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On A Class of δ-Supplemented Modules

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Abstract. Let R be an arbitrary ring with identity and M a right R-module. In this paper, we introduce a class of modules which is an analogous to that of δ-supplemented modules and principally $\oplus$-supplemented modules. The module M is called *principally* $\oplus$-δ-*supplemented* if for any $m \in M$ there exists a direct summand A of M such that $M = mR + A$ and $mR \cap A$ is δ-small in A. We prove that some results of principally $\oplus$-supplemented modules can be extended to principally $\oplus$-δ-supplemented modules for this general setting. Several properties of these modules are given and it is shown that the class of principally $\oplus$-δ-supplemented modules lies strictly between classes of principally $\oplus$-supplemented modules and principally δ-supplemented modules. We investigate conditions which ensure that any factor modules, direct summands and direct sums of principally $\oplus$-δ-supplemented modules are also principally $\oplus$-δ-supplemented. We give a characterization of principally $\oplus$-δ-supplemented modules over a semisimple ring and a new characterization of principally δ-semiperfect rings is obtained by using principally $\oplus$-δ-supplemented modules.

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1. Introduction

Throughout this paper all rings have an identity and all modules are unitary right modules. $N \leq M$ will mean N is a submodule of M. A submodule N of a module M is called *small* in M if for every $K \leq M$ the equality $M = N + K$ implies $M = K$. Let N and P be submodules of M. We call P *a supplement* of N in M if $M = P + N$ and $P \cap N$ is small in P. A module M is called *supplemented* if every submodule of M has a supplement in M [10]. In [18], Zhou introduced the concept of δ-small submodules as a generalization of small submodules. A submodule N of M is said to be δ-*small* in M if whenever $M = N + K$ and M/K is singular, we have $M = K$. Let N be a submodule of M. A submodule L of M is called a δ-*supplement* of N in M if $M = N + L$ and $N \cap L$ is δ-small in L (therefore in M), and M is called δ-*supplemented* in case every submodule of M has a δ-supplement in M (see [8]

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in detail). Note that every supplemented module is δ-supplemented. Following [10], the module M is called *$\oplus$-supplemented* if for any submodule N of M, there exists a direct summand K of M with $M = N + K$ and $N \cap K$ small in K, i.e., every submodule of M has a direct summand supplement in M, while in [14] M is called *principally $\oplus$-supplemented* if every cyclic submodule of M has a direct summand supplement in M. Let M be a module, K and L submodules of M. K is called *a $\oplus$-δ-supplement* of N in M if $M = K + N$, K is a direct summand of M and $K \cap N$ is δ-small in K. Also M is called *$\oplus$-δ-supplemented* if every submodule of M has a $\oplus$-δ-supplement in M. Clearly, $\oplus$-δ-supplemented modules are δ-supplemented and $\oplus$-supplemented modules are $\oplus$-δ-supplemented.

In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ we denote, respectively, integers, rational numbers, the ring of integers and the $\mathbb{Z}$-module of integers modulo n. $M_n(R)$ stands for the ring of all $n \times n$ matrices over R. For unexplained concepts and notations, we refer the reader to [1, 10].

2. δ-small submodules and δ-supplement submodules

We collect basic properties of δ-small submodules in the following lemma which is contained in [18].

Lemma 2.1. *Let M be a module. Then we have the following.*

(1) *If N is δ-small in M and $M = X + N$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$.*
(2) *If K is δ-small in M and $f : M \to N$ is a homomorphism, then $f(K)$ is δ-small in N. In particular, if K is δ-small in $M \subseteq N$, then K is δ-small in N.*
(3) *Let $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2$ is δ-small in $M_1 \oplus M_2$ if and only if K_1 is δ-small in M_1 and K_2 is δ-small in M_2.*
(4) *Let N, K be submodules of M with K δ-small in M and $N \leq K$. Then N is also δ-small in M.*

The next lemma is clear from definitions.

Lemma 2.2. *Let M be a module and $m \in M$. Then the following are equivalent.*

(1) *mR is not δ-small in M.*
(2) *There is a maximal submodule N of M such that $m \notin N$ and M/N is singular.*

Lemma 2.3. *Let M be a module and K, L, H submodules of M. If L is a δ-supplement of K in M and K is a δ-supplement of H in M, then K is a δ-supplement of L in M.*

Proof. By assumption $M = K + L = K + H$, $K \cap L$ is δ-small in L and $K \cap H$ is δ-small in K. We prove $K \cap L$ is δ-small in K. Let X be a submodule of M such that $(K \cap L) + X = K$ and K/X is singular. Then $M = (K \cap L) + X + H$. Since $K \cap L$ is δ-small in M, by Lemma 2.1(1), there exists a projective semisimple submodule Y in $K \cap L$ such that $M = Y \oplus (X + H)$. Hence $K = (Y \oplus X) + (K \cap H)$. Since $K/(X + Y)$ is singular as a homomorphic image of K/X and $K \cap H$ is δ-small in K, $K = X \oplus Y$. Thus $Y = 0$ as K/X is singular and Y is projective semisimple. ∎

Lemma 2.4. *Let M be a module and K, N, T submodules of M. If K is a $\oplus$-δ-supplement of N in M and T is δ-small in M, then K is a $\oplus$-δ-supplement of $N + T$ in M.*

Proof. Let K be a $\oplus$-δ-supplement of N in M. Then K is a direct summand of M such that $M = N + K$ and $N \cap K$ is δ-small in K. We prove $(N+T) \cap K$ is δ-small in K. For if $[(N+T) \cap K] + L = K$ and K/L is singular for some $L \leq K$, then $M = L + N + T$ and $M/(L+N) = (K+N)/(L+N) \cong K/(K+(L \cap N))$ is singular as a homomorphic image of K/L. Since T is δ-small in M, $M = L + N$. Hence $K = L + (K \cap N)$. Since $K \cap N$ is δ-small in K and K/L is singular, we have $K = L$. ∎

3. Principally $\oplus$-δ-supplemented modules

In this section we define principally $\oplus$-δ-supplemented modules. We study properties, characterizations and decompositions of principally $\oplus$-δ-supplemented modules. We investigate the conditions under which any factor modules, direct summands and direct sums of a principally $\oplus$-δ-supplemented module are principally $\oplus$-δ-supplemented. For modules over a semisimple ring R we obtain that every R-module is principally $\oplus$-δ-supplemented if and only if every R-module is principally δ-semiperfect. Principally $\oplus$-supplemented modules are investigated in [14] and principally δ-lifting modules are studied in [6]. Recently, principally δ-supplemented modules are done in [7]. In this vein we introduce principally $\oplus$-δ-supplemented modules generalizing principally $\oplus$-supplemented modules, principally δ-lifting modules and strengthening principally δ-supplemented modules. Now we define principally $\oplus$-δ-supplemented modules with the next lemma.

Lemma 3.1. *Let M be a module, $m \in M$ and L a direct summand of M. Then the following are equivalent.*

(1) *$M = mR + L$ and $mR \cap L$ is δ-small in L.*
(2) *$M = mR + L$ and for any proper submodule K of L with L/K singular, $M \neq mR + K$.*

Proof. $(1) \Rightarrow (2)$ Let $K \leq L$ and $M = mR + K$ where L/K is singular. Then $L = (L \cap mR) + K$. Since $L \cap mR$ is δ-small in L, we have $L = K$.

$(2) \Rightarrow (1)$ Let $M = mR + L$ and $K \leq L$ and L/K singular with $L = (mR \cap L) + K$. Then $M = mR + L = mR + K$. By (2), $K = L$. So $mR \cap L$ is δ-small in L. ∎

Let M be a module and $m \in M$. A submodule L is called a *principally $\oplus$-δ-supplement* of mR in M if mR and L satisfy Lemma 3.1 and the module M is called *principally $\oplus$-δ-supplemented* if every cyclic submodule of M has a principally $\oplus$-δ-supplement in M, that is, for each $m \in M$ there exists a submodule A of M such that $M = mR + A = B \oplus A$ for some $B \leq M$ with $mR \cap A$ δ-small in A, therefore in M. In [6], a module M is called *principally δ-lifting* if for each $m \in M$, M has a decomposition $M = A \oplus B$ with $A \leq mR$ and $mR \cap B$ δ-small in B (equivalently, in M). Every principally δ-lifting module is a principally $\oplus$-δ-supplemented module. Principally $\oplus$-supplemented modules are introduced and investigated in [14]. The module M is called *principally $\oplus$-supplemented* if every cyclic submodule has a supplement which is a direct summand of M. Hence every principally $\oplus$-supplemented module is also principally $\oplus$-δ-supplemented. In [7], M is said to be a *principally δ-supplemented module* if for every cyclic submodule of M has a δ-supplement in M. Note that, every principally $\oplus$-δ-supplemented module is principally δ-supplemented. We show that the class of principally $\oplus$-δ-supplemented modules lies strictly between classes of principally $\oplus$-supplemented modules (principally δ-lifting modules) and principally δ-supplemented modules.

In the same direction as preceding paragraph one may define principally δ-$\oplus$-supplemented modules. A module M is called *principally δ-$\oplus$-supplemented* if for every cyclic submodule mR of M, M has a direct summand which is a δ-supplement of mR in M, that is, for any $m \in M$ there exists a direct summand A of M such that $M = mR + A$ and $mR \cap A$ is δ-small in A. So a principally δ-$\oplus$-supplemented module is the same as a principally $\oplus$-δ-supplemented module.

Example 3.1.

(1) Let R be an incomplete rank one discrete valuation ring, with quotient field K. By [10, Lemma A.5], the module $M = K \oplus K$ is principally $\oplus$-δ-supplemented but not lifting.

(2) Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus (\mathbb{Z}/2\mathbb{Z})$. We prove M is a principally $\oplus$-δ-supplemented module but neither supplemented nor lifting. It is routine to show that $M = (1,\overline{1})\mathbb{Z} + (\mathbb{Q} \oplus (\overline{0}))$. Let $(u,\overline{v}) \in M$. Assume that $\overline{v} = \overline{1}$ and $u \neq 1$. In this case we prove $M = (u,\overline{v})\mathbb{Z} + (\mathbb{Q} \oplus (\overline{0}))$. Let $(x,\overline{y}) \in M$. We have two possibilities.
 (i) $\overline{y} = \overline{1}$. Then $(x,\overline{y}) = (x,\overline{1}) = (u,\overline{1}) + (x-u,\overline{0}) \in (u,\overline{1})\mathbb{Z} + (\mathbb{Q} \oplus (\overline{0}))$.
 (ii) $\overline{y} = \overline{0}$. Then $(x,\overline{y}) = (x,\overline{0}) = (u,\overline{1})0 + (x,\overline{0}) \in (u,\overline{1})\mathbb{Z} + (\mathbb{Q} \oplus (\overline{0}))$.
Hence $M = (u,\overline{1})\mathbb{Z} + (\mathbb{Q} \oplus (\overline{0}))$. Since $((u,\overline{v})\mathbb{Z}) \cap (\mathbb{Q} \oplus (\overline{0}))$ is either zero or isomorphic to $\mathbb{Z} \oplus (\overline{0})$ which is small in $\mathbb{Q} \oplus (\overline{0})$, M is principally $\oplus$-δ-supplemented $\mathbb{Z}$-module. If M were supplemented $\mathbb{Z}$-module, its direct summand $\mathbb{Q}$ would be supplemented $\mathbb{Z}$-module. A contradiction. So M is neither supplemented nor lifting.

Recall that a submodule N of a module of M is called *fully invariant* if $f(N) \leq N$ for all endomorphisms f of M, and M is said to be a *duo module* (or *weak-duo*) if every submodule (or direct summand) of M is fully invariant (see for detail [12]). The module M is called *distributive* if for all submodules K, L and N of M, $N \cap (K+L) = (N \cap K) + (N \cap L)$ or $N + (K \cap L) = (N+K) \cap (N+L)$. Lemma 3.2 is well known and it is obvious from definitions.

Lemma 3.2. *Let $M = M_1 \oplus M_2 = K + N$ and $K \leq M_1$. If M is distributive and $K \cap N$ is δ-small in N, then $K \cap N$ is δ-small in $M_1 \cap N$.*

Recall the definitions for some of the terms to be used in the sequel. An R-module M is said to be *π-projective* if for every two submodules U, V of M with $U + V = M$ there exists $f \in \mathrm{End}_R(M)$ with $\mathrm{Im}(f) \leq U$ and $\mathrm{Im}(1-f) \leq V$ and M is called *refinable* if for any submodules U and V of M with $M = U + V$ there is a direct summand U' of M such that $U' \subseteq U$ and $M = U' + V$ (see, namely [16]). The module M has *the summand intersection property* if the intersection of two direct summands of M is again a direct summand of M.

Theorem 3.1. *Every principally δ-lifting module is principally $\oplus$-δ-supplemented. The converse holds if M satisfies any of the following conditions.*

(1) *M is a distributive module.*
(2) *M is a π-projective module.*
(3) *M is a duo module.*
(4) *M is a refinable module with the summand intersection property.*
(5) *M is an indecomposable module.*

Proof. Let M be a principally δ-lifting module and $m \in M$. Then M has a decomposition $M = A \oplus B$ such that $B \leq mR$ and $mR \cap A$ is δ-small in A. Since $M = mR + A$, M is principally $\oplus$-δ-supplemented. Conversely,

(1) Let M be a distributive principally $\oplus$-δ-supplemented module and $m \in M$. There exists a direct summand A of M such that $M = mR + A$ with $mR \cap A$ δ-small in A. Let $M = A \oplus B$ for some submodule B of M. Then by distributivity of M, we have $mR = (mR \cap A) \oplus (mR \cap B)$. Hence $M = (mR \cap B) \oplus A$. Thus $B = mR \cap B \leq mR$. Therefore M is principally δ-lifting.

(2) Let M be a π-projective principally $\oplus$-δ-supplemented module and $m \in M$. Then we have $M = mR + A$ and $mR \cap A$ is δ-small in A for some direct summand A of M. Since M is π-projective, by [15, 41.14], there exists $N \leq mR$ with $M = A \oplus N$. Therefore M is principally δ-lifting.

(3) Similar to the case (1).

(4) Let M be a refinable principally $\oplus$-δ-supplemented module with the summand intersection property and $m \in M$. Then there exists a direct summand A of M such that $M = mR + A$ and $mR \cap A$ is δ-small in A. Since M is refinable, there exists a direct summand U of M such that U is contained in mR and $M = U + A$. By the summand intersection property of M, $U \cap A$ is a direct summand of M. Let $M = (U \cap A) \oplus K$ for some submodule K of M. Then $A = (U \cap A) \oplus (K \cap A)$, and so $M = U \oplus (K \cap A)$. On the other hand, $mR \cap (K \cap A)$ is δ-small in A. Since $K \cap A$ is a direct summand of A, $mR \cap (K \cap A)$ is also δ-small in $K \cap A$. This completes the proof.

(5) Let M be an indecomposable module and $m \in M$. Since M is principally $\oplus$-δ-supplemented, there exist submodules A and B of M such that $mR \cap A$ is δ-small in A and $M = A \oplus B = mR + A$. By hypothesis, $A = M$ and $B = 0$. So that $mR \cap A = mR$ is δ-small in M. Note that in this case, every cyclic submodule of M is δ-small in M. ∎

Next example shows that there exists a principally $\oplus$-δ-supplemented module which is not principally δ-lifting.

Example 3.2. Consider the $\mathbb{Z}$-module $M = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$. Then $N_1 = (\bar{1}, \bar{2})\mathbb{Z}$, $N_2 = (\bar{1}, \bar{1})\mathbb{Z}$, $N_3 = (\bar{0}, \bar{2})\mathbb{Z}$, $N_4 = (\bar{0}, \bar{4})\mathbb{Z}$, $N_5 = (\bar{1}, \bar{4})\mathbb{Z}$, $N_6 = \mathbb{Z}/2\mathbb{Z}$ and $N_7 = \mathbb{Z}/8\mathbb{Z}$ are nonzero cyclic submodules of M. Hence $M = N_6 \oplus N_7 = N_2 \oplus N_5$ and N_3, N_4 are small submodules of M. Thus M is a principally $\oplus$-supplemented module and so principally $\oplus$-δ-supplemented. On the other hand, M is not principally δ-lifting, by [6].

Every principally $\oplus$-δ-supplemented module need not be principally $\oplus$-supplemented, as Example 3.8 shows. But in some cases these modules coincide.

Proposition 3.1. *Let M be a singular module. Then M is principally $\oplus$-supplemented if and only if it is principally $\oplus$-δ-supplemented.*

Proof. The necessity is clear. For the sufficiency, let $m \in M$. Then there exists a direct summand A of M with $M = mR + A$ and $mR \cap A$ δ-small in A. Assume that $A = (mR \cap A) + K$ for some submodule K of A. Since M is singular, A/K is also singular. Hence we have $A = K$. Thus $mR \cap A$ is small in A. Therefore M is principally $\oplus$-supplemented. ∎

Proposition 3.2. *Let M be a principally $\oplus$-δ-supplemented module. If every cyclic submodule of M has a uniform principally $\oplus$-δ-supplement, then M is principally $\oplus$-supplemented.*

Proof. Let $m \in M$. By hypothesis, there exists a uniform direct summand A of M with $M = mR + A$ and $mR \cap A$ δ-small in A. Assume that $(mR \cap A) + K = A$ for some submodule K of A. If $K = 0$, then there is nothing to do. Let $K \neq 0$. Since K is essential in A, A/K

is singular. Then we have $K = A$. Hence $mR \cap A$ is small in A. Thus M is principally $\oplus$-supplemented. ∎

Proposition 3.3. *Every principally $\oplus$-δ-supplemented module is principally δ-supplemented. The converse is true for refinable modules.*

Proof. The first assertion is clear. Let M be a principally δ-supplemented module and $m \in M$. Let A be a submodule of M with $M = mR + A$ and $mR \cap A$ δ-small in A. Since M is refinable, there is a direct summand U of M such that $U \subseteq A$ and $M = U + mR$. Also U is a direct summand of A. This implies that $mR \cap U$ is δ-small in A. Hence $mR \cap U$ is δ-small in U. ∎

Next example shows that there exists a principally δ-supplemented module which is not principally $\oplus$-δ-supplemented.

Example 3.3. Let F be a field and x and y commuting indeterminates over F. Consider the polynomial ring $R = F[x,y]$, the ideals $I_1 = (x^2)$ and $I_2 = (y^2)$ of R, and the ring $S = R/(x^2, y^2)$. Let $M = \bar{x}S + \bar{y}S$. Then M is an indecomposable S-module, principally supplemented but not principally $\oplus$-supplemented. Hence M is principally δ-supplemented. On the other hand, since M is singular, it is not principally $\oplus$-δ-supplemented by Proposition 3.1.

Because of the following example it can be said that any submodule of a principally $\oplus$-δ-supplemented module may not be principally $\oplus$-δ-supplemented.

Example 3.4. Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Since every cyclic submodule of $\mathbb{Q}$ is small and so δ-small in $\mathbb{Q}$, $\mathbb{Q}$ is principally $\oplus$-δ-supplemented. But the submodule $\mathbb{Z}$ of $\mathbb{Q}$ is not principally $\oplus$-δ-supplemented as a $\mathbb{Z}$-module since $2\mathbb{Z}$ does not have any principally $\oplus$-δ-supplement in $\mathbb{Z}$.

Now we investigate conditions which ensure that a homomorphic image and so a direct summand of a principally $\oplus$-δ-supplemented module is principally $\oplus$-δ-supplemented.

Theorem 3.2. *Let M be a distributive principally $\oplus$-δ-supplemented module. Then every homomorphic image of M is principally $\oplus$-δ-supplemented.*

Proof. Let L be a submodule of M and $(mR+L)/L$ a cyclic submodule of M/L. Then there exists a direct summand A of M such that $M = A \oplus B = mR + A$ for some $B \leq M$ and $mR \cap A$ is δ-small in A. Now $M/L = (mR+L)/L + (A+L)/L$ and, since M is distributive, $(mR+L) \cap (A+L) = L + (mR \cap A)$. So $((mR+L)/L) \cap ((A+L)/L) = (L + (mR \cap A))/L$ is δ-small in $(A+L)/L$ as a homomorphic image of δ-small $mR \cap A$ in A under the natural map π from A onto $(A+L)/L$ by Lemma 2.1(2). Again by distributivity of M and $A \cap B = 0$, we have $(A+L) \cap (B+L) = L$. Hence $(A+L)/L$ is a direct summand of M/L. ∎

Corollary 3.1. *Every direct summand of a distributive principally $\oplus$-δ-supplemented module is principally $\oplus$-δ-supplemented.*

Proposition 3.4. *Let M be a module and N a submodule of M. If every cyclic submodule of M has a principally $\oplus$-δ-supplement which contains N, then M/N is principally $\oplus$-δ-supplemented.*

Proof. Let $m \in M$ and consider the submodule $\overline{m}R$ of M/N. By hypothesis, there exists a direct summand L of M such that $N \leq L$, $M = mR + L$ and $mR \cap L$ is δ-small in L. Let

$M = K \oplus L$ for some submodule K of M and π denote the natural epimorphism from M onto M/N. Then we have $M/N = (K+N)/N \oplus (L/N) = \overline{m}R + (L/N)$. On the other hand, $\pi(mR \cap L) = \pi(mR) \cap \pi(L) = \overline{m}R \cap (L/N)$ is δ-small in $\pi(L) = L/N$. Hence the proof is completed. ∎

Lemma 3.3. *Let M be a module and N a fully invariant submodule of M. If $M = M_1 \oplus M_2$ for some submodules M_1 and M_2 of M, then $M/N = (M_1+N)/N \oplus (M_2+N)/N$.*

Proof. Clearly, $M/N = (M_1+N)/N + (M_2+N)/N$. If $m_1 + N = m_2 + N$ with $m_i \in M_i$ $(i = 1,2)$, then $m_1 - m_2 \in N$. As N is a fully invariant submodule of M, we see that $m_1, m_2 \in N$. Hence $(M_1+N)/N \cap (M_2+N)/N = 0$, as required. ∎

Proposition 3.5. *Let M be a principally $\oplus$-δ-supplemented module. Then M/N is principally $\oplus$-δ-supplemented for every fully invariant submodule N of M.*

Proof. Let N be a fully invariant submodule of M and $\overline{m}R$ a submodule of M/N, where $m \in M$. Since M is principally $\oplus$-δ-supplemented, there exists a direct summand A of M such that $M = mR + A$ and $mR \cap A$ is δ-small in A. Let $M = A \oplus B$ for some submodule B of M. By Lemma 3.3, we have $M/N = (A+N)/N \oplus (B+N)/N$. Also $M/N = (A+N)/N + \overline{m}R$. It is clear that $(A+N)/N \cap \overline{m}R$ is δ-small in $(A+N)/N$. This completes the proof. ∎

As an immediate consequence of Proposition 3.5, we deduce that if M is principally $\oplus$-δ-supplemented, then so are $M/\mathrm{Rad}(M)$ and $M/\mathrm{Soc}(M)$.

Corollary 3.2. *Let M be a weak-duo and principally $\oplus$-δ-supplemented module. Then every direct summand of M is principally $\oplus$-δ-supplemented.*

Recall that a module M has D_3 if whenever M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, $M_1 \cap M_2$ is also a direct summand of M [10].

Proposition 3.6. *Let M be a principally $\oplus$-δ-supplemented module. If M has D_3, then every direct summand of M is also principally $\oplus$-δ-supplemented.*

Proof. Let N be a direct summand of M and $n \in N$. Since M is principally $\oplus$-δ-supplemented, there exists a direct summand A of M with $M = A + nR$ and $A \cap nR$ δ-small in A. Hence $M = A + N$ and $N = (A \cap N) + nR$. Due to D_3, $A \cap N$ is a direct summand of M, N and A. By Lemma 2.1(3), $(A \cap N) \cap nR$ is δ-small in $A \cap N$ because $A \cap N$ is a direct summand of A. Thus N is principally $\oplus$-δ-supplemented. ∎

Due to Proposition 3.6 and [5, Lemma 2.4] we obtain the following result.

Corollary 3.3. *Let M be a principally $\oplus$-δ-supplemented and UC extending module. Then every direct summand of M is principally $\oplus$-δ-supplemented.*

It is obvious that every module with the summand intersection property has D_3. Then the following result is an immediate consequence of Proposition 3.6 and [4, Theorem 4.6].

Corollary 3.4. *Let R be a right semihereditary ring and F a principally $\oplus$-δ-supplemented finitely generated free R-module. Then R is principally $\oplus$-δ-supplemented as an R-module.*

Next example shows that for a module M and a submodule N, if M/N is principally $\oplus$-δ-supplemented, then M need not be principally $\oplus$-δ-supplemented.

Example 3.5. Consider the $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z}$, where p is a prime number and n is a positive integer. Then $\mathbb{Z}/p^n\mathbb{Z}$ is principally δ-lifting and so principally $\oplus$-δ-supplemented, but $\mathbb{Z}$ is not principally $\oplus$-δ-supplemented.

Proposition 3.7. *Let $M = M_1 \oplus M_2$ be a distributive module. Then M is principally $\oplus$-δ-supplemented if and only if M_1 and M_2 are principally $\oplus$-δ-supplemented.*

Proof. Let M be a principally $\oplus$-δ-supplemented module. Due to Corollary 3.1, M_1 and M_2 are principally $\oplus$-δ-supplemented. Assume that M_1 and M_2 are principally $\oplus$-δ-supplemented modules and $m \in M$. By distributivity of M, we have $mR = (mR \cap M_1) \oplus (mR \cap M_2)$. Since $mR \cap M_1$ and $mR \cap M_2$ are cyclic submodules of M_1 and M_2 respectively, there exist direct summands A of M_1 and B of M_2 such that $M_1 = (mR \cap M_1) + A = A' \oplus A$ and $A \cap (mR \cap M_1) = A \cap mR$ is δ-small in A, and $M_2 = (mR \cap M_2) + B = B' \oplus B$ and $B \cap (mR \cap M_2) = B \cap mR$ is δ-small in B. Then $M = mR + A + B = (A' \oplus B') \oplus (A \oplus B)$. Again by distributivity, $mR \cap (A+B) = (mR \cap A) + (mR \cap B)$ is δ-small in $A+B$ by Lemma 2.1(3). This completes the proof. ∎

Proposition 3.8. *Let $M = M_1 \oplus M_2$ be a duo module. Then M is principally $\oplus$-δ-supplemented if and only if M_1 and M_2 are principally $\oplus$-δ-supplemented.*

Proof. Necessity is clear from Proposition 3.6 because duo modules satisfy the summand intersection property. Sufficiency is resemble to the proof of Proposition 3.7. ∎

Corollary 3.5. *Let M be a principally $\oplus$-δ-supplemented module and every finite direct sum of M a distributive (or duo) module. Then every finitely M-generated module is principally $\oplus$-δ-supplemented.*

Recall that a module M is called *regular* (in the sense of Zelmanowitz) [17] if for any $m \in M$ there exists a map $\alpha \in \mathrm{Hom}_R(M,R)$ such that $m = m\alpha(m)$ and it is known that every cyclic submodule of a regular module is a direct summand. Hence any regular module is principally $\oplus$-δ-supplemented. We give an example to show that principally $\oplus$-δ-supplemented modules need not be a regular module.

Example 3.6. Any cyclic submodule of $\mathbb{Q}$ as a $\mathbb{Z}$-module is a small submodule of $\mathbb{Q}$. Therefore $\mathbb{Q}$ is a principally $\oplus$-δ-supplemented $\mathbb{Z}$-module. On the other hand, $\mathbb{Q}$ can not be a regular $\mathbb{Z}$-module since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = 0$.

A module M is said to be *principally semisimple* if every cyclic submodule is a direct summand of M. Tuganbaev calls a principally semisimple module as a regular module in [13], and lifting modules are named as semiregular modules. Every semisimple module is principally semisimple. Every principally semisimple module is principally δ-lifting and so principally $\oplus$-δ-supplemented. A ring R is called *principally semisimple* if the right R-module R is principally semisimple. It is clear that every principally semisimple ring is von Neumann regular and vice versa. For a module M, we write $\mathrm{Rad}_\delta(M) = \sum\{L \mid L \text{ is a } \delta\text{-small submodule of } M\}$. Since every small submodule of M is δ-small, $\mathrm{Rad}(M) \leq \mathrm{Rad}_\delta(M)$. In the ring case, we shall denote $\mathrm{Rad}_\delta(M)$ by $\mathrm{J}_\delta(R)$ and usually $\mathrm{Rad}(M)$ by $\mathrm{J}(R)$ for a ring R. It is shown that $\mathrm{J}_\delta(R)$ is an ideal of R, and there are cases for a ring R such that $\mathrm{J}_\delta(R)$ strictly contains $\mathrm{J}(R)$ (see namely [18]). Also note that for any module M, $\mathrm{Rad}_\delta(M)$ is a δ-small submodule of M provided every proper submodule of M is contained in a maximal submodule of M, therefore $\mathrm{J}_\delta(R)$ is a δ-small right and δ-small left ideal of R.

Lemma 3.4. [10, Lemma 4.47] *Let $M = S \oplus T = N + T$ where S is T-projective. Then $M = S' \oplus T$ where $S' \leq N$.*

Lemma 3.5. *Let M be a principally $\oplus$-δ-supplemented module. Then $M/\mathrm{Rad}_\delta(M)$ is a principally semisimple module if M has one of the following conditions.*

(1) *M is a distributive module.*
(2) *M is a projective module.*

Proof.

(1) For any $m \in M$, there exists a direct summand A of M such that $M = mR + A$ and $mR \cap A$ is δ-small in A. So $mR \cap A$ is δ-small in M. By distributivity of M, we have $(mR + \mathrm{Rad}_\delta(M)) \cap (A + \mathrm{Rad}_\delta(M)) = \mathrm{Rad}_\delta(M) + (mR \cap A) = \mathrm{Rad}_\delta(M)$ since $mR \cap A$ is δ-small in M. Then

$$M/\mathrm{Rad}_\delta(M) = [(mR + \mathrm{Rad}_\delta(M))/\mathrm{Rad}_\delta(M)] \oplus [(A + \mathrm{Rad}_\delta(M))/\mathrm{Rad}_\delta(M)].$$

(2) Let $m \in M$. There exists a direct summand A of M such that $M = mR + A$ and $mR \cap A$ is δ-small in A. So $mR \cap A$ is δ-small in M. By projectivity of M, there exists a direct summand N of M such that $M = N \oplus A$ with $N \leq mR$ by Lemma 3.4. Then $(mR + \mathrm{Rad}_\delta(M))/\mathrm{Rad}_\delta(M) = (N + \mathrm{Rad}_\delta(M))/\mathrm{Rad}_\delta(M)$ and $\mathrm{Rad}_\delta(M) = \mathrm{Rad}_\delta(N) \oplus \mathrm{Rad}_\delta(A)$ imply

$$M/\mathrm{Rad}_\delta(M) = [(mR + \mathrm{Rad}_\delta(M))/\mathrm{Rad}_\delta(M)] \oplus [(A + \mathrm{Rad}_\delta(M))/\mathrm{Rad}_\delta(M)].$$

Hence every principal submodule of $M/\mathrm{Rad}_\delta(M)$ is a direct summand in either case. Therefore $M/\mathrm{Rad}_\delta(M)$ is principally semisimple. ∎

Proposition 3.9. *Let M be a principally $\oplus$-δ-supplemented module and N a submodule of M. If $N \cap Rad_\delta(M) = 0$, then N is principally semisimple.*

Proof. Let $x \in N$. By hypothesis, there exists a direct summand A of M with $M = A + xR$ and $A \cap xR$ δ-small in A. Hence $N = (A \cap N) + xR$ and $A \cap xR \leq \mathrm{Rad}_\delta(M)$. Since $(A \cap N) \cap xR \leq N \cap \mathrm{Rad}_\delta(M) = 0$, we have $N = (A \cap N) \oplus xR$. Therefore N is principally semisimple. ∎

Theorem 3.3 may be proved easily by making use of Lemma 3.5 for distributive modules. But we prove it in another way.

Theorem 3.3. *Let M be a principally $\oplus$-δ-supplemented module. Then M has a principally semisimple submodule M_1 such that M_1 has an essential socle and $Rad_\delta(M) \oplus M_1$ is essential in M.*

Proof. By Zorn's Lemma we may find a submodule M_1 of M such that $\mathrm{Rad}_\delta(M) \oplus M_1$ is essential in M. By Proposition 3.9, M_1 is principally semisimple. Next we show that M_1 has an essential socle. For this we prove for any $m \in M_1$, mR has a simple submodule. If mR is simple, we have done. Otherwise let $m_1 \in mR$ such that $m_1R \neq mR$. By hypothesis there exists a direct summand C of M such that $M = m_1R + C$ with $m_1R \cap C$ δ-small in C. Then $m_1R \cap C \leq M_1 \cap \mathrm{Rad}_\delta(M) = 0$. So $M = m_1R \oplus C$ and then $mR = m_1R \oplus (mR \cap C)$. Clearly, $mR \cap C = m_1'R$ for some $m_1' \in mR$ and $mR = m_1R \oplus m_1'R$. If m_1R and $m_1'R$ are simple, then we stop. Otherwise let $m_2 \in m_1R$ such that $m_2R \neq m_1R$. Similarly, there is $m_2' \in m_1R$ such that $m_1R = m_2R \oplus m_2'R$. Hence $mR = m_2R \oplus m_2'R \oplus m_1'R$. If m_2R is simple, then we stop. Otherwise we continue in this way. Since mR is cyclic, this process must terminate at a finite step, say n. At this step all direct summands of mR should be simple. This completes the proof. ∎

Theorem 3.4. *Let M be a principally $\oplus$-δ-supplemented module. Assume that M satisfies ascending chain condition on direct summands. Then M has a decomposition $M = M_1 \oplus M_2$, where M_1 is a semisimple module and M_2 is a module with* $\mathrm{Rad}_\delta(M_2)$ *essential in M_2.*

Proof. Let M_1 be a submodule of M such that $\mathrm{Rad}_\delta(M)\oplus M_1$ is essential in M and $m_1 \in M_1$. By Proposition 3.9, M_1 is principally semisimple. Since M is principally $\oplus$-δ-supplemented, there exists a direct summand A_1 of M such that $M = m_1R + A_1$ and $m_1R \cap A_1$ is δ-small in both A_1 and M. Hence $m_1R \cap A_1 = 0$ and $M = m_1R \oplus A_1$. Then $M_1 = m_1R \oplus (M_1 \cap A_1)$. If $M_1 \cap A_1 \neq 0$, let $0 \neq m_2 \in M_1 \cap A_1$. There exists a direct summand A_2 of M such that $M = m_2R + A_2$ and $m_2R \cap A_2$ is δ-small in both A_2 and M. Hence $m_2R \cap A_2 = 0$, $M = m_2R \oplus A_2 = m_1R \oplus m_2R \oplus (A_1 \cap A_2)$. So $M_1 \cap A_1 = m_2R \oplus (M_1 \cap A_1 \cap A_2)$ and $M_1 = m_1R \oplus (M_1 \cap A_1) = m_1R \oplus m_2R \oplus (M_1 \cap A_1 \cap A_2)$. If $M_1 \cap A_1 \cap A_2 \neq 0$, let $0 \neq m_3 \in M_1 \cap A_1 \cap A_2$. There exists a direct summand A_3 of M such that $M = m_3R \oplus A_3 = m_1R \oplus m_2R \oplus m_3R \oplus (A_1 \cap A_2 \cap A_3)$ and $M_1 \cap A_1 \cap A_2 = m_3R \oplus (M_1 \cap A_1 \cap A_2 \cap A_3)$ and $M_1 = m_1R \oplus m_2R \oplus m_3R \oplus (M_1 \cap A_1 \cap A_2 \cap A_3)$. By hypothesis this procedure stops at a finite number of steps, say t. At this stage we may have $M = m_tR \oplus A_t = m_1R \oplus m_2R \oplus m_3R \oplus \cdots \oplus m_tR \oplus (A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_t)$ and $M_1 = m_1R \oplus m_2R \oplus m_3R \oplus \cdots \oplus m_tR$. Let $M_2 = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_t$. Then $M = M_1 \oplus M_2$ with $\mathrm{Rad}_\delta(M) = \mathrm{Rad}_\delta(M_2)$. Since $M_1\oplus$ $\mathrm{Rad}_\delta(M)$ is essential in M, it follows that $\mathrm{Rad}_\delta(M_2)$ is essential in M_2. Since M has the ascending chain condition on direct summands, without loss of generality, we may assume that all cyclic submodules m_1R, m_2R, m_3R, ..., m_tR to be simple. This completes the proof. ∎

Theorem 3.5. *Let M be a module with $Rad_\delta(M) = 0$. Then the following conditions are equivalent.*

(1) *M is principally $\oplus$-δ-supplemented.*
(2) *M is principally $\oplus$-supplemented*
(3) *M is principally semisimple.*

Proof. We prove only (1) $\Rightarrow$ (3) since (2) $\Leftrightarrow$ (3) is proved in [14] and (3) $\Rightarrow$ (1) is clear. Let M be a principally $\oplus$-δ-supplemented module and $m \in M$. There exists a direct summand A of M such that $M = mR + A$ and $mR \cap A$ is δ-small in A. Since $mR \cap A$ is also δ-small in M and $\mathrm{Rad}_\delta(M) = 0$, mR is a direct summand of M. Therefore M is principally semisimple. ∎

It is known that every von Neumann regular ring has zero Jacobson radical. But there are von Neumann regular rings R with $J_\delta(R) \neq 0$ as the following example shows.

Example 3.7. Let $Q = \prod_{i=1}^{\infty} F_i$, where each $F_i = \mathbb{Z}_2$. Let R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Then R is von Neumann regular and $\bigoplus_{i=1}^{\infty} F_i = Soc(R) = J_\delta(R)$.

Corollary 3.6. *Let R be a ring. If R a is von Neumann regular ring, then R is a principally $\oplus$-δ-supplemented R-module. The converse holds if $J_\delta(R) = 0$.*

Definition 3.1. *Let M be a module. M is called a δ-hollow module (or a principally δ-hollow module) if every proper submodule (or cyclic submodule) is δ-small in M.*

Note that each hollow module is δ-hollow, and each δ-hollow module is principally δ-hollow and so principally $\oplus$-δ-supplemented. Let M be a module. Clearly, if $M = xR$ for every $x \in M \backslash \mathrm{Rad}_\delta(M)$, then M is principally δ-hollow.

Theorem 3.6. *Let M be a projective module having $Rad_\delta(M)$ finite uniform dimension. Consider the following statements.*

(1) *M is a direct sum of principally $\oplus$-δ-supplemented modules.*
(2) *M has a decomposition $M = M_1 \oplus M_2$ where M_1 is a direct sum of principally semisimple modules and M_2 is a finite direct sum of principally δ-hollow modules.*

Then (2) $\Rightarrow$ (1). (1) $\Rightarrow$ (2) *in case M satisfies ascending chain condition on direct summands.*

Proof. (2) $\Rightarrow$ (1) Assume that M has a decomposition $M = M_1 \oplus M_2$ with submodules M_1 and M_2 satisfying stated conditions in (2). Both M_1 and M_2 are direct sums of principally $\oplus$-δ-supplemented modules as M_1 is a direct sum of principally semisimple modules, and M_2 is a direct sum of principally δ-hollow modules and each principally δ-hollow module is principally $\oplus$-δ-supplemented.

(1) $\Rightarrow$ (2) Assume that $M = \bigoplus_{i\in I} M_i$, where each M_i is a principally $\oplus$-δ-supplemented module and $\mathrm{Rad}_\delta(M)$ has finite uniform dimension. Since $\mathrm{Rad}_\delta(M) = \bigoplus_{i\in I}\mathrm{Rad}_\delta(M_i)$, there is a finite subset J of I with $\mathrm{Rad}_\delta(M_i) = 0$ for all $i \in I \setminus J$. Therefore, by Theorem 3.5, M_i is principally semisimple for all $i \in I \setminus J$. Hence $M = M_1 \oplus \left(\bigoplus_{j\in J} M_j\right)$, where M_1 is a direct sum of principally semisimple modules. Due to Theorem 3.4, without loss of generality, we may assume that $\mathrm{Rad}_\delta(M_j)$ is essential in M_j, where $j \in J$. Then for $j \in J$, M_j has finite uniform dimension by [3, Proposition 3.20]. Now we prove each M_j is principally δ-hollow or a finite direct sum of principally δ-hollow modules, for $j \in J$. Let $j \in J$. Since M is projective, M_j is also projective. Then $\mathrm{Rad}_\delta(M_j) \neq M_j$ by [18, Lemma 1.9]. We complete the proof by induction on the uniform dimension. Suppose that M_j has uniform dimension 1, and let $x \in M_j \setminus \mathrm{Rad}_\delta(M_j)$. Since M_j is principally $\oplus$-δ-supplemented, there exists a direct summand K of M_j such that $M_j = xR + K$ and $xR \cap K$ is δ-small in K. Let $M_j = K \oplus K_1$ for some submodule K_1 of M_j. Since M_j has uniform dimension 1, we have $K = 0$ or $K_1 = 0$. If $K_1 = 0$, then xR is a submodule of $\mathrm{Rad}_\delta(M_j)$. This is a contradiction. Hence $K = 0$ and so $M_j = xR$. It follows that M_j is principally δ-hollow. Now suppose that $n > 1$ be a positive integer and assume each M_j having uniform dimension $k (1 \leq k < n)$ is principally δ-hollow or a finite direct sum of principally δ-hollow submodules. Let $j \in J$ and assume M_j has uniform dimension n. Suppose M_j is not principally δ-hollow. Let $x \in M_j \setminus \mathrm{Rad}_\delta(M_j)$ such that $M_j \neq xR$. Since M_j is principally $\oplus$-δ-supplemented, there exist submodules K, K_1 of M_j with $M_j = xR + K = K \oplus K_1$ and $xR \cap K$ δ-small in K. Note that $K_1 \neq 0$ and $K \neq 0$. Since projective modules have D_3 and then by Proposition 3.6, K and K_1 are principally $\oplus$-δ-supplemented modules by induction, K and K_1 are principally δ-hollow or a finite direct sum of principally δ-hollow submodules. So (1) $\Rightarrow$ (2) holds and this completes the proof. ∎

One may ask what happens to Theorem 3.5 in which the condition "$\mathrm{Rad}_\delta(M) = 0$" changes to "$\mathrm{Rad}_\delta(M)$ is δ-small in M".

Theorem 3.7. *Let M be a projective module with $\mathrm{Rad}_\delta(M)$ δ-small in M and consider the following conditions.*

(1) *M is principally $\oplus$-δ-supplemented.*

(2) $M/\mathrm{Rad}_\delta(M)$ *is principally semisimple.*

Then (1) $\Rightarrow$ (2). *If M is a refinable module, then* (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Since M is a principally $\oplus$-δ-supplemented module, $M/\mathrm{Rad}_\delta(M)$ is principally semisimple by Lemma 3.5.

(2) $\Rightarrow$ (1) Let mR be any cyclic submodule of M. By (2), there exists a submodule U of M such that $M/\mathrm{Rad}_\delta(M) = [(mR+\mathrm{Rad}_\delta(M))/\mathrm{Rad}_\delta(M)] \oplus [U/\mathrm{Rad}_\delta(M)]$. Then $M = mR + U$ and $(mR+\mathrm{Rad}_\delta(M)) \cap U = (mR \cap U) + \mathrm{Rad}_\delta(M) = \mathrm{Rad}_\delta(M)$. Hence $mR \cap U \leq \mathrm{Rad}_\delta(M)$ and it is δ-small in M. Since $M = mR + U$ and being M refinable, there exists a direct

summand A of M such that $A \leq U$ and $M = mR + A$. Since $mR \cap A \leq mR \cap U$ is δ-small in M and A is a direct summand of M, by Lemma 2.1(3), $mR \cap A$ is δ-small in A. Hence A is a principally $\oplus$-δ-supplement of mR in M. This completes the proof. ∎

Recall that R is called a *right V-ring* if every simple right R-module is injective, equivalently, by [9, Theorem 3.75], for any right R-module M, $\text{Rad}(M) = 0$. In this note we shall call the ring R is a *right δ-V-ring* if for any right R-module M, $\text{Rad}_\delta(M) = 0$. Since every small submodule is δ-small, $\text{Rad}(M) \leq \text{Rad}_\delta(M)$ for any module M.

We adopt the definition of a small projective module in [15, 19.10(8)] and we say an R-module M *δ-small projective* if $\text{Hom}(M, -)$ is exact with respect to the exact sequences of right R-modules $0 \to K \xrightarrow{i} L \to N \to 0$ with $i(K)$ a δ-small submodule of L. If R is a δ-V-ring, then every module is δ-small projective. In a subsequent paper the present authors study δ-small projective modules in detail. As is usual, to study δ-V-rings it is convenient to deal with an injective notion. A module M is called *δ-small injective* if $\text{Hom}(-, M)$ is exact with respect to the exact sequences of right R-modules $0 \to K \xrightarrow{i} L \to N \to 0$ with $i(K)$ a δ-small submodule of L. Clearly for a R right δ-V-ring, every right R-module is both δ-small projective and δ-small injective.

Lemma 3.6. *Let R be a ring and consider the following conditions.*

(1) *R is a right δ-V-ring.*
(2) *Every right R-module is δ-small projective.*
(3) *Every right R-module is δ-small injective.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Clear. (2) $\Rightarrow$ (3) Let M be a right R-module and an exact sequence of right R-modules with $i(K)$ a δ-small submodule of L

$$0 \to K \xrightarrow{i} L \xrightarrow{f} N \to 0 \tag{3.1}$$

Applying $\text{Hom}(N, -)$ to that sequence, by (2) we have an exact sequence

$$0 \to \text{Hom}(N,K) \xrightarrow{i^*} \text{Hom}(N,L) \xrightarrow{f^*} \text{Hom}(N,N) \to 0$$

For the identity map $1 \in \text{Hom}(N,N)$ we have a map $g \in \text{Hom}(N,L)$ such that $1 = f^* g$. Hence the sequence (3.1) splits and so any map from K to M extends from L to M. (3) $\Rightarrow$ (2) Dual to (2) $\Rightarrow$ (3). ∎

Theorem 3.8. *Let R be a right V-ring. If every right R-module is δ-small projective, then every principally $\oplus$-δ-supplemented module is a direct sum of a projective semisimple module and a principally semisimple module.*

Proof. Let R be a right V-ring and M any right R-module. We have $\text{Rad}(M) = 0$. By [2, Proposition 3.1] or [9, Theorem 3.75] every submodule of M is contained in a maximal submodule, and [18, Lemma 1.5(4)] implies $\text{Rad}_\delta(M)$ is δ-small in M. Since every right R-module is δ-small projective, we apply the functor $\text{Hom}(M/\text{Rad}_\delta(M), -)$ to the sequence $0 \to \text{Rad}_\delta(M) \to M \to M/\text{Rad}_\delta(M) \to 0$ we have $M = \text{Rad}_\delta(M) \oplus K$ for some submodule K of M. By Lemma 2.1(1), there exists a projective semisimple submodule Y of $\text{Rad}_\delta(M)$ such that $M = Y \oplus K$. Hence $Y = \text{Rad}_\delta(M)$. Due to Proposition 3.9, K is principally semisimple and this completes the proof. ∎

A ring R is called *δ-semiregular* if every cyclically presented R-module has a projective δ-cover. By combining Lemma 3.5, Theorem 3.5 and Theorem 3.8 we obtain the next result.

Theorem 3.9. *Let R be a right δ-V-ring and consider the following conditions.*

(1) *Every right R-module is principally $\oplus$-δ-supplemented.*
(2) *Every right R-module is principally $\oplus$-supplemented.*
(3) *Every right R-module is principally semisimple.*
(4) *R is von Neumann regular.*
(5) *Every projective R-module is principally $\oplus$-δ-supplemented.*
(6) *R is δ-semiregular.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) and (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6).

Proof. (4) $\Rightarrow$ (5) Let M be a projective right R-module. By [13, Proposition 1.25], M is principally semisimple. This implies that M is principally $\oplus$-δ-supplemented.

(5) $\Rightarrow$ (4) Since R is projective as a right R-module, R is principally $\oplus$-δ-supplemented. Being $J_\delta(R) = 0$, R is principally semisimple by Theorem 3.5. Hence R is von Neumann regular.

(4) $\Leftrightarrow$ (6) Clear by [18, Theorem 3.5] since $J_\delta(R) = 0$. ∎

Theorem 3.10. *Let R be a ring with $J_\delta(R) = 0$. Then the following are equivalent.*

(1) *Every projective R-module is principally $\oplus$-δ-supplemented.*
(2) *Every free R-module is principally $\oplus$-δ-supplemented.*
(3) *Every projective R-module is principally semisimple.*
(4) *Every free R-module is principally semisimple.*

Proof. (2) $\Rightarrow$ (1) Let every free R-module be principally $\oplus$-δ-supplemented and P a projective module. Then there exists a free module F such that P is a direct summand of F. By (2), F is principally $\oplus$-δ-supplemented with $\mathrm{Rad}_\delta(F) = 0$ since $\mathrm{J}_\delta(R) = 0$. Lemma 3.5 implies F is principally semisimple and then P is principally semisimple, therefore P is principally $\oplus$-δ-supplemented. The rest is clear. ∎

At the moment we have the following conjecture.

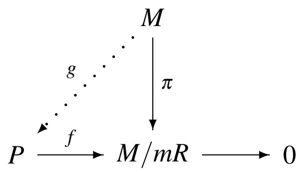
Conjecture 3.1. *Every right V-ring is right δ-V-ring.*

By [18], a projective module P is called *a projective δ-cover* of a module M if there exists an epimorphism $f : P \longrightarrow M$ with $Ker f$ δ-small in P, and a ring R is called *δ-perfect (δ-semiperfect)* if every R-module (simple R-module) has a projective δ-cover. Clearly, every δ-perfect ring is δ-semiperfect. A module M is said to be *principally δ-semiperfect* if every factor module of M by a cyclic submodule has a projective δ-cover. A ring R is called *principally δ-semiperfect* in case the right R-module R is principally δ-semiperfect. Every δ-semiperfect module is principally δ-semiperfect. Next we characterize projective principally $\oplus$-δ-supplemented modules.

Theorem 3.11. *Let M be a projective module. Then the following are equivalent.*

(1) *M is principally δ-semiperfect.*
(2) *M is principally $\oplus$-δ-supplemented.*

Proof. (1) ⇒ (2) Let $m \in M$ and $P \xrightarrow{f} M/mR$ be a projective δ-cover and $M \xrightarrow{\pi} M/mR$ the natural epimorphism.



Then there exists a map $M \xrightarrow{g} P$ such that $fg = \pi$. Hence $P = g(M) + \mathrm{Ker}f$. Since $\mathrm{Ker}f$ is δ-small, by Lemma 2.1(1), there exists a projective semisimple submodule Y of $\mathrm{Ker}f$ such that $P = g(M) \oplus Y$. So $g(M)$ is projective. Thus $M = K \oplus \mathrm{Ker}g$ for some submodule K of M. Let $x \in \mathrm{Ker}g$. Then $fg = \pi$ implies $\pi(x) = 0$. Hence $\mathrm{Ker}g \leq mR$. Next we show $g(K) \cap \mathrm{Ker}f = g(K \cap mR)$. Let $x \in K \cap mR$. Then $0 = \pi(x) = fg(x)$. So $x \in g^{-1}(\mathrm{Ker}f)$ and $K \cap mR \leq g^{-1}(\mathrm{Ker}f)$ and $K \cap mR \leq g^{-1}(\mathrm{Ker}f) \cap K$. Then $g(K \cap mR) \leq g(g^{-1}(\mathrm{Ker}f) \cap K) = \mathrm{Ker}f \cap g(K)$. Let $x \in \mathrm{Ker}f \cap g(K)$. There is $y \in K$ such that $g(y) = x$ and $f(x) = 0$. Then $\pi(y) = f(g(y)) = f(x) = 0$. So $y \in mR$ and $x = g(y) \in g(K \cap mR)$. Hence $g(K) \cap \mathrm{Ker}f = g(K \cap mR)$ and it is δ-small in P and therefore in $g(K)$. Since g is an isomorphism between K and $g(K)$, $g^{-1}(g(K) \cap \mathrm{Ker}f)$ is δ-small in K. Because $K \cap mR \leq g^{-1}(g(K) \cap \mathrm{Ker}f)$, $K \cap mR$ is δ-small in K by Lemma 2.1(4).

(2) ⇒ (1) Assume that M is a principally $\oplus$-δ-supplemented module. Let $m \in M$. There exists a direct summand A of M such that $M = mR + A$ with $mR \cap A$ δ-small in A. Consider the maps $A \xrightarrow{\pi} A/(mR \cap A) \xrightarrow{h} M/mR$ where π is the natural epimorphism and h is the isomorphism $A/(mR \cap A) \cong M/mR$. Since $\mathrm{Ker}(h\pi) = \mathrm{Ker}\pi = mR \cap A$ is δ-small in A, A is a projective δ-cover of M/mR. So M is principally δ-semiperfect. ∎

Now we can give a characterization of principally δ-semiperfect rings by using the notion of principally $\oplus$-δ-supplemented.

Corollary 3.7. *Let R be a ring. Then the following are equivalent.*

(1) *R is principally δ-semiperfect.*
(2) *R is principally $\oplus$-δ-supplemented.*

Proof. Clear by Theorem 3.11. ∎

It is known that a ring R is semisimple if and only if every R-module is projective. As a consequence of Theorem 3.11, we have the next result.

Corollary 3.8. *Let R be a semisimple ring. Then every R-module is principally $\oplus$-δ-supplemented if and only if every R-module is principally δ-semiperfect.*

We conclude this paper by giving the aforementioned example which shows that every principally $\oplus$-δ-supplemented module need not be principally $\oplus$-supplemented.

Example 3.8. Let F be a field, $I = \left[\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right]$, and consider the ring

$$R = \{(x_1, \ldots, x_n, x, x, \ldots) : n \in \mathbb{N}, x_i \in M_2(F), x \in I\}$$

with componentwise operations. By [11, Example 2.15], $J(R) = 0$ and R is not a von Neumann regular ring. Then R is not principally $\oplus$-supplemented as an R-module due to [14, Theorem 3.30]. On the other hand, it is known that, from [18, Example 4.3],

$J_\delta(R) = \{(x_1,\ldots,x_n,x,x,\ldots) : n \in \mathbb{N}, x_i \in M_2(F), x \in K\}$, where $K = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ and R is δ-perfect. Hence R is principally δ-semiperfect. By Corollary 3.7, R is principally $\oplus$-δ-supplemented.

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