(\mathbb{R}^2_+)$ for any $0\leq\theta_1\leq\alpha_1$, $0<\theta_2\leq\alpha_2$, $q=p/(p-1)$ and $0<\nu<1$. Indeed, from the uniform convergence of the integral (4.8) with respect to $(r,t)\in\mathbb{R}^2_+$ we see that $rtu(r,\theta_1,t,\theta_2)$ is a smooth function. Moreover, invoking (2.1), (2.7), (2.11), (3.2), (4.6), (4.7) and the generalized Minkowski inequality we obtain the estimate

$$\begin{aligned} &||A_{\cdot,\cdot}(\cdot\cdot u(\cdot,\theta_1,\cdot,\theta_2))||_{\xi,\omega} \\ &\leq C\int_{\mathbb{R}^2_+}\tau_1^{2b}\sqrt{\frac{\alpha_2}{\theta_2}}\frac{e^{\frac{\alpha_1^2 a}{4\alpha_2}+(\tau_1+\tau_2)\alpha_1}+e^{\frac{\theta_1^2 a}{4\theta_2}+\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}}}{e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1}e^{(\delta_1+\delta_3)\tau_1+(\delta_2+\delta_4)\tau_2-\frac{2(\tau_1^2+\tau_2^2)\alpha_2}{a}}d\tau_1 d\tau_2 \\ &\times\left(\int_{\mathbb{R}^2_+}K_0^{\omega}(r\sin(\delta_1))K_0^{\omega}(t\sin(\delta_2))(rt)^{\xi\omega-1}drdt\right)^{\frac{1}{\omega}} \\ &<+\infty, \end{aligned} \tag{4.9}$$

for $b>0$, $\omega>1$, $\xi>0$, and $\delta_i \in [0,\pi/2]$, $i=1,2,3,4$. Thus, in particular $rtu(r,\theta_1,t,\theta_2) \in \mathscr{A}_{\nu,p} \cap L_{1-\nu,p}(\mathbb{R}^2_+)$ with $q=p/(p-1)$ and $0<\nu<1$. Furthermore for any φ such that $\varphi(r,t)$, $rt\varphi(r,t) \in \mathscr{A}_{\nu,p}$ via (2.9) we write

$$
\begin{aligned}
&\langle \cdot\cdot u(\cdot,\theta_1,\cdot,\theta_2),\varphi(\cdot,\cdot)\rangle \\
&= \int_{\mathbb{R}^2_+} rtu(r,\theta_1,t,\theta_2)\varphi(r,t)drdt \\
&= \frac{4}{\pi^4}\int_{\mathbb{R}^2_+} \tau_1\tau_2\sinh(\pi\tau_1)\sinh(\pi\tau_2)K_{i\tau_1}(r)K_{i\tau_2}(t)\mathscr{K}L[v(r,t)](\tau_1,\tau_2) \\
&\quad\times \frac{\sqrt{\theta_2}e^{\frac{\alpha_1^2 a}{4\alpha_2}-\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}+\sqrt{\tau_1^2+\tau_2^2}(\alpha_1-\theta_1)}-\sqrt{\alpha_2}e^{\frac{\theta_1^2 a}{4\theta_2}-\frac{(\tau_1^2+\tau_2^2)\theta_2}{a}}}{\sqrt{\theta_2}e^{\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}}\left(e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1\right)}d\tau_1 d\tau_2\varphi(r,t)drdt \\
&= \frac{4}{\pi^4}\int_{\mathbb{R}^2_+} \tau_1\tau_2\sinh(\pi\tau_1)\sinh(\pi\tau_2)\mathscr{K}L[v(r,t)](\tau_1,\tau_2) \\
&\quad\times \frac{\sqrt{\theta_2}e^{\frac{\alpha_1^2 a}{4\alpha_2}-\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}+\sqrt{\tau_1^2+\tau_2^2}(\alpha_1-\theta_1)}-\sqrt{\alpha_2}e^{\frac{\theta_1^2 a}{4\theta_2}-\frac{(\tau_1^2+\tau_2^2)\theta_2}{a}}}{\sqrt{\theta_2}e^{\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}}\left(e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1\right)} \\
&\quad\times \int_{\mathbb{R}^2_+} K_{i\tau_1}(r)K_{i\tau_2}(t)\varphi(r,t)drdtd\tau_1 d\tau_2,
\end{aligned}
\tag{4.10}
$$

where the change of order of integration in (4.10) is due to the Fubini's theorem, and this fact is a consequence of (4.6), (4.7) and (4.9) together with the Hölder inequality. Precisely we appeal to the estimate

$$
\begin{aligned}
&\frac{4}{\pi^4}\int_{\mathbb{R}^2_+} \tau_1\tau_2\sinh(\pi\tau_1)\sinh(\pi\tau_2)\left|\mathscr{K}L[v(r,t)](\tau_1,\tau_2)\right| \\
&\quad\times \left|\frac{\sqrt{\theta_2}e^{\frac{\alpha_1^2 a}{4\alpha_2}-\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}+\sqrt{\tau_1^2+\tau_2^2}(\alpha_1-\theta_1)}-\sqrt{\alpha_2}e^{\frac{\theta_1^2 a}{4\theta_2}-\frac{(\tau_1^2+\tau_2^2)\theta_2}{a}}}{\sqrt{\theta_2}e^{\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}}\left(e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1\right)}\right| \\
&\quad\times \int_{\mathbb{R}^2_+} |K_{i\tau_1}(r)||K_{i\tau_2}(t)||\varphi(r,t)|drdtd\tau_1 d\tau_2 \\
&\le C\alpha_{0,\nu,p}(\varphi)\int_{\mathbb{R}^2_+}\tau_1\tau_2\sqrt{\frac{\alpha_2}{\theta_2}}\frac{e^{\frac{\alpha_1^2 a}{4\alpha_2}+(\tau_1+\tau_2)\alpha_1}+e^{\frac{\theta_1^2 a}{4\theta_2}+\frac{(\tau_1^2+\tau_2^2)\alpha_2}{a}}}{e^{\frac{\alpha_1^2 a}{4\alpha_2}+\sqrt{\tau_1^2+\tau_2^2}\alpha_1}-1} \\
&\quad\times e^{(\delta_1+\delta_3)\tau_1+(\delta_2+\delta_4)\tau_2-\frac{2(\tau_1^2+\tau_2^2)\alpha_2}{a}}d\tau_1 d\tau_2 \\
&\quad\times\left(\int_{\mathbb{R}^2_+}K_0^q(r\sin(\delta_1))K_0^q(t\sin(\delta_2))(rt)^{(1-\nu)q-1}drdt\right)^{\frac{1}{q}} \\
&< +\infty,
\end{aligned}
$$

where $q = p/(p-1)$ and $\delta_i \in]0, \pi/2]$, $i = 1,2,3,4$. Hence we conclude that the iterated integral (4.10) converges uniformly. Therefore we can take limits under the integral sign in (4.10) as $\theta_1 \to 0^+$, with $\theta_2 = \alpha_2$, and as $\theta_1 \to \alpha_1^-$, with $\theta_2 = \alpha_2$

$$\lim_{\theta_1 \to 0^+, \theta_2 = \alpha_2} \langle u(\cdot, \theta_1, \cdot, \theta_2), \cdot \cdot \varphi(\cdot, \cdot) \rangle = v(x,t), \qquad \lim_{\theta_1 \to \alpha_1^-, \theta_2 = \alpha_2} \langle u(\cdot, \theta_1, \cdot, \theta_2), \cdot \cdot \varphi(\cdot, \cdot) \rangle = 0,$$

i.e., the boundary conditions are verified.

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References

[1] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions. Vols. I & II*, McGraw-Hill Book Company, Inc., New York, 1953.

[2] J. H. Jeon, A. V. Chechkin and R. Metzler, First passage behavior of fractional Brownian motion in two-dimensional wedge domains, *Europhys. Lett.* **94** (2011), no. 2, Art. No. 20008.

[3] R. Menikoff, Analogies between Riemann problem for 1-D fluid dynamics and 2-D steady supersonic flow, in *Current Progress in Hyperbolic Systems: Riemann Problems and Computations (Brunswick, ME, 1988)*, 225–240, Contemp. Math., 100 Amer. Math. Soc., Providence, RI.

[4] G. Pelosi, G. Manara, A. Freni and M. L. Bernard, Current evaluation on the faces of an impedance wedge illuminated by an electromagnetic pulse, *IEEE Trans. Antennas Propagat.* **42** (1994), no. 12, 1663–1667.

[5] A. D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[6] I. N. Sneddon, *The Use of Integral Transforms*, McGraw-Hill, New York, 1972.

[7] E. White, A time-dependent solution of a wave traversing a wedge by a finite length ping ray trace technique, *J. Acoust. Soc. Am.* **96** (1994), no. 5, 3354–3354.

[8] S. Yakubovich and N. Vieira, The use of the Kontorovich-Lebedev transform in an analysis of regularized Schrödinger equation, *Integral Transforms Spec. Funct.* **24** (2013), no. 1, 9–22.

[9] S. Yakubovich, Multidimensional Kontorovich-Lebedev transforms, *Integral Transforms Spec. Funct.* **22** (2011), no. 2, 123–141.

[10] S. B. Yakubovich, On a testing-function space for distributions associated with the Kontorovich-Lebedev transform, *Collect. Math.* **57** (2006), no. 3, 279–293.

[11] S. B. Yakubovich, *Index Transforms*, World Sci. Publishing, River Edge, NJ, 1996.

[12] A. H. Zemanian, *Generalized Integral Transformations*, second edition, Dover, New York, 1987.

[13] A. H. Zemanian, The Kontorovich-Lebedev transformation on distributions of compact support and its inversion, *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 139–143.