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A Sufficient Condition on Group Connectivity of Graphs

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Abstract. Let A be an Abelian group, $n \geq 3$ be an integer, and $ex(n,A)$ be the maximum integer such that every n-vertex simple graph with at most $ex(n,A)$ edges is not A-connected. In this paper, we obtain a necessary condition for a graph being A-connected. Employing the condition we present a lower bound for $ex(n,Z_3)$ which improves some known result and prove that every cubic graph (not necessarily simple graph) with order at least 18 is not Z_3-connected.

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1. Introduction

In this paper,we consider undirected graphs without loops but multiple edges are allowed. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, and we follow Bondy and Murty [1], for undefined notations and terminology. For an orientation D of G and a vertex $v \in V(G)$, the set of all edges incident with v being the tail (or the head, respectively) is denoted by $E^+(v)$ (or $E^-(v)$, respectively).

Let $D = D(G)$ be an orientation of an undirected graph G. If an edge $e = uv \in E(G)$ is directed from u to v, then let $tail(e) = u$ and $head(e) = v$. For a vertex $v \in V(G)$, let

$$E_D^+(v) = \{e \in E(D) : v = head(e)\} \quad and \quad E_D^-(v) = \{e \in E(D) : v = tail(e)\}.$$

Integer flow was originally introduced by Tutte [4,5] as a generalization of map coloring problems. The following are some definitions about basic integer flow concepts.

Definition 1.1. *An Z_k-flow of a graph G is an ordered pair (D, f) such that*

$$\sum_{e \in E_D^+(v)} f(e) \equiv \sum_{e \in E_D^-(v)} f(e) \pmod{k},$$

for every vertex $v \in V(G)$.

A Z_k-flow is nowhere-zero Z_k-flow if $f(e) \neq 0$ for each $e \in E(G)$. The following is the well-known 3-flow conjecture

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Conjecture 1.1. (**3-flow conjecture, Tutte**) *Every 4-edge-connected graph admits a nowhere-zero Z_3-flow.*

Let A be an (additive) Abelian graoup with identity 0. If $f : E(G) \to A$ is a function, then the *boundary* of f is a map $\partial f : V(G) \mapsto A$ such that, for any vertex $v \in V(G)$,

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e).$$

For an integer $k \geq 1$, $\mathbf{Z}_m$ denotes the set of integers modulo k, as well as the additive cyclic group on k elements. For a graph G, a function $b : V(G) \mapsto \mathbf{Z}_k$ is a *zero sum function* in $\mathbf{Z}_k$ if $\sum_{v \in V(G)} \equiv 0 \pmod{k}$. The set of all zero sum function in $\mathbf{Z}_k$ of G is denoted by $Z(G, \mathbf{Z}_k)$.

Graph connectivity was introduced in [2] as a generalization of integer flow, and an inductive approach for flow problems. A graph G is Z_k-connected if for any function $b \in Z(G, \mathbf{Z}_k)$, G has a nowhere-zero Z_k-flow f such that $\partial f(v) = b(v)$ for each vertex $v \in V(G)$. Clearly, if G is Z_k-connected, then it has a nowhere-zero Z_k-flow. But the converse may not be true.

Conjecture 1.2. (Jaeger, Linial, Payan and Tarsi [2]) *Every 5-edge-connected graph is Z_3-connected.*

Let A be an Abelian group, $|A| \geq 3$. Denote $ex(n, A)$ to be the maximum integer such that every n-vertex simple graph with at most $ex(n, A)$ edges is not A-connected. This concept was introduced by Luo, Xu and Yu [3]. They obtained the upper bound and lower bound of $ex(n, Z_3)$ and conjectured that $ex(n, Z_3)$ equals the upper bound.

Theorem 1.1. (Luo, Xu and Yu [3]) *For an integer $n \geq 6$, $3n/2 \leq ex(n, Z_3) \leq 2n - 3$.*

In this paper, we improve the lower bound of $ex(n, Z_3)$ and prove that every cubic graph is not Z_3-connected if it contains at least 18 vertices.

2. Mail results

The following lemma plays a key role in this paper.

Lemma 2.1. *Let k be an integer, $k \geq 3$, and let G be a graph with n vertices and m edges. If G is Z_k-connected, then we have*

$$(k-1)^m \geq k^{n-1}.$$

Proof. Suppose G is Z_k-connected. Let $\mathscr{F}$ be the collection of all functions $f : E(G) \to Z_k$ such that $f(e) \neq 0$, for $\forall e \in E(G)$. Let $b \in Z(G, \mathbf{Z}_k)$ be a zero-sum function. Denote

$$\mathscr{F}_b = \{f \in \mathscr{F} \mid \partial f = b\}.$$

Since G is Z_k-connected, we have $\mathscr{F}_b \neq \emptyset$, for any function $b \in Z(G, \mathbf{Z}_k)$.

Let $b_1, b_2 \in Z(G, \mathbf{Z}_k)$ such that $b_1 \neq b_2$. Then $\forall f_1 \in \mathscr{F}_{b_1}$ and $\forall f_2 \in \mathscr{F}_{b_2}$, we have

$$\partial(f_1 - f_2) = \partial f_1 - \partial f_2 = b_1 - b_2 \neq 0.$$

So $f_1 \neq f_2$ and then $\mathscr{F}_{b_1} \cap \mathscr{F}_{b_2} = \emptyset$. Therefore $\{\mathscr{F}_b \mid b \in Z(G, \mathbf{Z}_k)\}$ is a partition of $\mathscr{F}$. Then

$$\mathscr{F} = \bigcup_{b \in Z(G, \mathbf{Z}_k)} \mathscr{F}_b$$

and

$$|\mathscr{F}| = \sum_{b \in Z(G,\mathbf{Z}_k)} |\mathscr{F}_b| \geq \sum_{b \in Z(G,\mathbf{Z}_k)} 1 = |Z(G,\mathbf{Z}_k)|.$$

Since $|\mathscr{F}| = (k-1)^m$ and $|Z(G,\mathbf{Z}_k)| = k^{n-1}$, we have $(k-1)^m \geq k^{n-1}$. ∎

We derive a necessary condition for a graph being Z_k-connected. The following theorem is an immediate corollary of Lemma 2.1.

Theorem 2.1. *If $(k-1)^m < k^{n-1}$, then G is not Z_k-connected, where $m = |E(G)|$ and $n = |V(G)|$.*

Corollary 2.1. *A cycle length of t is not Z_k-connected if $t > k$.*

Proof. Let C_t be a cycle length of t. We have $m = |E(C_t)| = t$ and $n = |V(C_t)| = t$. If $t > k$, then $(k-1)^t < k^{t-1}$. By Theorem 2.1, C_t is not Z_k-connected. ∎

A graph G is k-regular if $d(v) = k$ for all $v \in V(G)$, a 3-regular graph is called cubic graph.

Corollary 2.2. *Let G be a cubic graph. If $|V(G)| \geq 18$, then G is not Z_3-connected.*

Proof. Since G is cubic, we have $m = |E(G)| = 3|V(G)|/2 = 3n/2$. If $n \geq 18$, then $2^m < 3^{n-1}$. By Theorem 2.1, G is not Z_3-connected. ∎

Note that a cubic graph has a nowhere-zero Z_3-flow if and only if it is a bipartite graph. So there are infinite graphs having a nowhere-zero Z_3-flow. However the condition that a graph being Z_3-connected is much stronger. Corollary 2.2 tells us Z_3-connected cubic graphs are very few and the number of them is finite.

Theorem 2.2. *For an integer $n \geq 20$, $1.58(n-1) \leq ex(n,Z_3) \leq 2n-3$.*

Proof. Let G be a graph with $n = |V(G)| \geq 6$. If $m = |E(G)| < 1.58(n-1)$, then $2^m < 3^{n-1}$. By Theorem 2.1, G is not Z_3-connected. So $ex(n,Z_3) \geq 1.58(n-1)$. Combine the result with that of Theorem 1.1, we have $1.58(n-1) \leq ex(n,Z_3) \leq 2n-3$. ∎

If $n \geq 20$, then $1.58(n-1) > 3n/2$. So Theorem 2.2 is an improvement of Theorem 1.1.

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