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Asymptotic Behavior of Solutions of a Nonlinear Generalized Pantograph Equation with Impulses

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Abstract. Sufficient conditions are obtained on the asymptotic behavior of solutions of the nonlinear generalized pantograph equation with impulses

$$\begin{cases} x'(t)+p(t)f(x(\alpha t-\tau))=0, & t\geq t_0,\ t\neq t_k,\\ x(t_k)=b_kx(t_k^-)+\frac{1-b_k}{\alpha}\int_{\alpha t_k-\tau}^{t_k}p\left(\frac{s+\tau}{\alpha}\right)f(x(s))ds, & k=1,2,\ldots. \end{cases}$$

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1. Introduction

Functional differential equations with proportional delays are usually referred to as pantograph equations. The name pantograph originated from the work of Ockendon and Taylor [11] on the collection of current by the pantograph head of an electric locomotive. These equations arise in a variety of applications, such as number theory, electrodynamics, astrophysics, nonlinear dynamical systems, quantum mechanics and cell growth [2, 5, 11]. Therefore, the problems have attracted a great deal of attention. There are many papers devoted to the qualitative properties and numerical solutions of these equations (see, for example, [1–3,5,6,8,11,12,14] and the references cited therein). On the other hand, the theory of impulsive differential equations is now being recognized as being not only richer than the corresponding theory of differential equations without impulses, but also representing a more natural framework for mathematical model of many real-world phenomena [7, 13]. There has also been increasing interest in the oscillation and stability theory of impulsive delay differential equations and many results have been obtained (see [4, 9, 10, 15] and the references cited therein). In particular, there are some papers on the asymptotic behavior of solutions of impulsive differential equations with constant delays [9, 15]. However, to the best of our knowledge, there is very little in the way of results for the asymptotic behavior of solutions of the pantograph equations with impulses except for [4].

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In this paper, we consider the asymptotic behavior of solutions of the nonlinear generalized pantograph equations with impulsive perturbations

$$\begin{cases} x'(t)+p(t)f(x(\alpha t-\tau))=0, & t\geq t_0,\ t\neq t_k, \\ x(t_k)=b_k x(t_k^-)+\frac{1-b_k}{\alpha}\int_{\alpha t_k-\tau}^{t_k} p\left(\frac{s+\tau}{\alpha}\right)f(x(s))ds, & k=1,2,.... \end{cases} \tag{1.1}$$

where $0<\alpha\leq 1,\ \tau\geq 0,\ p(t)\in C([t_0,\infty),[0,+\infty)),\ f\in C(R,R),\ 0\leq t_0<t_1<...<t_k<t_{k+1}<...$, with $\lim_{t\to\infty} t_k=\infty$, and $b_k, k=1,2,...$, are constants, $x(t_k^-)$ denotes the left limit of $x(t)$ at $t=t_k$.

We note that when all $b_k=1, k=1,2,...$, system (1.1) reduces to the generalized pantograph differential equation with linear functional argument

$$x'(t)+p(t)f(x(\alpha t-\tau))=0, t\geq t_0. \tag{1.2}$$

The authors [14] presented a numerical method for solution of (1.2), but the qualitative properties of solutions for this equation have not been investigated. When $\alpha=1$, system (1.1) reduces to the impulsive delay differential equation

$$\begin{cases} x'(t)+p(t)f(x(t-\tau))=0, & t\geq t_0,\ t\neq t_k, \\ x(t_k)=b_k x(t_k^-)+(1-b_k)\int_{t_k-\tau}^{t_k} p(s+\tau)f(x(s))ds, & k=1,2,.... \end{cases} \tag{1.3}$$

The asymptotic behavior of solutions of (1.3) has been studied by Shen and Liu [15].

The main purpose of this paper is to investigate the asymptotic behavior of solutions of the system (1.1). As a consequence, some sufficient conditions are obtained for the asymptotic stability of solutions of (1.2) and (1.3), respectively. Our results generalize the known ones.

With the system (1.1), one associates an initial condition of the form

$$x(t)=\phi(t), \quad t\in[\alpha t_0-\tau,t_0], \tag{1.4}$$

where $\phi\in C([\alpha t_0-\tau,t_0],R)$.

A function $x(t)$ is said to be a solution of (1.1) satisfying the initial value condition (1.4) if $x(t)$ is defined on $[\alpha t_0-\tau,\infty)$ and satisfies

(i) $x(t)=\phi(t)$ for $\alpha t_0-\tau\leq t\leq t_0$, $x(t)$ is continuous for $t\geq t_0$ and $t\neq t_k$, $k=1,2,...$;
(ii) $x(t)$ is continuously differentiable for $t>t_0, t\neq t_k, k=1,2,...$, and $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^+)=x(t_k)$ for $k=1,2,...$;
(iii) $x(t)$ satisfies (1.1).

Using the method of steps as in the case without impulses, one can show the global existence and uniqueness of the solution of the initial problem (1.1) and (1.4).

As is customary, a solution of (1.1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

2. Main results

Theorem 2.1. *Assume that the following conditions are fulfilled:*

$$0<b_k\leq 1\ (k=1,2,...) \quad \textit{and} \quad \sum_{k=1}^{\infty}(1-b_k)<\infty; \tag{2.1}$$

there exists a positive number M such that

$$|x|\leq|f(x)|\leq M|x|, \quad x\in R, \quad xf(x)>0\ (x\neq 0); \tag{2.2}$$

$$\limsup_{t\to\infty}\int_{\alpha t-\tau}^{\frac{t+\tau}{\alpha}} p\left(\frac{s+\tau}{\alpha}\right)ds<\frac{2\alpha}{M}. \tag{2.3}$$

Then every solution of (1.1) *tends to a constant as* $t\to\infty$.

Proof. Let $x(t)$ be any solution of (1.1), we shall prove that $\lim_{t\to\infty}x(t)$ exists and is finite. For this purpose, we rewrite the system (1.1) in the form

$$\begin{cases}\left[x(t)-\frac{1}{\alpha}\int_{\alpha t-\tau}^{t}p\left(\frac{s+\tau}{\alpha}\right)f(x(s))ds\right]'+\frac{1}{\alpha}p\left(\frac{t+\tau}{\alpha}\right)f(x(t))=0, & t\ge t_0,\ t\ne t_k,\\ x(t_k)=b_kx(t_k^-)+\frac{1-b_k}{\alpha}\int_{\alpha t_k-\tau}^{t_k}p\left(\frac{s+\tau}{\alpha}\right)f(x(s))ds, & k=1,2,\ldots.\end{cases} \tag{2.4}$$

From (2.3), one can select $\delta>0$ sufficiently small and $T>t_0$ sufficiently large such that

$$\int_{\alpha t-\tau}^{\frac{t+\tau}{\alpha}} p\left(\frac{s+\tau}{\alpha}\right)ds<\frac{2\alpha}{M}-\delta,\quad \text{for } t\ge T. \tag{2.5}$$

Define two functionals as follows

$$V_1(t)=\left[x(t)-\frac{1}{\alpha}\int_{\alpha t-\tau}^{t}p\left(\frac{s+\tau}{\alpha}\right)f(x(s))ds\right]^2,$$

and

$$V_2(t)=\frac{1}{\alpha^3}\int_{\alpha t-\tau}^{t}p\left(\frac{s+(1+\alpha)\tau}{\alpha^2}\right)\int_s^t p\left(\frac{u+\tau}{\alpha}\right)f^2(x(u))duds.$$

As $t\ne t_k$, calculating $dV_1(t)/dt$ and $dV_2(t)/dt$ along the solution of (1.1) and using the inequality $2ab\le a^2+b^2$ yields

$$\begin{aligned}\frac{dV_1}{dt}&=-\frac{2}{\alpha}p\left(\frac{t+\tau}{\alpha}\right)f(x(t))\left[x(t)-\frac{1}{\alpha}\int_{\alpha t-\tau}^{t}p\left(\frac{s+\tau}{\alpha}\right)f(x(s))ds\right]\\&\le-\frac{1}{\alpha}p\left(\frac{t+\tau}{\alpha}\right)\left[2x(t)f(x(t))-\frac{1}{\alpha}f^2(x(t))\int_{\alpha t-\tau}^{t}p\left(\frac{s+\tau}{\alpha}\right)ds\right.\\&\quad\left.-\frac{1}{\alpha}\int_{\alpha t-\tau}^{t}p\left(\frac{s+\tau}{\alpha}\right)f^2(x(s))ds\right],\end{aligned}$$

and

$$\begin{aligned}\frac{dV_2}{dt}&=\frac{1}{\alpha^3}p\left(\frac{t+\tau}{\alpha}\right)f^2(x(t))\int_{\alpha t-\tau}^{t}p\left(\frac{s+(1+\alpha)\tau}{\alpha^2}\right)ds\\&\quad-\frac{1}{\alpha^2}p\left(\frac{t+\tau}{\alpha}\right)\int_{\alpha t-\tau}^{t}p\left(\frac{s+\tau}{\alpha}\right)f^2(x(s))ds.\end{aligned}$$

Let $V(t) = V_1(t) + V_2(t)$. For $t \neq t_k$, it follows from the above two inequalities and (2.5) that

$$\begin{aligned}
\frac{dV}{dt} &= \frac{dV_1}{dt} + \frac{dV_2}{dt} \\
&\leq -\frac{1}{\alpha} p\left(\frac{t+\tau}{\alpha}\right) f^2(x(t)) \left[\frac{2x(t)}{f(x(t))} - \frac{1}{\alpha}\int_{\alpha t-\tau}^{t} p\left(\frac{s+\tau}{\alpha}\right) ds \right.\\
&\quad \left. - \frac{1}{\alpha^2}\int_{\alpha t-\tau}^{t} p\left(\frac{s+(1+\alpha)\tau}{\alpha^2}\right) ds\right] \\
&= -\frac{1}{\alpha} p\left(\frac{t+\tau}{\alpha}\right) f^2(x(t)) \left[\frac{2x(t)}{f(x(t))} - \frac{1}{\alpha}\int_{\alpha t-\tau}^{\frac{t+\tau}{\alpha}} p\left(\frac{s+\tau}{\alpha}\right) ds\right] \\
&\leq -\frac{1}{\alpha} p\left(\frac{t+\tau}{\alpha}\right) f^2(x(t)) \left[\frac{2}{M} - \frac{1}{\alpha}\int_{\alpha t-\tau}^{\frac{t+\tau}{\alpha}} p\left(\frac{s+\tau}{\alpha}\right) ds\right] \\
&\leq -\frac{\delta}{\alpha^2} p\left(\frac{t+\tau}{\alpha}\right) f^2(x(t)).
\end{aligned} \tag{2.6}$$

As $t = t_k$, one can easily get

$$\begin{aligned}
V(t_k) &= \left[x(t_k) - \frac{1}{\alpha}\int_{\alpha t_k-\tau}^{t_k} p\left(\frac{s+\tau}{\alpha}\right) f(x(s)) ds\right]^2 \\
&\quad + \frac{1}{\alpha^3}\int_{\alpha t_k-\tau}^{t_k} p\left(\frac{s+(1+\alpha)\tau}{\alpha^2}\right) \int_s^{t_k} p\left(\frac{u+\tau}{\alpha}\right) f^2(x(u)) du ds. \\
&= b_k^2\left[x(t_k^-) - \frac{1}{\alpha}\int_{\alpha t_k-\tau}^{t_k} p\left(\frac{s+\tau}{\alpha}\right) f(x(s)) ds\right]^2 \\
&\quad + \frac{1}{\alpha^3}\int_{\alpha t_k-\tau}^{t_k} p\left(\frac{s+(1+\alpha)\tau}{\alpha^2}\right) \int_s^{t_k} p\left(\frac{u+\tau}{\alpha}\right) f^2(x(u)) du ds \\
&\leq V(t_k^-).
\end{aligned} \tag{2.7}$$

This and (2.6) shows that $V(t)$ is eventually decreasing. Since $V(t) \geq 0$, then $\lim_{t\to\infty} V(t) = \gamma$ exists and $\gamma \geq 0$. From (2.4), (2.6) and (2.7), it follows that

$$\int_T^{\infty} p\left(\frac{s+\tau}{\alpha}\right) f^2(x(s)) ds \leq \frac{\alpha}{\delta} V(T).$$

This implies

$$p\left(\frac{t+\tau}{\alpha}\right) f^2(x(t)) \in L^1(t_0, \infty),$$

and hence, for $0 < \alpha \leq 1$ and $\tau \geq 0$, we have

$$\lim_{t\to\infty} \int_{\alpha t-\tau}^{t} p\left(\frac{s+\tau}{\alpha}\right) f^2(x(s)) ds = 0. \tag{2.8}$$

It follows from (2.8) that

$$\begin{aligned}
0 \leq V_2(t) &= \frac{1}{\alpha^3}\int_{\alpha t-\tau}^{t} p\left(\frac{s+(1+\alpha)\tau}{\alpha^2}\right) \int_s^t p\left(\frac{u+\tau}{\alpha}\right) f^2(x(u)) du ds \\
&\leq \frac{1}{\alpha^3}\int_{\alpha t-\tau}^{t} p\left(\frac{s+(1+\alpha)\tau}{\alpha^2}\right) ds \int_{\alpha t-\tau}^{t} p\left(\frac{u+\tau}{\alpha}\right) f^2(x(u)) du
\end{aligned}$$

$$= \frac{1}{\alpha^2} \int_t^{\frac{t+\tau}{\alpha}} p\left(\frac{u+\tau}{\alpha}\right) ds \int_{\alpha t-\tau}^t p\left(\frac{u+\tau}{\alpha}\right) f^2(x(u))du$$
$$\leq \frac{2}{M\alpha} \int_{\alpha t-\tau}^t p\left(\frac{u+\tau}{\alpha}\right) f^2(x(u))du \to 0, \quad \text{as} \quad t \to \infty.$$

We then have $\lim_{t\to\infty} V_1(t) = \lim_{t\to\infty} V(t) = \gamma$, that is,

$$\lim_{t\to\infty} \left[x(t) - \frac{1}{\alpha} \int_{\alpha t-\tau}^t p\left(\frac{s+\tau}{\alpha}\right) f(x(s))ds\right]^2 = \gamma. \tag{2.9}$$

Let $u(t) = x(t) - 1/\alpha \int_{\alpha t-\tau}^t p((s+\tau)/\alpha) f(x(s))ds$. Then (2.9) shows that

$$\lim_{t\to\infty} u^2(t) = \gamma. \tag{2.10}$$

From (1.1), we can easily get

$$\begin{aligned} u(t_k) &= x(t_k) - \frac{1}{\alpha} \int_{\alpha t_k-\tau}^{t_k} p\left(\frac{s+\tau}{\alpha}\right) f(x(s))ds \\ &= b_k \left[x(t_k^-) - \frac{1}{\alpha} \int_{\alpha t_k-\tau}^{t_k} p\left(\frac{s+\tau}{\alpha}\right) f(x(s))ds\right] = b_k u(t_k^-). \end{aligned} \tag{2.11}$$

Thus, from (2.4) and (2.11), we have

$$\begin{cases} u'(t) + \frac{1}{\alpha} p\left(\frac{t+\tau}{\alpha}\right) f(x(t)) = 0, & t \geq t_0,\ t \neq t_k, \\ u(t_k) = b_k x(t_k^-), & k = 1, 2, \ldots. \end{cases} \tag{2.12}$$

If $\gamma = 0$, then $\lim_{t\to\infty} u(t) = 0$. If $\gamma > 0$, then there exists a sufficiently large T_1 such that $u(t) \neq 0$ for $t \geq T_1$. Otherwise, there is a sequence $\{\tau_k\}$ with $\lim_{k\to\infty} \tau_k = \infty$ such that $u(\tau_k) = 0$, and so $u^2(\tau_k) \to 0$ as $k \to \infty$. This contradicts with $\gamma > 0$. Therefore, for any $t_k > T_1, t \in [t_k, t_{k+1})$, we have $u(t) > 0$ or $u(t) < 0$ because $u(t)$ is continuous on $[t_k, t_{k+1})$. Without lost of generality, we assume that $u(t) > 0$ on $[t_k, t_{k+1})$, it follows that $u(t_{k+1}) = b_k u(t_{k+1}^-) > 0$, and thus $u(t) > 0$ on $[t_{k+1}, t_{k+2})$. By induction, we can conclude that $u(t) > 0$ on $[t_k, \infty)$. This and (2.10) imply that

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} \left(x(t) - \frac{1}{\alpha} \int_{\alpha t-\tau}^t p\left(\frac{s+\tau}{\alpha}\right) f(x(s))ds\right) = \nu, \tag{2.13}$$

where $\nu = \sqrt{\gamma}$ and is finite. In view of (2.12), we have

$$\begin{aligned} \frac{1}{\alpha} \int_{\alpha t-\tau}^t p\left(\frac{s+\tau}{\alpha}\right) f(x(s))ds &= u(\alpha t-\tau) - u(t) - \sum_{\alpha t < t_k < t} (u(t_k) - u(t_k^-)) \\ &= u(\alpha t-\tau) - u(t) - \sum_{\alpha t < t_k < t} (1-b_k) u(t_k^-). \end{aligned}$$

Letting $t \to \infty$ and noticing that $\sum_{k=1}^\infty (1-b_k) < \infty$, we have

$$\lim_{t\to\infty} \frac{1}{\alpha} \int_{\alpha t-\tau}^t p\left(\frac{s+\tau}{\alpha}\right) f(x(s))ds = 0. \tag{2.14}$$

It follows from (2.13) and (2.14) that $\lim_{t\to\infty} x(t) = \nu$ and the proof is completed. ∎

By Theorem 2.1, we have the the following asymptotic behavior result immediately.

Theorem 2.2. *The conditions of Theorem 2.1 imply that every oscillatory solution of* (1.1) *tends to zero as $t \to \infty$.*

In Theorem 2.1, taking $b_k \equiv 1$, $k = 1, 2, \ldots$, we have

Corollary 2.1. *Assume that* (2.2) *and* (2.3) *hold. Then every solution of* (1.2) *tends to a constant as* $t \to \infty$.

Theorem 2.3. *The conditions of Theorem 2.1 together with*

$$\int_{t_0}^{\infty} p(t)dt = \infty \tag{2.15}$$

imply that every solution of (1.1) *tends to zero as* $t \to \infty$.

Proof. By Theorem 2.2, we only have to prove that every nonoscillatory solution of (1.1) tends to zero as $t \to \infty$. Without lost of generality, let $x(t)$ be an eventually positive solution of (1.1), we shall prove that $\lim_{t\to\infty} x(t) = 0$. As in the proof of Theorem 2.1, we can rewrite (1.1) in the form (2.12). Integrating from t_0 to t both sides of (2.12) yields

$$\int_{t_0}^{t} p\left(\frac{s+\tau}{\alpha}\right) f(x(s))ds = \alpha\left[u(t_0) - u(t) - \sum_{t_0<t_k<t}(1-b_k)u(t_k^-)\right].$$

Using (2.13) and noticing $\sum_{k=1}^{\infty}(1-b_k) < \infty$, we can conclude that

$$\int_{t_0}^{\infty} p\left(\frac{s+\tau}{\alpha}\right) f(x(s))ds < \infty.$$

This, together with (2.15), implies that $\liminf_{t\to\infty} f(x(t)) = 0$. Let $\{s_n\}$ be a sequence such that $s_n \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} f(x(s_n)) = 0$. Then we must have $\liminf_{n\to\infty} x(s_n) = c = 0$. Otherwise, there exists a subsequence $\{s_{n_k}\}$ such that $x(s_{n_k}) \geq c/2$ for k sufficiently large. From (2.2), it follows that $f(x(s_{n_k})) \geq c/2$, which yields a contradiction because of $\lim_{k\to\infty} f(x(s_{n_k})) = 0$. Therefore, $\liminf_{t\to\infty} x(t) = 0$. This and Theorem 2.1 imply that $\lim_{t\to\infty} x(t) = 0$ and so the proof is completed. ∎

The following corollary follows from Corollary 2.1 and Theorem 2.3.

Corollary 2.2. *Assume that* (2.2), (2.3) *and* (2.15) *hold. Then every solution of* (1.2) *tends to zero as* $t \to \infty$.

Taking $\alpha = 1$ in Theorem 2.3, we have the following

Corollary 2.3. [15] *Assume that the following conditions hold:*

(i) $0 < b_k \leq 1$ $(k = 1, 2, \ldots)$ and $\sum_{k=1}^{\infty}(1-b_k) < \infty$;
(ii) *there exists a positive number* M *such that* $|x| \leq |f(x)| \leq M|x|$, $x \in R$, $xf(x) > 0$ $(x \neq 0)$;
(iii) $\limsup_{t\to\infty} \int_{t-\tau}^{t+\tau} p(s+\tau)ds < 2/M$;
(iv) $\int_{t_0}^{\infty} p(t)dt = \infty$.

Then every solution of (1.3) *tends to zero as* $t \to \infty$.

3. Examples

In this section, we give two examples to illustrate the usefulness of our main results.

Example 3.1. Consider the pantograph differential equation

$$x'(t) + \frac{2(2t-1)}{(2t+1)^2} x(t/2-1) = 0, t \geq 1, \tag{3.1}$$

where $\alpha = 1/2$, $p(t) = (2(2t-1))/((2t+1)^2)$ and $f(x) = x$.

One can easily see that

$$\int_{t_0}^{\infty} p(t)dt = \int_1^{\infty} \frac{2(2t-1)}{(2t+1)^2}dt = \infty,$$

and

$$\limsup_{t\to\infty} \int_{\alpha t-\tau}^{\frac{t+\tau}{\alpha}} p\left(\frac{s+\tau}{\alpha}\right) ds = \limsup_{t\to\infty} \int_{t/2-1}^{2(t+1)} \frac{4s+3}{2(4s+5)^2}ds \le \limsup_{t\to\infty} \int_{t/2-1}^{2(t+1)} \frac{1}{2s+1}ds$$
$$= \lim_{t\to\infty} \frac{1}{2}\ln\left(\frac{4t+5}{t-1}\right) = \ln 2 < 1.$$

Thus, the conditions (2.2), (2.3) and (2.15) hold. By Corollary 2.2, every solution of (3.1) tends to zero as $t \to \infty$. Indeed, $x(t) = 1/(2t+1)$ $(t \ge 2)$ is such a solution.

Example 3.2. Consider the impulsive differential equation

$$\begin{cases} x'(t) + \frac{1}{4t}\left[1+\cos^2 x(t/e-1)\right] x(t/e-1) = 0, & t \ge t_0 = 1,\ t \ne k, \\ x(k) = \frac{k^2-1}{k^2}x(k^-) + \frac{1}{k^2}\int_{k/e-1}^{k} \frac{x(s)}{4(s+1)}(1+\cos^2 x(s))ds, & k = 2,3,..., \end{cases} \tag{3.2}$$

where $\alpha = 1/e$, $p(t) = \frac{1}{4t}$, $f(x) = (1+\cos^2 x)x$, $b_k = \frac{k^2-1}{k^2}, k = 1,2,....$

One can easily find that

$$\sum_{k=1}^{\infty}(1-b_k) = \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty, \qquad |x| \le \left|(1+\cos^2 x)x\right| \le 2|x|,$$
$$(1+\cos^2 x)x^2 > 0\ (x \ne 0), \qquad \int_{t_0}^{\infty} p(s)ds = \int_1^{\infty} 1/(4s)ds = \infty,$$

and

$$\limsup_{t\to\infty} \int_{\alpha t-\tau}^{(t+\tau)/\alpha} p\left(\frac{s+\tau}{\alpha}\right) ds = \limsup_{t\to\infty} \int_{t/e-1}^{e(t+1)} \frac{1}{4e(s+1)}ds$$
$$= \frac{1}{4e}\limsup_{t\to\infty}\left(1+\ln\frac{e(t+1)+1}{t}\right) = 1/(2e) < 1/e.$$

By Theorem 2.3, every solution of (3.2) tends to zero as $t \to \infty$.

4. Conclusion

In this paper, Lyapunov functional method was developed to investigate the asymptotic behavior of solutions of a nonlinear generalized pantograph equation with certain impulses and sufficient conditions were obtained under which every solution of the equation tends to a constant or zero as $t \to \infty$. We believe that the method can be developed to investigate nonlinear neutral generalized pantograph equation with impulses. This is left for future investigations.

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