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# A General Iterative Method for Multi-Valued Mappings

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**Abstract.** Our purpose in this paper is to introduce a new general iterative approximation method for a family of multi-valued mappings in reflexive Banach spaces. Under suitable conditions, some strong convergence theorems for approximating a common fixed point of a family of multi-valued mappings are obtained. The main result extends various results existing in the current literature.

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## 1. Introduction

In a Banach space $E$, a mapping $T$ of $E$ into itself is said to be *Lipschitzian* if there exists $L \geq 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for each $x, y \in E$. $T$ is called a *nonexpansive* if $L = 1$. A mapping $f : E \to E$ is said to be *contraction* if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall\, x, y \in E.$$

We use $\Pi_E$ to denote the collection of all contractions on $E$, that is,

$$\Pi_E = \{f : E \to E : f \text{ is a contraction on E}\}.$$

The set $C \subseteq E$ is called *proximinal* if, for each $x \in E$, there exists an element $y \in C$ such that $\|x - y\| = d(x, C)$, where $d(x, C) = \inf\{\|x - z\| : z \in C\}$. We use $\mathscr{P}(E)$ to denote the family of nonempty proximal bounded subsets of $E$, i.e.,

$$\mathscr{P}(E) = \{C \subseteq E : C \text{ is nonempty proximal and bounded}\}.$$

Let $\mathscr{CB}(E), \mathscr{C}(E)$ denote the family of nonempty closed bounded subsets of $E$, and the family of nonempty compact subsets of $E$, respectively. $H(\cdot,\cdot)$ denotes the Hausdorff matric on $\mathscr{CB}(E)$, defined by

$$H(A, B) := \max\left\{\sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\|\right\}, \quad \forall\, A, B \in \mathscr{CB}(E).$$

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A multi-valued mapping $T : E \to \mathscr{CB}(E)$ is said to be *nonexpansive* if

$$H(Tx,Ty) \leq \|x-y\|, \quad \forall\, x,y \in E;$$

$T$ is said to be *quasi-nonexpansive* if $F(T) \notin \emptyset$ and

$$H(Tx,Tp) \leq \|x-p\|, \quad \forall\, x \in E, p \in F(T).$$

The multi-valued mapping $T : E \to \mathscr{CB}(E)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in $E$ such that $d(x_n, T(x_n)) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in E$. It is noted that, if $C$ is compact, then every multi-valued mapping $T : C \to \mathscr{CB}(C)$ is hemicompact. $\{x_n\}$ is said to satisfy Condition $(A')$ if, for any subsequence $x_{n_k} \rightharpoonup x$ and $d(x_{n+1}, T_n(x_n)) \to 0$ implies that $x \in F$, where $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ is the common fixed point set of the family of multi-valued mappings $T_i, i = 1,2,\ldots$.

Since Banach's Contraction Mapping Principle was extended nicely to multi-valued mappings by Nadler [10] in 1969, many authors have studied the fixed point theory for multi-valued mappings (see, e.g., [1,6,7,17,23]).

Let $E^*$ be the dual space of $E$, by a gauge function $\varphi$ we mean a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. The duality mapping $J_\varphi : E \to 2^{E^*}$ associated to a gauge function $\varphi$ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|\varphi(\|x\|), \|f^*\| = \varphi(\|x\|)\}, \quad \forall\, x \in E.$$

In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by $J$, is referred to as the normalized duality mapping. Clearly, there holds the relation $J_\varphi(x) = \varphi(\|x\|)/\|x\|J(x)$ for all $x \neq 0$ (see [2]). Browder [2] initiated the study of certain classes of nonlinear operators by means of the duality mapping $J_\varphi$. Following Browder [2], we say that a Banach space $E$ has a *weakly continuous duality mapping* if there exists a gauge $\varphi$ for which the duality mapping $J_\varphi(x)$ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J_\varphi(x_n)\}$ converges weakly* to $J_\varphi(x)$. It is known that $l^p$ has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \forall\, t \geq 0,$$

then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad \forall\, x \in E,$$

where $\partial$ denotes the sub-differential in the sense of convex analysis.

Sastry and Babu [15] defined the Mann iteration schemes for multi-valued mappings $T : C \to \mathscr{P}(C)$ and fixed point $p \in F(T)$. The sequence of Mann iterates is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \quad \forall\, n \geq 0, \end{cases} \tag{1.1}$$

where $y_n \in T(x_n)$ such that $\|z_n - p\| = d(p, T(x_n))$ and $\alpha_n \in [0,1]$ for all $n \geq 0$. They proved that the Mann iteration schemes for a multi-valued map $T$ with a fixed point $p$ converge to a fixed point of $T$ under certain conditions. Panyanak [11] extended the above result of Sastry and Babu [15] to uniformly convex Banach spaces but the domain of $T$ remains compact. Furthermore, he also obtained the strong convergence theorem for nonexpansive multi-valued mapping. In 2009, Shahzad and Zegeye [16] obtained first the strong convergence theorem of the Ishikawa iterative scheme for a multi-valued quasi-nonexpansive mapping $T : C \to \mathscr{CB}(C)$ such that $F(T)$ is nonempty and

$$T(p) = \{p\} \quad \text{for any} \quad p \in F(T).$$

Furthermore, for removing the rigid restriction on $F(T)$, that is $T(p) = \{p\}$ for any $p \in F(T)$, Shahzad and Zegeye [16] proved the second strong convergence theorem of the Ishikawa iterative scheme for a multi-valued quasi-nonexpansive mapping $T : C \to \mathscr{P}(C)$ such that $F(T)$ is nonempty.

In 2011, Song and Cho [18] modified and improved the proofs of the main results in [16]. Khan and Yildirim [5] further improved and generalized the results of [28] and [16].

Recently, Zuo [28] introduced two viscosity approximation sequence for a family of multi-valued nonexpansive mappings in a Banach space. Let $C$ be a nonempty closed convex subset of Banach space $E$ and $T_i : C \to \mathscr{P}(C), i = 1,2,\dots$, be a family of multi-valued nonexpansive mappings with $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ which is sunny nonexpansive retract of $C$, with $Q$ a nonexpansive retract of $C$ onto $F$. For each $n \in \mathbb{N}$,

$$P_{T_n}(x) = \{y \in T_n(x) : \|x - y\| = d(x, T_n(x))\}$$

and $f : C \to C$ is a contraction mapping with constant $\alpha \in (0,1)$. Let $\alpha_n \in (0,1), \beta_n \in (0,1)$; for any given $x_0 \in C$,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad n \geq 0, \tag{1.2}$$

where $y_n \in P_{T_n}(x_n)$, for each $n \in \mathbb{N}$. Furthermore, Zuo introduced the following multi-valued version of the modified Mann iteration:

$$x_{n+1} = \beta_n f(x_n) + \alpha_n x_n + (1 - \alpha_n - \beta_n) y_n, \quad n \geq 0, \tag{1.3}$$

where $y_n \in P_{T_n}(x_n)$. It is proved in [28] that both sequences generated by (1.2) and (1.3) converge strongly to a common fixed point $\tilde{x} = Q(f(\tilde{x}))$ of a family of multi-valued nonexpansive mappings in a Banach spaces $E$ which admits weakly sequentially continuous duality mapping $J_\varphi$ with a gauge function $\varphi$. Moreover, $\tilde{x}$ is the unique solution of the variational inequlity:

$$\langle f(\tilde{x}) - \tilde{x}, j_\varphi(y - \tilde{x}) \rangle \leq 0, \quad \forall\, y \in \bigcap_{n=1}^{\infty} F(T_n). \tag{1.4}$$

On the other hand, iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [4,24–26] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space:

$$\theta(x) = \min_{x \in C} 1/2 \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.5}$$

where $A$ is a linear bounded operator, $C$ is the fixed point set of a nonexpansive mapping $T$ and $b$ is a given point in $H$. In [9] Marino and Xu considered a general iterative method for a nonexpansive mapping in a Hilbert space $H$. Starting with arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad n \geq 0, \tag{1.6}$$

where $A$ is a strongly positive bounded linear operator on $H$, i.e.,

(1.7) $$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \quad \text{for all} \quad x \in H.$$

They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies the appropriate conditions, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to the unique solution $x^*$ in $F(T)$ of the variational inequality

(1.8) $$\langle (A-\gamma f)x^*, x-x^*\rangle \geq 0, \quad x \in F(T),$$

which is the optimality condition for the minimization problem: $\min_{x\in C} 1/2\langle Ax, x\rangle - h(x)$, where $h$ is a potential function for $\gamma f(i.e., h'(x) = \gamma f(x)$ for $x \in H)$. Many authors have improved and extended the results of [9] in the framework of Hilbert spaces (see, e.g., [12–14, 22]).

In a Banach space $E$ having a weakly continuous duality mapping $J_\varphi$ with a gauge function $\varphi$, an operator $A$ is said to be *strongly positive* [21] if there exists a constant $\bar{\gamma} > 0$ with the property

(1.9) $$\langle Ax, J_\varphi(x)\rangle \geq \bar{\gamma}\|x\|\varphi(\|x\|)$$

and

(1.10) $$\|\alpha I - \beta A\| = \sup_{\|x\|\leq 1} \left|\langle (\alpha I - \beta A)x, J_\varphi(x)\rangle\right|, \quad \alpha \in [0,1], \ \beta \in [-1,1],$$

where $I$ is the identity mapping. If $E := H$ is a real Hilbert space, then the inequality (1.9) reduce to (1.7).

In this paper, inspired and motivated by Sastry and Babu [15], Panyanak [11], Shahzad and Zegeye [16], Song and Cho [18], Zuo [28], Marino and Xu [9], we consider the following two general iterative schemes for a family of multi-valued mappings in a Banach space. Let $T_i : E \to \mathscr{P}(E), i = 1, 2, \ldots$, be a family of multi-valued mappings. Let $f \in \Pi_E$ with coefficient $\alpha \in (0,1)$, $A$ a strongly positive bounded linear operator on $E$ with coefficient $\bar{\gamma} > 0$, $0 < \gamma < \bar{\gamma}/\alpha$ and $\alpha_n \in (0,1), \beta_n \in (0,1)$. For any given $x_0 \in E$,

(1.11) $$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \quad n \geq 0,$$

where $y_n \in P_{T_n}(x_n)$. Furthermore, we introduce the following general multi-valued version of the modified Mann iteration:

(1.12) $$x_{n+1} = \beta_n \gamma f(x_n) + \alpha_n x_n + ((1-\alpha_n)I - \beta_n A)y_n, \quad n \geq 0,$$

where $y_n \in P_{T_n}(x_n)$. Some strong convergence theorems are proved in the framework of a reflexive Banach space which admits a weakly continuous duality mapping $J_\varphi$, where $\varphi : [0,\infty) \to [0,\infty)$ is a gauge function. The results presented in this paper improve and extend the corresponding results announced by Zuo [28], Marino and Xu [9], and many others.

## 2. Preliminaries

Throughout this paper, let $E$ be a real Banach space and $E^*$ be its dual space. We write $x_n \rightharpoonup x$ (respectively $x_n \overset{*}{\rightharpoonup} x$) to indicate that the sequence $\{x_n\}$ weakly (respectively weak*) converges to $x$; as usual $x_n \to x$ will symbolize strong convergence.

Now we collect some useful lemmas for proving the convergence result of this paper.

**Lemma 2.1.** [21, Lemma 3.1] *Assume that a Banach space $E$ has a weakly continuous duality mapping $J_\varphi$ with gauge $\varphi$. Let $A$ be a strong positive linear bounded operator on $E$ with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \varphi(1)\|A\|^{-1}$. Then $\|I - \rho A\| \le \varphi(1)(1 - \rho\bar{\gamma})$.*

The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [8].

**Lemma 2.2.** [8] *Assume that a Banach space $E$ has a weakly continuous duality mapping $J_\varphi$ with gauge $\varphi$.*

(i) *For all $x, y \in E$, the following inequality holds:*

$$\Phi(\|x+y\|) \le \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

*In particular, for all $x, y \in E$,*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

Let $Q$ be a mapping of $E$ onto $C$. Then $Q$ is said to be *sunny* if $Q(Q(x) + t(x - Q(x))) = Q(x)$ for all $x \in E$ and $t \ge 0$. A mapping $Q$ of $E$ into $E$ is said to be *retraction* if $Q^2 = Q$. If a mapping $Q$ is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is a range of $Q$. A subset $C$ of $E$ is said to be a sunny nonexpansive retract of $E$ if there exists a sunny nonexpansive retraction of $E$ onto $C$, and it is said to be a nonexpansive retract of $E$ if there exists a nonexpansive retraction of $E$ onto $C$.

**Lemma 2.3.** [3] *Let $C$ be a nonempty convex subset of a smooth Banach space $E$, $J : E \to E^*$ be the (normalized) duality mapping of $E$, and $Q : E \to C$ be a retraction. Then the following are equivalent.*

(1) $\langle x - Qx, j(y - Qx)\rangle \le 0$ *for all $x \in E$ and $y \in C$.*
(2) *$Q$ is both sunny and nonexpansive.*

It is noted that Lemma 2.3 still holds if the normalized duality map $J$ is replaced by the general duality map $J_\varphi$, where $\varphi$ is gauge function.

**Lemma 2.4.** [24, Lemma 2.1] *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the property*

$$a_{n+1} \le (1 - \gamma_n)a_n + \delta_n, \quad n \ge 0,$$

*where $\{\gamma_n\} \subseteq (0,1)$ and $\{\delta_n\} \subseteq \mathbb{R}$ such that*

$$\sum_{n=1}^{\infty} \gamma_n = \infty, \quad \text{and either} \quad \limsup_{n\to\infty} \delta_n/\gamma_n \le 0 \quad \text{or} \quad \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

*Then* $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.5.** [19] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $E$ and $\{\beta_n\}$ a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \ge 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

*Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

## 3. Main results

In this section, we prove strong convergence theorems for a countable family of multi-valued mappings.

**Theorem 3.1.** *Let $E$ be a reflexive Banach space which admits weakly sequentially continuous duality mapping $J_\varphi$ with gauge $\varphi$ such that $\varphi(1) = 1$. Let $f \in \Pi_E$ with coefficient $\alpha \in (0,1)$, $A$ a strongly positive bounded linear operator on $E$ with coefficient $\bar{\gamma} > 0$ such that $\|A\| = 1$, and $0 < \gamma < \bar{\gamma}/\alpha$. Let $T_i : E \to \mathscr{P}(E), \forall\, i = 0,1,2,\ldots$ be a family of multi-valued mappings with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ which is sunny nonexpansive retract of $E$, with $Q_F$ a nonexpansive retraction and $P_{T_i}$ a nonexpansive mapping. Let the sequence $\{x_n\}$ be defined by* (1.11)*, where a real sequence $\{\alpha_n\}$ satisfies the following conditions :*

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

*If $\{x_n\}$ satisfies condition $(A')$, then $\{x_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x} = Q_F((I - A + \gamma f)\tilde{x})$ of a family $T_i, i = 1,2,\ldots$. Moreover, $\tilde{x}$ is a unique solution of the variational inequality:*

$$\langle (A - \gamma f)\tilde{x}, J_\varphi(z - \tilde{x})\rangle \leq 0, \quad \forall\, z \in F. \tag{3.1}$$

*Proof.* We first show that the uniqueness of a solution of the variational inequality (3.1). Suppose both $\tilde{x} \in F$ and $x^* \in F$ are solutions to (3.1), then

$$\langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - x^*)\rangle \leq 0 \tag{3.2}$$

and

$$\langle (A - \gamma f)x^*, J_\varphi(x^* - \tilde{x})\rangle \leq 0. \tag{3.3}$$

Adding (3.2) and (3.3), we obtain

$$\langle (A - \gamma f)\tilde{x} - (A - \gamma f)x^*, J_\varphi(\tilde{x} - x^*)\rangle \leq 0. \tag{3.4}$$

Noticing that for any $x, y \in E$,

$$\begin{aligned}
\langle (A - \gamma f)x - (A - \gamma f)y, J_\varphi(x - y)\rangle &= \langle A(x-y), J_\varphi(x-y)\rangle - \gamma \langle f(x) - f(y), J_\varphi(x-y)\rangle \\
&\geq \bar{\gamma}\|x - y\|\varphi(\|x-y\|) - \gamma\|f(x) - f(y)\|\|J_\varphi(x-y)\| \\
&\geq \bar{\gamma}\Phi(\|x-y\|) - \gamma\alpha\Phi(\|x-y\|) \\
&= (\bar{\gamma} - \gamma\alpha)\Phi(\|x-y\|) \geq 0. \qquad (3.5)
\end{aligned}$$

Therefore $\tilde{x} = x^*$ and the uniqueness is proved. Below we use $\tilde{x}$ to denote the unique solution of (3.1).

Note that $p \in P_{T_i}(p) = \{p\}$ for any fixed point $p \in F(T_i), i = 1,2,\ldots$. Then we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n - p\| \\
&= \|\alpha_n \gamma f(x_n) - \alpha_n \gamma f(p) + \alpha_n \gamma f(p) - \alpha_n A(p) + (I - \alpha_n A)y_n - (I - \alpha_n A)p\| \\
&\leq \alpha_n \gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - A(p)\| + \varphi(1)(1 - \alpha_n\bar{\gamma})\|y_n - p\| \\
&= \alpha_n \gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - A(p)\| + \varphi(1)(1 - \alpha_n\bar{\gamma})d(y_n, P_{T_n}(p)) \\
&\leq \alpha_n \gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - A(p)\| + \varphi(1)(1 - \alpha_n\bar{\gamma})H(P_{T_n}(x_n), P_{T_n}(p)) \\
&\leq \alpha_n \gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - A(p)\| + \varphi(1)(1 - \alpha_n\bar{\gamma})\|x_n - p\|
\end{aligned}$$

$$\leq \alpha_n \|\gamma f(p) - A(p)\| + (1 - \alpha_n(\varphi(1)\bar{\gamma} - \gamma\alpha))\|x_n - p\|$$
$$\leq \max\{\|x_n - p\|, \|\gamma f(p) - A(p)\|/(\bar{\gamma} - \gamma\alpha)\}.$$

By induction, we obtain that $\|x_n - p\| \leq \max\{\|x_0 - p\|, \|\gamma f(p) - A(p)\|/(\bar{\gamma} - \gamma\alpha)\}\ \forall n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded, and so are $\{A(y_n)\}$ and $\{f(x_n)\}$. Then we have

$$d(x_{n+1}, T_n(x_n)) \leq \|x_{n+1} - y_n\| = \alpha_n \|\gamma f(x_n) - A(y_n)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.6}$$

We observe that $Q_F(I - A + \gamma f)$ is a contraction. Indeed, for all $x, y \in E$, we have

$$\begin{aligned}\|Q_F(\gamma f + (I-A))(x) - Q_F(\gamma f + (I-A))(y)\| &\leq \|(\gamma f + (I-A))(x) - (\gamma f + (I-A))(y)\| \\ &\leq \gamma \|f(x) - f(y)\| + \|I - A\|\|x - y\| \\ &\leq \gamma\alpha\|x - y\| + (1 - \overline{\gamma})\|x - y\| \\ &\leq (1 - (\overline{\gamma} - \alpha\gamma))\|x - y\|.\end{aligned}$$

Banach's Contraction Mapping Principle guarantees that $Q_F(\gamma f + (I-A))$ has a unique fixed point, say $\tilde{x} \in E$. That is, $\tilde{x} = Q_F(\gamma f + (I-A))(\tilde{x})$.

Next, we shall show that

$$\langle \gamma f(\tilde{x}) - A(\tilde{x}), j_\varphi(x_n - \tilde{x})\rangle \leq 0. \tag{3.7}$$

Since $E$ is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z \in E$ and

$$\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A(\tilde{x}), j_\varphi(x_{n+1} - \tilde{x})\rangle = \lim_{k\to\infty} \langle \gamma f(\tilde{x}) - A(\tilde{x}), j_\varphi(x_{n_k} - \tilde{x})\rangle.$$

From (3.6), and since $\{x_n\}$ satisfies condition (A$'$), we obtain that $z \in F$. On the other hand, the assumption that the duality mapping $J_\varphi$ is weakly continuous, we then have

$$\begin{aligned}\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A(\tilde{x}), j_\varphi(x_{n+1} - \tilde{x})\rangle &= \lim_{k\to\infty} \langle \gamma f(\tilde{x}) - A(\tilde{x}), j_\varphi(x_{n_k} - \tilde{x})\rangle \\ &= \langle \gamma f(\tilde{x}) - A(\tilde{x}), j_\varphi(z - \tilde{x})\rangle \leq 0.\end{aligned}$$

Finally, we will show that $x_n \to \tilde{x}$ as $n \to \infty$. In fact, since $\Phi(t) = \int_0^t \varphi(\tau)d\tau, \forall t \geq 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is a gauge function, then for $1 \geq k \geq 0$, $\varphi(kx) \leq \varphi(x)$ and

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(kx)dx \leq k\int_0^t \varphi(x)dx = k\Phi(t).$$

Using Lemma 2.2, we get that

$$\begin{aligned}&\Phi(\|x_{n+1} - \tilde{x}\|) \\ &= \Phi\big(\|\alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n - \tilde{x}\|\big) \\ &\leq \Phi\big(\|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(\tilde{x}) + (I - \alpha_n A)y_n - (I - \alpha_n A)\tilde{x} + \alpha_n\gamma f(\tilde{x}) - \alpha_n A(\tilde{x})\|\big) \\ &\leq \Phi\big(\|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(\tilde{x}) + (I - \alpha_n A)y_n - (I - \alpha_n A)\tilde{x}\|\big) + \alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), j_\varphi(x_{n+1} - \tilde{x})\rangle \\ &\leq \Phi\big(\alpha_n\gamma\|f(x_n) - f(\tilde{x})\| + \varphi(1)(1 - \alpha_n\bar{\gamma})\|y_n - \tilde{x}\|\big) + \alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), j_\varphi(x_{n+1} - \tilde{x})\rangle \\ &\leq \Phi\big(\alpha_n\gamma\alpha\|x_n - \tilde{x}\| + (1 - \alpha_n\bar{\gamma})H(P_{T_n}(x_n), P_{T_n}(\tilde{x}))\big) + \alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), j_\varphi(x_{n+1} - \tilde{x})\rangle \\ &\leq \Phi\big(\alpha_n\gamma\alpha\|x_n - \tilde{x}\| + (1 - \alpha_n\bar{\gamma})\|x_n - \tilde{x}\|\big) + \alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), j_\varphi(x_{n+1} - \tilde{x})\rangle \\ &\leq \Phi\big((1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x_n - \tilde{x}\|\big) + \alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), j_\varphi(x_{n+1} - \tilde{x})\rangle \\ &\leq (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\Phi(\|x_n - \tilde{x}\|) + \alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), j_\varphi(x_{n+1} - \tilde{x})\rangle.\end{aligned}$$

From Lemma 2.4 we get that $x_n \to \tilde{x}$ which satisfies the variational inequality follows from the property of $Q_F$. ∎

We will give some example of a family of multi-valued mappings and a sequence satisfying condition $(A')$ as following :

**Example 3.1.** Let $E = \mathbb{R}$ and $C = [0,1]$. For each $n \in \mathbb{N}$, define a multi-valued mapping $T_n : C \to \mathscr{C}(C)$ by

$$T_n(x) = [0, x/n], \quad \text{for all} \quad x \in C.$$

It is easy to see that $\{0\} = F(T_n)$ for all $n \in \mathbb{N}$. If $\{x_n\} = \{1/n\}_{n\in\mathbb{N}}$, then $\{x_n\} \subseteq C$ and $x_n \to 0$ as $n \to \infty$. From definition of $T$, we have that

$$\begin{aligned} d(x_{n+1}, T_n(x_n)) &= d\left(1/n+1, \left[0, 1/n^2\right]\right) \\ &= 1/n+1-1/n^2 \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence $\{x_n\}$ satisfies the condition $(A')$.

**Remark 3.1.** In general case, if $A$ is any strongly positive bounded linear operator with coefficient $\bar{\gamma}$ and $0 < \gamma < \bar{\gamma}/\alpha$. We define a bounded linear operator $\overline{A}$ on $E$ by

$$\overline{A} = \|A\|^{-1} A.$$

It is easy to see that $\overline{A}$ is a strongly positive with coefficient $\|A\|^{-1}\bar{\gamma} > 0$ such that $\|\overline{A}\| = 1$ and

$$0 < \|A\|^{-1}\gamma < \|A\|^{-1}\bar{\gamma}/\alpha.$$

Let the sequence $\{x_n\}$ be defined by, for any $x_0 \in E$,

$$x_{n+1} = \alpha_n \|A\|^{-1}\gamma f(x_n) + (I - \alpha_n \overline{A}) y_n, \quad n \geq 0, \tag{3.8}$$

where $y_n \in P_{T_n}(x_n)$. Replacing $A$ with $\overline{A}$ in Theorem 3.1, we obtain the following result.

**Theorem 3.2.** *Let $E$ be a reflexive Banach space which admits weakly sequentially continuous duality mapping $J_\varphi$ with gauge $\varphi$ such that $\varphi(1) = 1$. Let $f \in \Pi_E$ with coefficient $\alpha \in (0,1)$, $A$ a strongly positive bounded linear operator on $E$ with coefficient $\bar{\gamma} > 0$, and $0 < \gamma < \bar{\gamma}/\alpha$. Let $T_i : E \to \mathscr{P}(E), \forall i = 0,1,2,\ldots$ be a family of multi-valued mappings with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, which is sunny nonexpansive retract of $E$, with $Q_F$ a nonexpansive retraction and $P_{T_i}$ a nonexpansive mapping. Let the sequence $\{x_n\}$ be defined by* (3.8)*, where $\{\alpha_n\}$ is a real sequence in* $(0,1)$ *satisfying the following conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty.$$

*If $\{x_n\}$ satisfies condition $(A')$, $\{x_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x} = Q_F\left((I - \|A\|^{-1}(A + \gamma f)\tilde{x})\right)$ of a family $T_i, i = 0,1,2,\ldots$. Moreover, $\tilde{x}$ is a unique solution of the variational inequality:* (3.1).

*Proof.* From Theorem 3.1, we have that $\{x_n\}$ is generated by (3.8) converges strongly as $n \to \infty$ to a common fixed point $\tilde{x} = Q_F((I - \|A\|^{-1}(A + \gamma f)\tilde{x}))$ is a unique solution of the variational inequality:

$$\|A\|^{-1}\langle (A - \gamma f)\tilde{x}, J_\varphi(z - \tilde{x})\rangle \leq 0, \quad z \in F. \tag{3.9}$$

It is easy to see that (3.9) is equivalent to (3.1). Hence $\tilde{x}$ is a unique solution of the variational inequality (3.1). ∎

If $\gamma = 1$ and $A \equiv I$, the identity mapping, in Theorem 3.2, then the requirement that $\varphi(1) = 1$ is not necessary. In fact, Theorem 3.3 can be obtain from Theorem 3.1 immediately.

**Theorem 3.3.** *Let $E$ be a reflexive Banach space which admits weakly sequentially continuous duality mapping $J_\varphi$ with gauge $\varphi$. Let $f \in \Pi_E$ with coefficient $\alpha \in (0,1)$. Let $T_i : E \to \mathscr{P}(E), i = 0,1,2,\dots$ be a family of multi-valued mappings with $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, which is sunny nonexpansive retract of $E$, with $Q_F$ a nonexpansive retraction and $P_{T_i}$ a nonexpansive mapping. Let the sequence $\{x_n\}$ be defined by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \quad n \geq 0,$$

*where $y_n \in T_n(x_n)$, $0 < \alpha_n < 1$ satisfying the following conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

*If $\{x_n\}$ satisfies condition (A′), then $\{x_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x} = Q_F(f(\tilde{x}))$ of a family $T_i, i = 0,1,2,\dots$. Moreover, $\tilde{x}$ is a unique solution of the variational inequality:*

$$\langle (I-f)\tilde{x}, J_\varphi(z-\tilde{x})\rangle \leq 0, \quad z \in F.$$

Next, we prove the strong convergence theorem for a general multi-valued version of the modified Mann iteration.

**Theorem 3.4.** *Let $E$ be a reflexive Banach space with weakly sequentially continuous duality mapping $J_\varphi$ with gauge $\varphi$ such that $\varphi(1) = 1$. Let $f \in \Pi_E$ with coefficient $\alpha \in (0,1)$, $A$ a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ such that $\|A\| = 1$, and $0 < \gamma < \bar{\gamma}/\alpha$. Let $T_i : E \to \mathscr{P}(E), i = 0,1,2,\dots$ be a family of multi-valued mappings such that*

$$H(T_{n+1}x, T_n y) \leq \|x-y\| \quad \textit{for all} \quad n \in \mathbb{N}$$

*with $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ which is sunny nonexpansive retract of $E$, with $Q_F$ a nonexpansive retraction and $P_{T_i}$ a nonexpansive mapping. Let the sequence $\{x_n\}$ be defined by (1.12), where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions:*

$$\lim_{n\to\infty} \beta_n = 0, \sum_{n=0}^{\infty} \beta_n = \infty$$

*and*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

*If $\{x_n\}$ satisfies condition (A′), then $\{x_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x} = Q_F((I - A + \gamma f)\tilde{x})$ of a family $T_i, i = 0,1,2,\dots$. Moreover, $\tilde{x}$ is a unique solution of the variational inequality* (3.1).

*Proof.* Notice that $\beta_n \to 0$, we may assume, without loss of generality, that $\beta_n \leq 1 - \alpha_n$. Since $A$ is a strongly positive bounded linear operator, then

$$\|A\| = \sup\left\{ |\langle Ax, J_\varphi(x)\rangle| : x \in E, \|x\| = 1 \right\},$$

Now for any $x \in E$ with $\|x\| = 1$, we see that

$$\langle ((1-\alpha_n)I - \beta_n A)x, J_\varphi(x)\rangle = (1-\alpha_n)\langle x, J_\varphi(x)\rangle - \beta_n \langle Ax, J_\varphi(x)\rangle$$

$$= (1-\alpha_n)\varphi(1) - \beta_n\langle Ax, J_\varphi(x)\rangle$$
$$\geq (1-\alpha_n)\varphi(1) - \beta_n\|A\| = 1-\alpha_n-\beta_n \geq 0.$$

That is $(1-\alpha_n)I-\beta_nA$ is positive. It follows that

$$\begin{aligned}\|(1-\alpha_n)I-\beta_nA\| &= \sup\big\{\langle((1-\alpha_n)I-\beta_nA)x, J_\varphi(x)\rangle : x\in E, \|x\|=1\big\}\\ &= \sup\big\{\varphi(1)(1-\alpha_n)-\beta_n\langle Ax, J_\varphi(x)\rangle : x\in E, \|x\|=1\big\}\\ &\leq \varphi(1)(1-\alpha_n)-\beta_n\bar{\gamma}\varphi(1) = 1-\alpha_n-\beta_n\bar{\gamma}.\end{aligned}$$

Firstly, we show that $\{x_n\}$ is bounded. Indeed, take $p\in T_i(p)=\{p\}$, then

$$\begin{aligned}&\|x_{n+1}-p\|\\ &= \|\beta_n\gamma f(x_n)+\alpha_nx_n+((1-\alpha_n)I-\beta_nA)y_n-p\|\\ &= \|\beta_n\gamma f(x_n)-\beta_nA(p)+\alpha_nx_n-\alpha_np+((1-\alpha_n)I-\beta_nA)y_n-p+\alpha_np+\beta_nA(p)\|\\ &\leq \beta_n\|\gamma f(x_n)-A(p)\|+\alpha_n\|x_n-p\|+\|((1-\alpha_n)I-\beta_nA)y_n-((1-\alpha_n)I-\beta_nA)p\|\\ &\leq \beta_n\|\gamma f(x_n)-\gamma f(p)\|+\beta_n\|\gamma f(p)-A(p)\|+\alpha_n\|x_n-p\|+\|((1-\alpha_n)I-\beta_nA)\|\|y_n-p\|\\ &\leq \beta_n\gamma\|f(x_n)-f(p)\|+\beta_n\|\gamma f(p)-A(p)\|+\alpha_n\|x_n-p\|\\ &\quad+\varphi(1)(1-\alpha_n-\beta_n\bar{\gamma})d(y_n,P_{T_n}(p))\\ &\leq \beta_n\gamma\alpha\|x_n-p\|+\beta_n\|\gamma f(p)-A(p)\|+\alpha_n\|x_n-p\|\\ &\quad+\varphi(1)(1-\alpha_n-\beta_n\bar{\gamma})H\left(P_{T_n}(x_n),P_{T_n}(p)\right)\\ &\leq \beta_n\gamma\alpha\|x_n-p\|+\beta_n\|\gamma f(p)-A(p)\|+\alpha_n\|x_n-p\|+(1-\alpha_n-\beta_n\bar{\gamma})\|x_n-p\|\\ &= \beta_n\gamma\alpha\|x_n-p\|+\beta_n\|\gamma f(p)-A(p)\|+\alpha_n\|x_n-p\|+\varphi(1)(1-\alpha_n)\|x_n-p\|-\beta_n\bar{\gamma}\|x_n-p\|\\ &\leq \beta_n\gamma\alpha\|x_n-p\|+\beta_n\|\gamma f(p)-A(p)\|+\alpha_n\|x_n-p\|+(1-\alpha_n)\|x_n-p\|-\beta_n\bar{\gamma}\|x_n-p\|\\ &= \beta_n\gamma\alpha\|x_n-p\|+\beta_n\|\gamma f(p)-A(p)\|+\|x_n-p\|-\beta_n\bar{\gamma}\|x_n-p\|\\ &= (1-\beta_n(\varphi(1)\bar{\gamma}-\gamma\alpha))\|x_n-p\|+\beta_n\|\gamma f(p)-A(p)\|\\ &\leq \max\{\|x_n-p\|, \|\gamma f(p)-A(p)\|/(\bar{\gamma}-\gamma\alpha)\}.\end{aligned}$$

It follows from induction that

$$\|x_n-p\| \leq \max\{\|x_0-p\|, \|\gamma f(p)-A(p)\|/(\bar{\gamma}-\gamma\alpha)\}, \quad n\geq 0. \tag{3.10}$$

Hence $\{x_n\}$ is bounded, and so are $\{A(y_n)\}$ and $\{f(x_n)\}$. Next, we show that

$$\lim_{n\to\infty} d(x_{n+1}, T_n(x_n)) = 0. \tag{3.11}$$

Set $\lambda_n = \beta_n/(1-\alpha_n)$ and $z_n = \lambda_n\gamma f(x_n)+(1-\lambda_nA)y_n$. Then

$$x_{n+1} = \alpha_nx_n+(1-\alpha_n)z_n \quad \text{and} \quad \lim_{n\to\infty}\lambda_n = 0.$$

Thus

$$\begin{aligned}d(x_{n+1}, T_n(x_n)) &\leq \|x_{n+1}-y_n\| \leq \|x_{n+1}-z_n\|+\|z_n-y_n\|\\ &\leq \alpha_n\|x_n-z_n\|+\lambda_n\|\gamma f(x_n)-A(y_n)\|.\end{aligned} \tag{3.12}$$

It suffices to prove that

$$\lim_{n\to\infty}\|x_n-z_n\| = 0. \tag{3.13}$$

From definition of $z_n$, we have that

$$\begin{aligned}
&\|z_{n+1}-z_n\| \\
&= \|\lambda_{n+1}\gamma f(x_{n+1})+(1-\lambda_{n+1}A)y_{n+1}-\lambda_n\gamma f(x_n)-(1-\lambda_n A)y_n\| \\
&= \|\lambda_{n+1}\gamma f(x_{n+1})-\lambda_n\gamma f(x_n)\|+\|y_{n+1}-y_n\|+\lambda_n\|y_n\|+\lambda_{n+1}\|y_{n+1}\| \\
&\le \lambda_{n+1}\gamma\|f(x_{n+1})\|+\lambda_n\gamma\|f(x_n)\|+\lambda_n\|y_n\|+\lambda_{n+1}\|y_{n+1}\|+H(P_{T_{n+1}}(x_{n+1}),P_{T_n}(x_n)) \\
&\le \|x_{n+1}-x_n\|+\lambda_{n+1}\gamma\|f(x_{n+1})\|+\lambda_n\gamma\|f(x_n)\|+\lambda_n\|y_n\|+\lambda_{n+1}\|y_{n+1}\|.
\end{aligned}$$

Thus

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le\limsup_{n\to\infty}2(\lambda_n+\lambda_{n+1})M=0.$$

Lemma 2.5 implies that $\lim_{n\to\infty}\|x_n-z_n\|=0$ as required. It follow from (3.12) that

$$\lim_{n\to\infty}d(x_{n+1},T_n(x_n))=0.$$

Applying the condition $(A')$ of $\{x_n\}$ and using the same argument as in the proof of Theorem 3.1, there exists $\tilde{x}\in F$ such that

$$\langle\gamma f(\tilde{x})-A(\tilde{x}),j_\varphi(x_{n+1}-\tilde{x})\rangle\le 0. \tag{3.14}$$

Finally, we will show that $x_n\to\tilde{x}$ as $n\to\infty$. Applying (1.11) and Lemma 2.2, we have

$$\begin{aligned}
&\Phi(\|x_{n+1}-\tilde{x}\|) \\
&=\Phi\big(\|\beta_n\gamma f(x_n)+\alpha_n x_n+((1-\alpha_n)I-\beta_n A)y_n-\tilde{x}\|\big) \\
&=\Phi\big(\|\beta_n\gamma f(x_n)-\beta_n A(\tilde{x})+\alpha_n x_n-\alpha_n\tilde{x}+((1-\alpha_n)I-\beta_n A)y_n-\tilde{x}+\alpha_n\tilde{x}+\beta_n A(\tilde{x})\|\big) \\
&=\Phi\big(\|\beta_n\gamma f(x_n)-\beta_n\gamma f(\tilde{x})+\beta_n\gamma f(\tilde{x})-\beta_n A(\tilde{x})+\alpha_n x_n-\alpha_n\tilde{x}+((1-\alpha_n)I-\beta_n A)y_n \\
&\quad-((1-\alpha_n)I-\beta_n A)\tilde{x}\|\big) \\
&=\Phi\big(\|\beta_n\gamma(f(x_n)-f(\tilde{x}))+\alpha_n(x_n-\tilde{x})+((1-\alpha_n)I-\beta_n A)(y_n-\tilde{x})+\beta_n\gamma f(\tilde{x})-\beta_n A(\tilde{x})\|\big) \\
&\le\Phi\big(\|\beta_n\gamma(f(x_n)-f(\tilde{x}))+\alpha_n(x_n-\tilde{x})+((1-\alpha_n)I-\beta_n A)(y_n-\tilde{x})\|\big) \\
&\quad+\beta_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_{n+1}-\tilde{x})\rangle \\
&\le\Phi\big(\beta_n\gamma\alpha\|x_n-\tilde{x}\|+\alpha_n\|x_n-\tilde{x}\|+\|(1-\alpha_n)I-\beta_n A\|\|y_n-\tilde{x}\|\big) \\
&\quad+\beta_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_{n+1}-\tilde{x})\rangle \\
&\le\Phi\big(\beta_n\gamma\alpha\|x_n-\tilde{x}\|+\alpha_n\|x_n-\tilde{x}\|+\|(1-\alpha_n)I-\beta_n A\|H(P_{T_n}(x_n),P_{T_n}(\tilde{x}))\big) \\
&\quad+\beta_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_{n+1}-\tilde{x})\rangle \\
&\le\Phi\big(\beta_n\gamma\alpha\|x_n-\tilde{x}\|+\alpha_n\|x_n-\tilde{x}\|+\varphi(1)(1-\alpha_n-\beta_n\bar{\gamma})\|x_n-\tilde{x}\|\big) \\
&\quad+\beta_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_{n+1}-\tilde{x})\rangle \\
&\le\Phi\big((1-\beta_n(\varphi(1)\bar{\gamma}-\gamma\alpha))\|x_n-\tilde{x}\|\big)+\beta_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_{n+1}-\tilde{x})\rangle.
\end{aligned}$$

Applying Lemma 2.4 to the above inequality, we can conclude that $x_n\to\tilde{x}$ as $n\to\infty$. This completes the proof. ∎

In general case, if $A$ is any strongly positive bounded linear operator with coefficient $\bar{\gamma}$ and $0<\gamma<\bar{\gamma}/\alpha$, we obtain the following result.

**Theorem 3.5.** *Let $E$ be a reflexive Banach space with weakly sequentially continuous duality mapping $J_\varphi$ with gauge $\varphi$ such that $\varphi(1)=1$. Let $f\in\Pi_E$ with coefficient $\alpha\in(0,1)$,* *A a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$, and $0 < \gamma < \bar{\gamma}/\alpha$. Let $T_i : E \to \mathscr{P}(E), i = 0,1,2,\ldots$ be a family of multi-valued mappings such that*

$$H(T_{n+1}x, T_n y) \leq \|x - y\| \quad \text{for all} \quad n \in \mathbb{N}$$

*with $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, which is sunny nonexpansive retract of E, with $Q_F$ a nonexpansive retraction and $P_{T_i}$ a nonexpansive. Let the sequence $\{x_n\}$ be defined by*

$$x_{n+1} = \beta_n \|A\|^{-1} \gamma f(x_n) + \alpha_n x_n + ((1-\alpha_n)I - \beta_n \overline{A}) y_n, \quad n \geq 0, \tag{3.15}$$

*where $y_n \in T_n(x_n)$ and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions:*

$$\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=0}^{\infty} \beta_n = \infty$$

*and*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

*If $\{x_n\}$ satisfies condition $(A')$, then $\{x_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x} = Q_F((I - A + \gamma f)\tilde{x})$ of a family $T_i, i = 1,2,\ldots$. Moreover, $\tilde{x}$ is a unique solution of the variational inequality* (3.1).

*Proof.* From Remark 3.1 and Theorem 3.4, we have that $\{x_n\}$ is generated by (3.15) converges strongly as $n \to \infty$ to a common fixed point $\tilde{x} = Q_F((I - \|A\|^{-1}(A + \gamma f)\tilde{x}))$ is a unique solution of the variational inequality:

$$\|A\|^{-1} \langle (A - \gamma f)\tilde{x}, J_\varphi(z - \tilde{x}) \rangle \leq 0, \quad z \in F. \tag{3.16}$$

This is equivalent to (3.1). ∎

Setting $\gamma = 1$ and $A \equiv I$, the identity mapping, the requirement $\varphi(1) = 1$ is not necessary. In fact, the following theorem can be obtain from Theorem 3.5 immediately.

**Theorem 3.6.** *Let E be a reflexive Banach space with weakly sequentially continuous duality mapping $J_\varphi$ with gauge $\varphi$. Let $f \in \Pi_E$ with coefficient $\alpha \in (0,1)$. Let $T_i : E \to \mathscr{P}(E), i = 0,1,2,\ldots$ be a family of multi-valued mappings such that $H(T_{n+1}x, T_n y) \leq \|x - y\|$ for all $n \in \mathbb{N}$ with $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, which is sunny nonexpansive retract of E, with $Q_F$ a nonexpansive retraction and $P_{T_i}$ a nonexpansive mapping. Let the sequence $\{x_n\}$ be defined by*

$$x_{n+1} = \beta_n f(x_n) + \alpha_n x_n + (1 - \alpha_n - \beta_n) y_n, \quad n \geq 0,$$

*where $y_n \in P_{T_n}(x_n)$ and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0,1)$ satisfying $\alpha_n + \beta_n < 1$ and the following conditions:*

$$\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=0}^{\infty} \beta_n = \infty$$

*and*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

*If $\{x_n\}$ satisfies condition $(A')$, then $\{x_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x} = Q_F(f(\tilde{x}))$ of a family $T_i, i = 1,2,\ldots$. Moreover, $\tilde{x}$ is a unique solution of the variational inequality*

$$\langle (I - f)\tilde{x}, J_\varphi(z - \tilde{x}) \rangle \leq 0, \quad z \in F.$$

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