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On a Class of Degenerate Nonlocal Problems with Sign-Changing Nonlinearities

[1]NGUYEN THANH CHUNG AND [2]HOANG QUOC TOAN
[1]Department of Mathematics and Informatics, Quang Binh University, 312 Ly Thuong Kiet, Dong Hoi, Quang Binh, Vietnam
[2]Department of Mathematics, Hanoi University of Science, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam
[1]ntchung82@yahoo.com, [2]hq_toan@yahoo.com

Abstract. Using variational techniques, we study the nonexistence and multiplicity of solutions for the degenerate nonlocal problem

$$\begin{cases} -M\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) & = \lambda |x|^{-p(a+1)+c} f(x,u) \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain, $0 \in \Omega$, $0 \leq a < \frac{N-p}{p}$, $1 < p < N$, $c > 0$, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function that may be degenerate at zero, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a sign-changing Carathéodory function and λ is a parameter.

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1. Introduction and preliminaries

In this paper, we are concerned with the problem

$$\begin{cases} -M\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) & = \lambda |x|^{-p(a+1)+c} f(x,u) \quad \text{in } \Omega, \\ u & = 0 \quad \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain, $0 \in \Omega$, $0 \leq a < \frac{N-p}{p}$, $1 < p < N$, $c > 0$, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a sign-changing Carathéodory function, and λ is a parameter. It should be noticed that if $a = 0$ and $c = p$ then problem (1.1) becomes

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p dx\right)\Delta_p u & = \lambda f(x,u) \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Since the first equation in (1.2) contains an integral over Ω, it is no longer a pointwise identity; therefore it is often called nonlocal problem. This problem models several physical

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and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [5]. Moreover, problem (1.2) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.3}$$

presented by Kirchhoff in 1883, see [11]. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.3) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [2, 7, 8, 10, 12, 13, 16, 17, 19], in which the authors have used different methods to get the existence of solutions for (1.2). In [15, 21], Z. Zhang *et al.* studied the existence of nontrivial solutions and sign-changing solutions for (1.2). One of the important hypotheses in these papers is that the Kirchhoff function M is non-degenerate, i.e.,

$$M(t) \geq m_0 > 0 \text{ for all } t \in \mathbb{R}^+. \tag{1.4}$$

Motivated by the ideas introduced in [6, 9, 14, 20], the goal of this paper is to study the existence of solutions for problem (1.1) without condition (1.4). More exactly, we consider problem (1.1) in the case when f is a sign-changing Carathéodory function and the Kirchhoff function M is allowed to take the value 0 at 0. Using the minimum principle combined with the mountain pass theorem, we show that problem (1.1) has at least two distinct, non-negative nontrivial weak solutions for λ large enough. We also prove that (1.1) has no nontrivial solution if λ is small enough. Our results supplement the previous ones in the non-degenerate case. Moreover, we consider problem (1.1) in the general case $0 \leq a < \frac{N-p}{p}$, $1 < p < N$, $c > 0$. To our best knowledge, the present paper is the first contribution related to a Kirchhoff equation in this direction.

In order to state the main results, let us introduce the following conditions:

(M_0) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and satisfies

$$M(t) \geq m_0 t^{\alpha - 1} \text{ for all } t \in \mathbb{R}^+,$$

where $m_0 > 0$ and $1 < \alpha < \min\left\{ \frac{N}{N-p}, \frac{N-p(a+1)+c}{N-p(a+1)} \right\}$;

(F_1) $f : \Omega \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function, such that

$$|f(x,t)| \leq C t^{\alpha p - 1} \text{ for all } t \in [0, +\infty) \text{ and } x \in \Omega,$$

where α is given in (M_0);

(F_2) There exist $t_0, t_1 > 0$ such that $F(x,t) \leq 0$ for all $0 \leq t \leq t_0$ and $F(x,t_1) > 0$ for all $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s) ds$;

(F_3) It holds that

$$\limsup_{t \to \infty} \frac{F(x,t)}{t^{\alpha p}} \leq 0 \text{ uniformly in } x \in \Omega.$$

We point out that if $a = 0$, $c = p$ and $M(t) \equiv 1$, problem (1.1) has been studied by K. Perera [14]. We emphasize that the main difference between the local case ($M \equiv 1$) and the present paper ($M \not\equiv 1$) is that the operator appears in problem (1.1) is not homogeneous. Moreover, from the physical point of view, nonlocal coefficient $M\left(\int_\Omega |x|^{-ap} |\nabla u|^p dx\right)$ of the divergence term in (1.1) is a function (may be degenerate at zero) depending on the average of the kinetic energy. It should be noticed that since $0 \le a < \frac{N-p}{p}$, $1 < p < N$, $c > 0$, our results are better than those in [14] even in the case $M \equiv 1$. Finally, with the same arguments used in this work, we can deal with the case $\alpha = 1$. Thus, our paper is a natural extension from [14] and recent results on p-Kirchhoff type problems.

We start by recalling some useful results in [3, 4, 20]. We have known that for all $u \in C_0^\infty(\mathbb{R}^N)$, there exists a constant $C_{a,b} > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-bq}|u|^q dx\right)^{\frac{p}{q}} \le C_{a,b} \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx, \tag{1.5}$$

where $-\infty < a < \frac{N-p}{p}$, $a \le b \le a+1$, $q = p^*(a,b) = \frac{Np}{N-dp}$, $d = 1+a-b$.

Let $W_0^{1,p}(\Omega, |x|^{-ap})$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{a,p} = \left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{\frac{1}{p}}.$$

Then $W_0^{1,p}(\Omega, |x|^{-ap})$ is a reflexive Banach space. From the boundedness of Ω and the standard approximation argument, it is easy to see that (1.5) holds for any $u \in W_0^{1,p}(\Omega, |x|^{-ap})$ in the sense that

$$\left(\int_{\mathbb{R}^N} |x|^{-\alpha}|u|^r dx\right)^{\frac{p}{r}} \le C_{a,b} \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx, \tag{1.6}$$

for $1 \le r \le p^* = \frac{Np}{N-p}$, $\alpha \le (1+a)r + N\left(1 - \frac{r}{p}\right)$, that is, the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is continuous, where $L^r(\Omega, |x|^{-\alpha})$ is the weighted $L^r(\Omega)$ space with the norm

$$|u|_{r,\alpha} := |u|_{L^r(\Omega,|x|^{-\alpha})} = \left(\int_\Omega |x|^{-\alpha}|u|^r dx\right)^{\frac{1}{r}}.$$

In fact, we have the following compact embedding result which is an extension of the classical Rellich-Kondrachov compactness theorem (see [20]).

Lemma 1.1 (Compact embedding theorem)**.** *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and that $0 \in \Omega$, and $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $1 \le r < \frac{Np}{N-p}$ and $\alpha < (1+a)r + N\left(1 - \frac{r}{p}\right)$. Then the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is compact.*

From Lemma 1.1, B. Xuan proved in [20] that the first eigenvalue λ_1 of the singular quasilinear equation

$$\begin{cases} -\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) &= \lambda |x|^{-p(a+1)+c}|u|^{p-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega, \end{cases}$$

is isolated, unique (up to a multiplicative constant), that is, the first eigenvalue is simple and it is given by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega,|x|^{-ap})\setminus\{0\}} \frac{\int_\Omega |x|^{-ap}|\nabla u|^p dx}{\int_\Omega |x|^{-p(a+1)+c}|u|^p dx} > 0.$$

This is a natural extension from the previous results on the case $a = 0$ and $c = p$ relying esstentially on the Caffarelli-Kohn-Nirenberg inequalities.

Definition 1.1. *We say that $u \in X = W_0^{1,p}(\Omega, |x|^{-ap})$ is a weak solution of problem (1.1) if for all $\varphi \in X$, it holds that*

$$M\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dx - \lambda\int_\Omega |x|^{-p(a+1)+c} f(x,u)\varphi dx = 0.$$

Our main results of this paper can be described as follows.

Theorem 1.1. *Assume that the conditions (M_0) and (F_1) hold. Then there exists a positive constant λ_* such that for any $\lambda < \lambda_*$, problem (1.1) has no nontrivial weak solution.*

Theorem 1.2. *Assume that the conditions (M_0) and (F_1)-(F_3) hold. Then there exists a positive constant λ^* such that for any $\lambda \geq \lambda^*$, problem (1.1) has at least two distinct non-negative, nontrivial weak solutions.*

2. Proof of the main results

For simplicity, we denote $X = W_0^{1,p}(\Omega, |x|^{-ap})$. In the following, when there is no misunderstanding, we always use C_i to denote positive constants.

Proof of Theorem 1.1. First, since $1 < \alpha < \min\left\{\frac{N}{N-p}, \frac{N-p(a+1)+c}{N-p(a+1)}\right\}$, the embedding $X \hookrightarrow L^{\alpha p}(\Omega, |x|^{-p(a+1)+c})$ is compact, see Lemma 1.1. Then there exists $C_1 > 0$ such that

$$C_1\|u\|_{L^{\alpha p}(\Omega,|x|^{-p(a+1)+c})} \leq \|u\|_{a,p} \text{for all } u \in X$$

or

$$C_1^{\alpha p}\int_\Omega |x|^{-p(a+1)+c}|u|^{\alpha p}dx \leq \left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^\alpha \quad \text{for all } u \in X.$$

It follows that the number

$$\lambda_\alpha := \inf_{u\in X\setminus\{0\}} \frac{\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^\alpha}{\int_\Omega |x|^{-p(a+1)+c}|u|^{\alpha p}dx} > 0. \tag{2.1}$$

If $u \in X$ is a nontrivial weak solution, then multiplying (1.1) by u, integrating by parts and using (M_0), (F_1) gives

$$\begin{aligned} m_0\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^\alpha &\leq M\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)\int_\Omega |x|^{-ap}|\nabla u|^p dx \\ &= \lambda\int_\Omega |x|^{-p(a+1)+c} f(x,u)u dx \\ &\leq C\lambda\int_\Omega |x|^{-p(a+1)+c}|u|^{\alpha p}dx. \end{aligned} \tag{2.2}$$

From (2.2), choosing $\lambda_* = \frac{\lambda_\alpha m_0}{C}$, where λ_α is given by (2.1), we conclude the proof of Theorem 1.1. ∎

We will prove Theorem 1.2 using critical point theory. Set $f(x,t) = 0$ for $t < 0$. For all $\lambda \in \mathbb{R}$, we consider the functional $T_\lambda : X \to \mathbb{R}$ given by

$$\begin{aligned} T_\lambda(\omega) &= \frac{1}{p}\widehat{M}\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right) - \lambda\int_\Omega |x|^{-p(a+1)+c}F(x,u)dx \\ &= J(u) - \lambda I(u), \end{aligned} \tag{2.3}$$

where

$$J(u) = \frac{1}{p}\widehat{M}\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right), \tag{2.4}$$

$$I(u) = \int_\Omega |x|^{-p(a+1)+c} F(x,u)dx, \quad u \in X.$$

By Lemma 1.1 and the condition (F_1), a simple computation implies that T_λ is well-defined and of C^1 class in X. Thus, weak solutions of problem (1.1) correspond to the critical points of the functional T_λ.

Lemma 2.1. *The functional T_λ given by (2.3) is weakly lower semicontinuous X.*

Proof. Let $\{u_m\}$ be a sequence that converges weakly to u in X. Then, by the continuity of norm, we have

$$\liminf_{m\to\infty} \int_\Omega |x|^{-ap}|\nabla u_m|^p \, dx \geq \int_\Omega |x|^{-ap}|\nabla u|^p \, dx.$$

Combining this with the continuity and monotonicity of the function $\psi : \mathbb{R}^+ \to \mathbb{R}$, $t \mapsto \psi(t) = \frac{1}{p}\widehat{M}(t)$, we get

$$\begin{aligned}
\liminf_{m\to\infty} J(u_m) &= \liminf_{m\to\infty} \frac{1}{p}\widehat{M}\left(\int_\Omega |x|^{-ap}|\nabla u_m|^p \, dx\right) = \liminf_{m\to\infty} \psi\left(\int_\Omega |x|^{-ap}|\nabla u_m|^p \, dx\right) \\
&\geq \psi\left(\liminf_{m\to\infty} \int_\Omega |x|^{-ap}|\nabla u_m|^p \, dx\right) \geq \psi\left(\int_\Omega |x|^{-ap}|\nabla u|^p \, dx\right) \\
&= \frac{1}{p}\widehat{M}\left(\int_\Omega |x|^{-ap}|\nabla u|^p \, dx\right) = J(u).
\end{aligned} \tag{2.5}$$

We shall show that

$$\lim_{m\to\infty} \int_\Omega F(x,u_m)dx = \int_\Omega F(x,u)dx. \tag{2.6}$$

Using (F_1) and Hölder's inequality, it follows that

$$\begin{aligned}
&\left|\int_\Omega |x|^{-p(a+1)+c}[F(x,u_m) - F(x,u)]dx\right| \\
&\leq \int_\Omega |x|^{-p(a+1)+c}|f(x,u+\theta_m(u_m-u))||u_m-u|dx \\
&\leq C\int_\Omega |x|^{-p(a+1)+c}|u+\theta_m(u_m-u)|^{\alpha p-1}|u_m-u|dx \\
&\leq C\|u+\theta_m(u_m-u)\|^{\alpha p-1}_{L^{\alpha p}(\Omega,|x|^{-p(a+1)+c})}\|u_m-u\|_{L^{\alpha p}(\Omega,|x|^{-p(a+1)+c})},
\end{aligned} \tag{2.7}$$

where $0 \leq \theta_m(x) \leq 1$ for all $x \in \Omega$.

On the other hand, since $1 < \alpha < \min\left\{\frac{N}{N-p}, \frac{N-p(a+1)+c}{N-p(a+1)}\right\}$, $X \hookrightarrow L^{\alpha p}(\Omega,|x|^{-p(a+1)+c})$ is compact, the sequence $\{u_m\}$ converges strongly to u in the space $L^{\alpha p}(\Omega,|x|^{-p(a+1)+c})$. It is easy to see that the sequence $\{\|u+\theta_m(u_m-u)\|_{L^{\alpha p}(\Omega,|x|^{-p(a+1)+c})}\}$ is bounded. Thus, it follows from (2.7) that relation (2.6) holds true. The proof of Lemma 2.1 is proved. ∎

Lemma 2.2. *The functional T_λ is coercive and bounded from below.*

Proof. By the conditions (F_1) and (F_3), there exists $C_\lambda > 0$ such that for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$, one has

$$\lambda F(x,t) \le \frac{m_0 \lambda_\alpha}{2\alpha p}|t|^{\alpha p} + C_\lambda, \tag{2.8}$$

where λ_α is given by (2.1). Hence, using (M_0) and the fact that

$$0 < \int_\Omega |x|^{-p(a+1)+c} dx < \infty$$

we get

$$\begin{aligned} T_\lambda(u) &\ge \frac{m_0}{\alpha p}\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^\alpha - \lambda \int_\Omega |x|^{-p(a+1)+c} F(x,u)dx \\ &\ge \frac{m_0}{\alpha p}\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^\alpha - \int_\Omega |x|^{-p(a+1)+c}\left(\frac{m_0\lambda_\alpha}{2\alpha p}|u|^{\alpha p} + C_\lambda\right)dx \\ &\ge \frac{m_0}{2\alpha p}\|u\|_{a,p}^{\alpha p} - \overline{C}_\lambda, \end{aligned} \tag{2.9}$$

where $\overline{C}_\lambda > 0$ is a constant. So, T_λ is coercive and bounded from below. ∎

Lemma 2.3. *If $u \in X$ is a weak solution of problem (1.1) then $u \ge 0$ in Ω.*

Proof. Indeed, if $u \in X$ is a weak solution of problem (1.1), then we have

$$\begin{aligned} 0 &= \langle T'_\lambda(u), u^-\rangle \\ &= M\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u \cdot \nabla u^- dx - \lambda\int_\Omega |x|^{-p(a+1)+c} f(x,u)u^- dx \\ &\ge m_0\left(\int_\Omega |x|^{-ap}|\nabla u^-|^p dx\right)^\alpha, \end{aligned}$$

where $u^- = \min\{u(x), 0\}$ is the negative part of u. It follows that $u \ge 0$ in Ω. ∎

By Lemmas 2.1-2.3, applying the minimum principle (see [18, p. 4, Theorem 1.2]), the functional T_λ has a global minimum and thus problem (1.1) admits a non-negative weak solution $u_1 \in X$. The following lemma shows that the solution u_1 is not trivial provided that λ is large enough.

Lemma 2.4. *There exists $\lambda^* > 0$ such that for all $\lambda \ge \lambda^*$, $\inf_{u\in X} T_\lambda(u) < 0$ and hence the solution $u_1 \not\equiv 0$.*

Proof. Indeed, let Ω' be a sufficiently large compact subset of Ω and a function $u_0 \in C_0^\infty(\Omega)$, such that $u_0(x) = t_0$ on Ω', $0 \le u_0(x) \le t_0$ on $\Omega\backslash\Omega'$, where t_0 is as in (F_2). Then we have

$$\begin{aligned} \int_\Omega |x|^{-p(a+1)+c}F(x,u_0)dx &= \int_{\Omega'} |x|^{-p(a+1)+c}F(x,u_0)dx + \int_{\Omega\backslash\Omega'} |x|^{-p(a+1)+c}F(x,u_0)dx \\ &\ge \int_{\Omega'} |x|^{-p(a+1)+c}F(x,t_0)dx - C\int_{\Omega\backslash\Omega'} |x|^{-p(a+1)+c}|u_0|^p dx \\ &\ge \int_{\Omega'} |x|^{-p(a+1)+c}F(x,t_0)dx - Ct_0^p\int_{\Omega\backslash\Omega'} |x|^{-p(a+1)+c} dx > 0, \end{aligned}$$

provided that $|\Omega\backslash\Omega'| > 0$ is small enough. So, we deduce that

$$T_\lambda(u_0) = \frac{1}{p}\widehat{M}\left(\int_\Omega |x|^{-ap}|\nabla u_0|^p dx\right) - \lambda\int_\Omega |x|^{-p(a+1)+c}F(x,u_0)dx$$
$$\le \frac{1}{p}\widehat{M}\left(\int_\Omega |x|^{-ap}|\nabla u_0|^p dx\right) - \lambda\left(\int_{\Omega'} |x|^{-p(a+1)+c}F(x,t_0)dx - Ct_0^p\int_{\Omega\backslash\Omega'}|x|^{-p(a+1)+c}dx\right).$$

Hence, if Ω' is large enough, there exists λ^* such that for all $\lambda \ge \lambda^*$ we have $T_\lambda(u_0) < 0$ and thus $u_1 \not\equiv 0$. Moreover, $T_\lambda(u_1) < 0$ for all $\lambda \ge \lambda^*$. ∎

Our idea is to obtain the second weak solution $u_2 \in X$ by applying the mountain pass theorem in [1]. To this purpose, we first show that for all $\lambda \ge \lambda^*$, the functional T_λ has the geometry of the mountain pass theorem.

Lemma 2.5. *There exist a constant $\rho \in (0, \|u_1\|_{a,p})$ and a constant $r > 0$ such that $T_\lambda(u) \ge r$ for all $u \in X$ with $\|u\|_{a,p} = \rho$.*

Proof. For each $u \in X$, we set

$$\Omega_u := \{x \in \Omega : \ u(x) > t_0\}, \tag{2.10}$$

where t_0 is given by (F_2). Then, we have $F(x,u(x)) \le 0$ on $\Omega\backslash\Omega_u$, so

$$T_\lambda(u) \ge \frac{m_0}{\alpha p}\Big(\int_\Omega |x|^{-ap}|\nabla u|^p dx\Big)^\alpha - \int_{\Omega_u} F(x,u)dx$$
$$= \frac{m_0}{\alpha p}\|u\|_{a,p}^{\alpha p} - \int_{\Omega_u} F(x,u)dx. \tag{2.11}$$

Using the Hölder inequality and Lemma 1.1, we get

$$\int_{\Omega_u} |x|^{-p(a+1)+c}F(x,u)dx \le C\int_{\Omega_u}|x|^{-p(a+1)+c}|u|^{\alpha p}dx$$
$$\le C\left(\int_{\Omega_u}|x|^{-p(a+1)+c}|u|^q dx\right)^{\frac{\alpha p}{q}}\left(\int_{\Omega_u}|x|^{-p(a+1)+c}dx\right)^{1-\frac{\alpha p}{q}}$$
$$\le C_2\|u\|_{a,p}^{\alpha p}\left(\int_{\Omega_u}|x|^{-p(a+1)+c}dx\right)^{1-\frac{\alpha p}{q}}, \tag{2.12}$$

where $\alpha p < q < \min\left\{\frac{Np}{N-p}, \frac{p(N-p(a+1)+c)}{N-p(a+1)}\right\}$.

From (2.11) and (2.12), it implies that

$$T_\lambda(u) \ge \|u\|_{a,p}^{\alpha p}\left[\frac{m_0}{\alpha p} - C_2\left(\int_{\Omega_u}|x|^{-p(a+1)+c}dx\right)^{1-\frac{\alpha p}{q}}\right]. \tag{2.13}$$

From (2.13), in order to prove Lemma 2.5, it is enough to show that

$$\int_{\Omega_u}|x|^{-p(a+1)+c}dx \to 0 \text{ as } \|u\|_{a,p} \to 0.$$

Given $\varepsilon > 0$, take a compact subset Ω_ε of Ω such that

$$\int_{\Omega\backslash\Omega_\varepsilon}|x|^{-p(a+1)+c}dx < \varepsilon$$

and let $\Omega_{u,\varepsilon} = \Omega_u \cap \Omega_\varepsilon$. Then

$$\int_\Omega |x|^{-ap}|\nabla u|^p \geq C_3 \int_{\Omega_{u,\varepsilon}} |x|^{-p(a+1)+c}|u|^p dx \geq C_3 t_0^p \int_{\Omega_{u,\varepsilon}} |x|^{-p(a+1)+c} dx, \tag{2.14}$$

so

$$\int_{\Omega_{u,\varepsilon}} |x|^{-p(a+1)+c} dx \to 0 \text{ as } \|u\|_{a,p} \to 0.$$

But since $\Omega_u \subset \Omega_{u,\varepsilon} \cup (\Omega \backslash \Omega_\varepsilon)$, we have

$$\int_{\Omega_u} |x|^{-p(a+1)+c} dx < \int_{\Omega_{u,\varepsilon}} |x|^{-p(a+1)+c} dx + \varepsilon,$$

and ε is arbitrary. This shows that

$$\int_{\Omega_u} |x|^{-p(a+1)+c} dx \to 0 \text{ as } \|u\|_{a,p} \to 0$$

and thus, Lemma 2.5 is proved. ∎

Lemma 2.6. *The functional T_λ satisfies the Palais-Smale condition in X.*

Proof. By Lemma 2.2, we deduce that T_λ is coercive on X. Let $\{u_m\}$ be a sequence such that

$$T_\lambda(u_m) \to \overline{c} < \infty, \quad T'_\lambda(u_m) \to 0 \text{ in } X^* \text{ as } m \to \infty, \tag{2.15}$$

where X^* is the dual space of X.

Since T_λ is coercive on X, relation (2.15) implies that the sequence $\{u_m\}$ is bounded in X. Since X is reflexive, there exists $u \in X$ such that, passing to a subsequence, still denoted by $\{u_m\}$, it converges weakly to u in X. Hence, $\{\|u_m - u\|\}$ is bounded. This and (2.15) imply that $T'_\lambda(u_m)(u_m - u)$ converges to 0 as $m \to \infty$. Using the condition (F_1) combined with Hölder's inequality, we conclude that

$$\begin{aligned}\int_\Omega |x|^{-p(a+1)+c}|f(x,u_m)||u_m - u| dx &\leq C \int_\Omega |x|^{-p(a+1)+c}|u_m|^{\alpha p}|u_m - u| dx \\ &\leq C_4 \|u_m\|^{\alpha p}_{L^{\alpha p}(\Omega,|x|^{-p(a+1)+c})} \|u_m - u\|_{L^{\alpha p}(\Omega,|x|^{-p(a+1)+c})},\end{aligned}$$

which shows that

$$\lim_{m\to\infty} \left\langle I'(u_m), u_m - u \right\rangle = 0. \tag{2.16}$$

Combining this with (2.15) and the fact that

$$\left\langle J'(u_m), u_m - u \right\rangle = \left\langle T'_\lambda(u_m), u_m - u \right\rangle + \lambda \left\langle I'(u_m), u_m - u \right\rangle$$

imply that

$$\lim_{m\to\infty} M\left(\int_\Omega |x|^{-ap}|\nabla u_m|^p dx\right) \int_\Omega |x|^{-ap}|\nabla u_m|^{p-2}\nabla u_m \cdot (\nabla u_m - \nabla u) dx = 0. \tag{2.17}$$

Since $\{u_m\}$ is bounded in X, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega |x|^{-ap}|\nabla u_m|^p dx \to t_0 \geq 0 \text{ as } m \to \infty.$$

If $t_0 = 0$ then $\{u_m\}$ converges strongly to $u = 0$ in X and the proof is finished. If $t_0 > 0$ then by (M_0) and the continuity of M, we get

$$M\left(\int_\Omega |x|^{-ap}|\nabla u_m|^p dx\right) \to M(t_0) > 0 \mathrm{as} m \to \infty.$$

Thus, for m sufficiently large, we have

$$0 < C_5 \le M\left(\int_\Omega |x|^{-ap}|\nabla u_m|^p dx\right) \le C_6. \tag{2.18}$$

From (2.17) and (2.18), we have

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}|\nabla u_m|^{p-2}\nabla u_m\cdot(\nabla u_m - \nabla u)dx = 0. \tag{2.19}$$

On the other hand, since $\{u_m\}$ converges weakly to u in X, we have

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\cdot(\nabla u_m - \nabla u)dx = 0. \tag{2.20}$$

By (2.19) and (2.20),

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}\left(|\nabla u_m|^{p-2}\nabla u_m - |\nabla u|^{p-2}\nabla u\right)\cdot(\nabla u_m - \nabla u)dx = 0.$$

or

$$\lim_{m\to\infty}\int_\Omega \left(|\nabla v_m|^{p-2}\nabla v_m - |\nabla v|^{p-2}\nabla v\right)\cdot(\nabla v_m - \nabla v)dx = 0, \tag{2.21}$$

where $\nabla v_m = |x|^{-a}\nabla u_m$, $\nabla v = |x|^{-a}\nabla u$.

We recall that the following inequalities hold

$$\begin{aligned} &\left\langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\right\rangle \ge C_7(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \text{ if } 1 < p < 2,\\ &\left\langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\right\rangle \ge C_8|\xi - \eta|^p \text{ if } p \ge 2, \end{aligned} \tag{2.22}$$

for all $\xi, \eta \in \mathbb{R}^N$, where $\langle .,. \rangle$ denotes the usual product in $\mathbb{R}^N$.

If $1 < p < 2$, using the Hölder inequality, by (2.21), (2.22) we have

$$\begin{aligned} 0 &\le \|u_m - u\|_{a,p}^p = \||\nabla v_m - \nabla v|\|_{L^p(\Omega)}^p\\ &\le \int_\Omega |\nabla v_m - \nabla v|^p(|\nabla v_m| + |\nabla v|)^{\frac{p(p-2)}{2}}(|\nabla v_m| + |\nabla v|)^{\frac{p(2-p)}{2}}dx\\ &\le \left(\int_\Omega |\nabla v_m - \nabla v|^2(|\nabla v_m| + |\nabla v|)^{p-2}dx\right)^{\frac{p}{2}}\left(\int_\Omega (|\nabla v_m| + |\nabla v|)^p dx\right)^{\frac{2-p}{2}}\\ &\le C_9\left(\int_\Omega \left\langle |\nabla v_m|^{p-2}\nabla v_m - |\nabla v|^{p-2}\nabla v, \nabla v_m - \nabla v\right\rangle dx\right)^{\frac{p}{2}} \times \left(\int_\Omega (|\nabla v_m| + |\nabla v|)^p dx\right)^{\frac{2-p}{2}}\\ &\le C_{10}\left(\int_\Omega \left\langle |\nabla v_m|^{p-2}\nabla v_m - |\nabla v|^{p-2}\nabla v, \nabla v_m - \nabla v\right\rangle dx\right)^{\frac{p}{2}}, \end{aligned}$$

which converges to 0 as $m \to \infty$. If $p \ge 2$, one has

$$\begin{aligned} 0 &\le \|u_m - u\|_{a,p}^p = \||\nabla v_m - \nabla v|\|_{L^p(\Omega)}^p\\ &\le C_{11}\int_\Omega \left\langle |\nabla v_m|^{p-2}\nabla v_m - |\nabla v|^{p-2}\nabla v, \nabla v_m - \nabla v\right\rangle dx, \end{aligned}$$

which converges to 0 as $m \to \infty$. So we conclude that $\{u_m\}$ converges strongly to u in X and the functional T_λ satisfies the Palais-Smale condition. ∎

Proof of Theorem 1.2. By Lemmas 2.1-2.4, problem (1.1) admits a non-negative, nontrivial weak solution u_1 as the global minimizer of T_λ. Setting

$$\overline{c} := \inf_{\chi \in \Gamma} \max_{u \in \chi([0,1])} T_\lambda(u), \tag{2.23}$$

where $\Gamma := \{\chi \in C([0,1], X) : \chi(0) = 0, \chi(1) = u_1\}$.

Lemmas 2.5, 2.6 show that all assumptions of the mountain pass theorem in [1] are satisfied, $T_\lambda(u_1) < 0$ and $\|u_1\|_{a,p} > \rho$. Then, $\overline{c}$ is a critical value of T_λ, i.e. there exists $u_2 \in X$ such that $T'_\lambda(u_2)(\varphi) = 0$ for all $\varphi \in X$ or u_2 is a weak solution of (1.1). Moreover, u_2 is not trivial and $u_2 \not\equiv u_1$ since $T_\lambda(u_2) = \overline{c} > 0 > T_\lambda(u_1)$. Theorem 1.2 is completely proved. ∎

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