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On Opial-Dan's Type Inequalities

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Abstract. In the present paper we establish some new Opial-type inequalities involving higher order partial derivatives. Our results in special cases yield some of the recent results on Opial's inequality and provide new estimates on inequalities of these types.

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1. Introduction

In 1960, Opial [21] established the following integral inequality:

Theorem 1.1. *Suppose $f \in C^1[0,h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0,h)$. Then the integral inequality holds*

$$\int_0^h \left| f(x) f'(x) \right| dx \leq \frac{h}{4} \int_0^h (f'(x))^2 dx, \tag{1.1}$$

where this constant $h/4$ is best possible.

Opial's inequality and its generalizations, extensions and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [2,4,7,18,19]. The inequality (1.1) has received considerable attention and a large number of papers dealing with new proofs, extensions, generalizations, variants and discrete analogues of Opial's inequality have appeared in the literature [9–13,15,16,20,22–28,30]. For an extensive survey on these inequalities, see [2, 19]. For Opial type integral inequalities involving high-order partial derivatives see [1,3,5,6,17,32].

The main purpose of the present paper is to establish some new Opial-type inequalities involving higher order partial derivatives by a extension of Das's idea [14]. Our results in special cases yield some of the recent results on Opial's type inequalities and provide some new estimates on such types of inequalities.

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2. Main results

Let $n \geq 1, k \geq 1$. Our main results are given in the following theorems.

Theorem 2.1. *Let $x(s,t) \in C^{(n-1,m-1)}([0,a] \times [0,b])$ be such that $(\partial^{\kappa}/\partial s^{\kappa})x(s,t)|_{s=0} = 0, 0 \leq \kappa \leq n-1$ and $(\partial^{\lambda}/\partial t^{\lambda})x(s,t)|_{t=0} = 0, 0 \leq \lambda \leq m-1$. Further, let $(\partial^{n}/\partial s^{n})((\partial^{m-1}/\partial t^{m-1})x(s,t))$ and $(\partial^{n-1}/\partial s^{n-1})((\partial^{m}/\partial t^{m})x(s,t))$ are absolutely continuous on $[0,a] \times [0,b]$, and let $1/p+1/q=1, p>1$ and $\int_0^a \int_0^b |(\partial^{n+m}/\partial s^n \partial t^m)x(s,t)|^q\, dsdt$, exist. Then*

$$\begin{aligned}
&\int_0^a \int_0^b \left| \frac{\partial^{\kappa+\lambda}}{\partial s^{\kappa}\partial t^{\lambda}} x(s,t) \frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t) \right| dsdt \\
&\leq C_{n,m,\kappa,\lambda,p,q}\, a^{n-\kappa-1+2/p} b^{m-\lambda-1+2/p} \left(\int_0^a \int_0^b \left| \frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t) \right|^q dsdt \right)^{(q+1)/q},
\end{aligned} \tag{2.1}$$

where

$$C_{n,m,\kappa,\lambda,p,q} = \frac{[(p(n-\kappa-1)+1)(p(n-\kappa-1)+2)(p(m-\lambda-1)+1)(p(m-\lambda-1)+2)]^{-1/p}}{2^{1/q}(n-\kappa-1)!(m-\lambda-1)!}.$$

Proof. From the hypotheses of the Theorem 2.1, we have for $0 \leq \kappa \leq n-1, 0 \leq \lambda \leq m-1$

$$\begin{aligned}
\left| \frac{\partial^{\kappa+\lambda}}{\partial s^{\kappa}\partial t^{\lambda}} x(s,t) \right| &\leq \frac{1}{(n-\kappa-1)!(m-\lambda-1)!} \\
&\quad \times \int_0^s \int_0^t (s-\sigma)^{n-\kappa-1}(t-\tau)^{m-\lambda-1} \left| \frac{\partial^{n+m}}{\partial \sigma^n \partial \tau^m} x(\sigma,\tau) \right| d\sigma d\tau.
\end{aligned} \tag{2.2}$$

Multiplying both sides of (2.2) by $|(\partial^{n+m}/\partial s^n \partial t^m)x(s,t)|$ and using the Hölder inequality, we have

$$\begin{aligned}
&\left| \frac{\partial^{\kappa+\lambda}}{\partial s^{\kappa}\partial t^{\lambda}} x(s,t) \frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t) \right| \\
&\leq \frac{\left| \frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t) \right|}{(n-\kappa-1)!(m-\lambda-1)!} \left(\int_0^s \int_0^t (s-\sigma)^{p(n-\kappa-1)}(t-\tau)^{p(m-\lambda-1)} d\sigma d\tau \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^s \int_0^t \left| \frac{\partial^{n+m}}{\partial \sigma^n \partial \tau^m} x(\sigma,\tau) \right|^q d\sigma d\tau \right)^{\frac{1}{q}} \\
&= \frac{s^{n-\kappa-1+1/p} t^{m-\lambda-1+1/p}}{(n-\kappa-1)!(m-\lambda-1)![(p(n-\kappa-1)+1)(p(m-\lambda-1)+1)]^{1/p}} \\
&\quad \times \left| \frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t) \right| \left(\int_0^s \int_0^t \left| \frac{\partial^{n+m}}{\partial \sigma^n \partial \tau^m} x(\sigma,\tau) \right|^q d\sigma d\tau \right)^{\frac{1}{q}}.
\end{aligned} \tag{2.3}$$

Thus, integrating both sides of (2.3) over t from 0 to b first and then integrating the resulting inequality over s from 0 to a and applying the Hölder inequality again, we obtain

$$\begin{aligned}
&\int_0^a\int_0^b\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x(s,t)\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|dsdt\\
&\le\frac{1}{(n-\kappa-1)!(m-\lambda-1)![(p(n-\kappa-1)+1)(p(m-\lambda-1)+1)]^{1/p}}\\
&\quad\times\left(\int_0^a\int_0^b s^{p(n-\kappa-1)+1}t^{p(m-\lambda-1)+1}dsdt\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|^q\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x(\sigma,\tau)\right|^q d\sigma d\tau\right)dsdt\right)^{\frac{1}{q}}.
\end{aligned}\tag{2.4}$$

On the other hand, from the hypotheses of Theorem 2.1 and in view of the following facts

$$\begin{aligned}
&\frac{\partial^2}{\partial s\partial t}\left[\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x(\sigma,\tau)\right|^q d\sigma d\tau\right)^2\right]\\
&=2\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|^q\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x(\sigma,\tau)\right|^q d\sigma d\tau\right)
\end{aligned}\tag{2.5}$$

and

$$\left(\int_0^a\int_0^b s^{p(n-\kappa-1)+1}t^{p(m-\lambda-1)+1}dsdt\right)^{1/p}=\frac{a^{n-\kappa-1+2/p}b^{m-\lambda-1+1/p}}{[(p(n-\kappa-1)+2)(p(m-\lambda-1)+2)]^{1/p}}.\tag{2.6}$$

From (2.4), (2.5) and (2.6), we have

$$\begin{aligned}
&\int_0^a\int_0^b\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x(s,t)\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|dsdt\\
&\le C_{n,m,\kappa,\lambda,p,q}a^{n-\kappa-1+2/p}b^{m-\lambda-1+2/p}\left(\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|^q dsdt\right)^{2/q},
\end{aligned}$$

where

$$C_{n,m,\kappa,\lambda,p,q}=\frac{[(p(n-\kappa-1)+1)(p(n-\kappa-1)+2)(p(m-\lambda-1)+1)(p(m-\lambda-1)+2)]^{-1/p}}{2^{1/q}(n-\kappa-1)!(m-\lambda-1)!}.$$

This completes the proof. ∎

Remark 2.1. Taking for $p=q=2,\kappa=\lambda=0$ in (2.1), (2.1) becomes to

$$\int_0^a\int_0^b\left|x(s,t)\cdot\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|dsdt\le c_{n,m}\cdot a^nb^m\cdot\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|^2dsdt,\tag{2.7}$$

where

$$c_{n,m}=\frac{1}{4n!m!}\left(\frac{2nm}{(2n-1)(2m-1)}\right)^{\frac{1}{2}}.$$

Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, then (2.7) becomes the following inequality:

$$\int_0^a |x(t)x^{(n)}(t)|dt \leq \frac{1}{2n!}\cdot\left(\frac{n}{2n-1}\right)^{\frac{1}{2}} a^n \int_0^a |x^{(n)}(t)|^2 dt. \tag{2.8}$$

This is just an inequality established by Das [14]. Obviously, for $n \geq 2$, (2.8) is sharper than the following inequality established by Willett [29].

$$\int_0^a |x(t)x^n(t)|dt \leq \frac{1}{2}a^n \int_0^a |x^n(t)|^2 dt.$$

Remark 2.2. Taking for $n = m = 1, \kappa = \lambda = 0$ and $p = q = 2$ in (2.1), (2.1) reduces to

$$\int_0^a \int_0^b \left| x(s,t)\cdot\frac{\partial^2}{\partial s \partial t}x(s,t)\right| dsdt \leq \frac{\sqrt{2}}{4}ab \int_0^a \int_0^b \left|\frac{\partial^2}{\partial s\partial t}x(s,t)\right|^2 dsdt. \tag{2.9}$$

Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, then (2.9) becomes the following inequality:

If $x(t)$ is absolutely continuous in $[0,a]$ and $x(0) = 0$, then

$$\int_0^a |x(t)x'(t)|dt \leq \frac{a}{2}\int_0^a |x'(t)|^2 dt.$$

This is just an inequality established by Beesack [8].

Theorem 2.2. *Let l and m be positive numbers satisfying $l+m>1$. Further, let $x(s,t) \in C^{(n-1,m-1)}([0,a]\times[0,b])$ be such that $(\partial^\kappa/\partial s^\kappa)x(s,t)|_{s=0} = 0, 0 \leq \kappa \leq n-1$ and $(\partial^\lambda/\partial t^\lambda)$ $x(s,t)|_{t=0} = 0, 0 \leq \lambda \leq m-1$, $(\partial^n/\partial s^n)\left((\partial^{m-1}/\partial t^{m-1})x(s,t)\right)$ and $(\partial^{n-1}/\partial s^{n-1})\left((\partial^m/\partial t^m)\right.$ $\left.x(s,t)\right)$ are absolutely continuous on $[0,a]\times[0,b]$, and $\int_0^a\int_0^b |(\partial^{n+m}/\partial s^n\partial t^m)x(s,t)|^{l+m} dsdt$, exist. Then*

$$\begin{aligned}&\int_0^a\int_0^b \left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa \partial t^\lambda}x(s,t)\right|^l \left|\frac{\partial^{n+m}}{\partial s^n \partial t^m}x(s,t)\right|^m dsdt\\ &\leq C^*_{n,m,\kappa,\lambda}a^{(n-\kappa)l}b^{(m-\lambda)l}\int_0^a\int_0^b \left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|^{l+m} dsdt,\end{aligned}$$

where

$$C^*_{n,m,\kappa,\lambda} = \xi^{1+l\xi}m^{m\xi}\left(\frac{(n-\kappa)(m-\lambda)(1-\xi)^2}{(n-\kappa-1)(m-\lambda-1)}\right)^{l(1-\xi)}\frac{1}{[(n-\kappa)!(m-\lambda)!]^l},\ \xi = \frac{1}{l+m}.$$

Proof. From the hypotheses of the Theorem , we have for $0 \leq \kappa \leq n-1, 0 \leq \lambda \leq m-1$

$$\begin{aligned}\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x(s,t)\right| &\leq \frac{1}{(n-\kappa-1)!(m-\lambda-1)!}\\ &\times\int_0^s\int_0^t (s-\sigma)^{n-\kappa-1}(t-\tau)^{m-\lambda-1}\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x(\sigma,\tau)\right| d\sigma d\tau.\end{aligned} \tag{2.10}$$

By Hölder's inequality with indices $l+m$ and $(l+m)/(l+m-1)$, it follows that

$$\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x(s,t)\right| \leq \frac{1}{(n-\kappa-1)!(m-\lambda-1)!}$$

$$\times\left(\int_0^s\int_0^t[(s-\sigma)^{n-\kappa-1}(t-\tau)^{m-\lambda-1}]^{\frac{l+m}{l+m-1}}d\sigma d\tau\right)^{\frac{l+m-1}{l+m}}$$
$$\times\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x(\sigma,\tau)\right|^{l+m}d\sigma d\tau\right)^{\frac{1}{l+m}}$$
$$=Ds^{n-\kappa-\xi}t^{m-\lambda-\xi}\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x(\sigma,\tau)\right|^{l+m}d\sigma d\tau\right)^{\xi},$$

where

$$D=\left(\frac{(1-\xi)^2}{(n-\kappa-\xi)(m-\lambda-\xi)}\right)^{1-\xi}\frac{1}{(n-\kappa-1)!(m-\lambda-1)!}.$$

Multiplying the both sides of above inequality by $|(\partial^{n+m}/\partial s^n\partial t^m)x(s,t)|^m$ and integrating both sides over t from 0 to b first and then integrating the resulting inequality over s from 0 to a, we obtain

$$\int_0^a\int_0^b\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x(s,t)\right|^l\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|^m dsdt$$
$$\leq D^l\int_0^a\int_0^b s^{l(n-\kappa-\xi)}t^{l(m-\lambda-\xi)}\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|^m$$
$$\times\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x(\sigma,\tau)\right|^{l+m}d\sigma d\tau\right)^{l\xi}dsdt.$$

Now, applying Hölder's inequality with indices $(l+m)/l$ and $(l+m)/m$ to the integral on the right side, we obtain

$$\int_0^a\int_0^b\left|\frac{\partial^{\kappa+\lambda}}{\partial s^\kappa\partial t^\lambda}x(s,t)\right|^l\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|^m dsdt$$
$$\leq D^l\left(\int_0^a\int_0^b s^{(n-\kappa-\xi)(l+m)}t^{(m-\lambda-\xi)(l+m)}dsdt\right)^{\frac{l}{l+m}}$$
$$\times\left(\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|^{m+l}\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x(\sigma,\tau)\right|^{l+m}d\sigma d\tau\right)^{\frac{l}{m}}dsdt\right)^{\frac{m}{l+m}}$$
$$=D^l\left(\int_0^a\int_0^b s^{(n-\kappa-\xi)(l+m)}t^{(m-\lambda-\xi)(l+m)}dsdt\right)^{\frac{l}{l+m}}$$
$$\times\left(\frac{m}{l+m}\int_0^a\int_0^b\frac{\partial^2}{\partial s\partial t}\left\{\left(\int_0^s\int_0^t\left|\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}x(\sigma,\tau))\right|^{l+m}d\sigma d\tau\right)^{\frac{l}{m}+1}\right\}dsdt\right)^{\frac{m}{l+m}}$$
$$=D^l\left(\frac{\xi^2}{(n-\kappa)(m-\lambda)}\right)^{\xi l}(m\xi)^{m\xi}a^{(n-\kappa)l}b^{(m-\lambda)l}\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|^{l+m}dsdt$$

$$= C^*_{n,m,\kappa,\lambda} a^{(n-\kappa)l} b^{(m-\lambda)l} \int_0^a \int_0^b \left| \frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t) \right|^{l+m} dsdt,$$

where

$$C^*_{n,m,\kappa,\lambda} = \xi^{1+l\xi} m^{m\xi} \left(\frac{(n-\kappa)(m-\lambda)(1-\xi)^2}{(n-\kappa-1)(m-\lambda-1)} \right)^{l(1-\xi)} \frac{1}{[(n-\kappa)!(m-\lambda)!]^l}, \ \xi = \frac{1}{l+m}.$$

This completes the proof. ∎

Remark 2.3. Taking for $\kappa = \lambda = 0$ in (2.10), (2.10) reduces to

$$\int_0^a \int_0^b |x(s,t)|^l \left| \frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t) \right|^m dsdt \le c^*_{n,m} a^{nl} b^{ml} \int_0^a \int_0^b \left| \frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t) \right|^{l+m} dsdt, \tag{2.11}$$

where

$$c^*_{n,m} = \xi^{l\xi+1} m^{\xi m} \left(\frac{mn(1-\xi)^2}{(n-\xi)(m-\xi)} \right)^{l(1-\xi)} \frac{1}{(n!m!)^l}, \quad \xi = \frac{1}{l+m}.$$

Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, then (2.11) becomes the following inequality:

$$\int_0^a |x(t)|^l \left| x^{(n)}(t) \right|^m dt \le \xi m^{m\xi} \left(\frac{n(1-\xi)}{n-\xi} \right)^{l(1-\xi)} (n!)^{-l} a^{nl} \int_0^a \left| x^{(n)}(t) \right|^{l+m} dt. \tag{2.12}$$

This is an inequality given by Das [14]. Taking for $n = 1$ in (2.12), we have

$$\int_0^a |x(t)|^l |x'(t)|^m dt \le \frac{m^{m/(l+m)}}{l+m} a^l \int_0^a |x'(t)|^{m+l} dt. \tag{2.13}$$

For $m, l \ge 1$ Yang [31] established the following inequality:

$$\int_0^a |x(t)|^l |x'(t)|^m dt \le \frac{m}{l+m} a^l \int_0^a |x'(t)|^{m+l} dt. \tag{2.14}$$

Obviously, for $m, l \ge 1$, (2.13) is sharper than (2.14).

Remark 2.4. Taking for $n = m = 1$ and $\kappa = \lambda = 0$ in (2.10), (2.10) reduces to

$$\int_0^a \int_0^b |x(s,t)|^l \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^m dsdt \le c^*_{1,1} (ab)^l \int_0^a \int_0^b \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^{m+l} dsdt. \tag{2.15}$$

Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, (2.15) becomes the following inequality:

$$\int_0^a |x(t)|^l |x'(t)|^m dt \le \xi m^{m\xi} a^l \int_0^a |x'(t)|^{m+l} dt, \quad \xi = (l+m)^{-1}.$$

This is just an inequality established by Yang [31].

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