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Positive Integer Powers of One Type of Complex Tridiagonal Matrix

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Abstract. In this paper, we firstly present a general expression for the entries of the rth ($r \in \mathbb{N}$) power of a certain n-square complex tridiagonal matrix, in terms of the Chebyshev polynomials of the first kind. Secondly, we obtain two complex factorizations for Fibonacci and Pell numbers. We also give some Maple 13 procedures in order to verify our calculations.

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1. Introduction

In recent years, computing the integer powers of tridiagonal matrices has been a very popular problem. Rimas investigated positive integer powers of certain tridiagonal matrices of odd and even order depending on the Chebyshev polynomials [10, 11]. Gutierrez generalized some papers of Rimas [3–7]. Eigenvalues of certain tridiagonal matrices are investigated in many papers [2, 8].

In this paper, we obtain the entries of positive integer powers of an n-square complex tridiagonal matrix, which is a generalization for rth power of certain tridiagonal matrices given in [10, 11],

$$B = \begin{bmatrix} a & 2b & & & & \\ -b & a & b & & 0 & \\ & -b & a & \ddots & & \\ & & \ddots & \ddots & b & \\ & 0 & & -b & a & b \\ & & & & -2b & a \end{bmatrix}, \tag{1.1}$$

where $b \neq 0$ and $a, b \in \mathbb{C}$. We also give complex factorization formulas for the Fibonacci and Pell numbers.

Now, we begin with following lemma.

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Lemma 1.1. [1] *Let* $\{H(n),\ n=1,2,\ldots\}$ *be sequence of tridiagonal matrices of the form*

$$H(n)=\begin{bmatrix} h_{1,1} & h_{1,2} & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & 0 & \\ & h_{3,2} & h_{3,3} & \ddots & \\ & 0 & \ddots & \ddots & h_{n-1,n} \\ & & & h_{n,n-1} & h_{n,n} \end{bmatrix}.$$

Then the succesive determinants of $H(n)$ *are given by the recursive formula:*

$$\begin{aligned} |H(1)| &= h_{1,1}, \\ |H(2)| &= h_{1,1}h_{2,2}-h_{1,2}h_{2,1}, \\ |H(n)| &= h_{n,n}\,|H(n-1)|-h_{n-1,n}h_{n,n-1}\,|H(n-2)|. \end{aligned}$$

Let $H^{\dagger}(n)$ and $H^{*}(n)$ be n-square two tridiagonal matrices as the following:

$$\begin{aligned} H^{\dagger}(n) &= \begin{bmatrix} h_{1,1} & h_{1,2} & & & \\ -h_{2,1} & h_{2,2} & h_{2,3} & & \\ & -h_{3,2} & h_{3,3} & \ddots & \\ & & \ddots & \ddots & h_{n-1,n} \\ & & & -h_{n,n-1} & h_{n,n} \end{bmatrix}, \\ H^{*}(n) &= \begin{bmatrix} h_{1,1} & -h_{1,2} & & & \\ h_{2,1} & h_{2,2} & -h_{2,3} & & \\ & h_{3,2} & h_{3,3} & \ddots & \\ & & \ddots & \ddots & -h_{n-1,n} \\ & & & h_{n,n-1} & h_{n,n} \end{bmatrix} \end{aligned}$$

From Lemma 1.1, since the matrices $H^{\dagger}(n)$ and $H^{*}(n)$ have the same recursive formula, it can be written that

$$\left|H^{\dagger}(n)\right| = |H^{*}(n)|. \tag{1.2}$$

2. Main results

In this section, we firstly give the eigenvalues and eigenvectors of the matrix B given by (1.1). Secondly, we present a general expression for the entries of B^r for $r\in\mathbb{N}$.

Let U be the following n-square tridiagonal matrix

$$U := \begin{bmatrix} 0 & 2 & & & & \\ -1 & 0 & 1 & & 0 & \\ & -1 & 0 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & -1 & 0 & 1 \\ & & & & -2 & 0 \end{bmatrix}.$$

By using (1.2), we write the characteristic polynomial of U as the following:

$$|tI-U| = \begin{vmatrix} t & 2 & & & & \\ -1 & t & 1 & & 0 & \\ & -1 & t & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & -1 & t & 1 \\ & & & & -2 & t \end{vmatrix}, \tag{2.1}$$

and from [11], the eigenvalues of U are

$$t_k = -2\mathbf{i}\cos\frac{(k-1)\pi}{n-1}, \text{ for } k=1,2,\dots,n \tag{2.2}$$

where t_k denotes kth eigenvalue of the matrix U and $i := \sqrt{-1}$.

Lemma 2.1. *Let Q be the following n-square tridiagonal matrix*

$$Q = \begin{bmatrix} a & 2 & & & & \\ -1 & a & 1 & & 0 & \\ & -1 & a & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & -1 & a & 1 \\ & & & & -2 & a \end{bmatrix} \tag{2.3}$$

where $a \in \mathbb{C}$. Then the eigenvalues of Q are

$$\mu_k = a - 2\mathbf{i}\cos\frac{(k-1)\pi}{n-1}, \textit{ for } k=1,2,\dots,n \tag{2.4}$$

where μ_k denotes kth eigenvalue of the matrix Q.

Proof. Eigenvalues of Q are the roots of its characteristic polynomial. From (1.2), we write the characteristic polynomial of Q to be

$$|\mu I - Q| = \begin{vmatrix} \mu-a & 2 & & & & \\ -1 & \mu-a & 1 & & 0 & \\ & -1 & \mu-a & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & -1 & \mu-a & 1 \\ & & & & -2 & \mu-a \end{vmatrix}.$$

Substituting $t = \mu - a$ and taking (2.1) and (2.2) into account, we find the eigenvalues of the matrix Q as

$$\mu_k = a - 2\mathbf{i}\cos\frac{(k-1)\pi}{n-1}, \text{ for } k=1,2,\dots,n. \qquad \blacksquare$$

Theorem 2.1. *Let us consider the matrix B given by (1.1). Then the eigenvalues of B are*

$$\lambda_k = a - 2\mathbf{i}b\cos\frac{(k-1)\pi}{n-1}, \textit{ for } k=1,2,\dots,n \tag{2.5}$$

where λ_k denotes kth eigenvalue of the matrix B and $b \neq 0$.

Proof. In order to prove the theorem, we need a relation between the matrices B and Q. Dividing all entries of B by nonzero b, we get a new matrix M normalized the upper and lower sub-diagonals such that

$$M = \begin{bmatrix} a/b & 2 & & & & \\ -1 & a/b & 1 & & 0 & \\ & -1 & a/b & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & -1 & a/b & 1 \\ & & & & -2 & a/b \end{bmatrix}.$$

Taking (2.3) and (2.4) into account, we find the eigenvalues of M to be

$$\frac{a}{b} - 2\mathbf{i}\cos\frac{(k-1)\pi}{n-1}, \text{ for } k = 1,2,\ldots,n.$$

Since the eigenvalues of B are just b times the eigenvalues of M, we get

$$\lambda_k = a - 2\mathbf{i}b\cos\frac{(k-1)\pi}{n-1}, \text{ for } k = 1,2,\ldots,n,$$

and the proof is completed. It is easy to see that all eigenvalues are simple. ∎

Each eigenvector of the matrix B is the solution of the following homogeneous linear equations system

$$(\lambda_j I - B)x = 0, \tag{2.6}$$

where λ_j is the *jth eigenvalue of the matrix* B $(1 \le j \le n)$. We clearly write the expression (2.6) as follows:

$$\begin{aligned} (\lambda_j - a)x_1 - 2bx_2 &= 0 \\ -bx_1 + (\lambda_j - a)x_2 - bx_3 &= 0 \\ -bx_2 + (\lambda_j - a)x_3 - bx_4 &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ -bx_{n-2} + (\lambda_j - a)x_{n-1} - bx_n &= 0 \\ -2bx_{n-1} + (\lambda_j - a)x_n &= 0. \end{aligned} \tag{2.7}$$

Since all eigenvalues are simple, each eigenvector corresponds to a different eigenvalue and rank of the coefficient matrix can be written as $rank(\lambda_j I - B) = n-1$. Dividing all terms of the each equations in (2.7) by $b \neq 0$, substituting $\delta_j = (\lambda_j - a)/b$, choosing $x_1 = 1$ arbitrarily and solving the set of systems (2.7) according to x_1, we find the eigenvectors of the matrix B as

$$x_{jk} = e(j-1)T_{k-1}\left(\frac{\mathbf{i}\delta_j}{2}\right) \text{ for } j,k = 1,2,\ldots,n, \tag{2.8}$$

where $T_k(x)$ is the *kth* degree Chebyshev polynomial of the first kind [9]:

$$T_k(x) = \cos k(\arccos x), \; -1 \le x \le 1,$$

and

$$e(k) = e^{-\mathbf{i}\frac{k\pi}{n}} \text{ for } k = 0,1,2,3,\ldots.$$

General expression for the entries of B^r

Consider the relation $B = NJN^{-1}$, where J is the Jordan's form of B and N is the transforming matrix. In order to get the general expression for the entries of B^r, we firstly find the matrices J and N.

Since all the eigenvalues λ_k $(k = 1,2,\ldots,n)$ are simple, each eigenvalue λ_k corresponds single Jordan cell $J_i(\lambda_k)$ in the matrix J. Taking this into account we write down the Jordan's form of the matrix B

$$J = diag(\lambda_1, \lambda_2, \ldots, \lambda_n). \tag{2.9}$$

Let us find the transforming matrix N and its inverse N^{-1}. Denoting jth column of N by N_j, we have $N = (N_1, N_2, \ldots, N_n)$. From (2.8) we get

$$N_j = \begin{bmatrix} e(0)\,T_0\left(\frac{\mathbf{i}\delta_j}{2}\right) \\ e(1)\,T_1\left(\frac{\mathbf{i}\delta_j}{2}\right) \\ \vdots \\ e(n-1)\,T_{n-1}\left(\frac{\mathbf{i}\delta_j}{2}\right) \end{bmatrix}, \; j = 1,2,\ldots,n. \tag{2.10}$$

By (2.10), we obtain the transforming matrix N as:

$$N = \begin{bmatrix} e(0)\,T_0\left(\frac{\mathbf{i}\delta_1}{2}\right) & e(0)\,T_0\left(\frac{\mathbf{i}\delta_2}{2}\right) & \ldots & e(0)\,T_0\left(\frac{\mathbf{i}\delta_n}{2}\right) \\ e(1)\,T_1\left(\frac{\mathbf{i}\delta_1}{2}\right) & e(1)\,T_1\left(\frac{\mathbf{i}\delta_2}{2}\right) & \ldots & e(1)\,T_1\left(\frac{\mathbf{i}\delta_n}{2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ e(n-1)\,T_{n-1}\left(\frac{\mathbf{i}\delta_1}{2}\right) & e(n-1)\,T_{n-1}\left(\frac{\mathbf{i}\delta_2}{2}\right) & \ldots & e(n-1)\,T_{n-1}\left(\frac{\mathbf{i}\delta_n}{2}\right) \end{bmatrix}. \tag{2.11}$$

Denoting the jth column of the inverse matrix N^{-1} by τ_j $\left(N^{-1} = (\tau_1, \tau_2, \ldots, \tau_n)\right)$, from [10], we get

$$\tau_j = \alpha_j \begin{bmatrix} f_1\bar{e}(j-1)\,T_{j-1}\left(\frac{\mathbf{i}\delta_1}{2}\right) \\ f_2\bar{e}(j-1)\,T_{j-1}\left(\frac{\mathbf{i}\delta_2}{2}\right) \\ \vdots \\ f_n\bar{e}(j-1)\,T_{j-1}\left(\frac{\mathbf{i}\delta_n}{2}\right) \end{bmatrix}, \; j = 1,2,\ldots,n \tag{2.12}$$

where

$$\beta_k = \begin{cases} 1, & \text{if } k = 1,n, \\ 2, & \text{if } 1 < k < n, \end{cases}, \; f_k = \frac{\beta_k}{2n-2} \text{ and } \alpha_j = \begin{cases} 1, & \text{if } j = 1,n, \\ 2, & \text{if } 1 < j < n. \end{cases}$$

Taking these expressions into account, we write down the matrix N^{-1} as
(2.13)

$$N^{-1}=\begin{bmatrix} \alpha_1 f_1\bar{e}(0)\,T_0\left(\frac{\mathbf{i}\delta_1}{2}\right) & \alpha_2 f_1\bar{e}(1)\,T_1\left(\frac{\mathbf{i}\delta_1}{2}\right) & \cdots & \alpha_n f_1\bar{e}(n-1)\,T_{n-1}\left(\frac{\mathbf{i}\delta_1}{2}\right) \\ \alpha_1 f_2\bar{e}(0)\,T_0\left(\frac{\mathbf{i}\delta_2}{2}\right) & \alpha_2 f_2\bar{e}(1)\,T_1\left(\frac{\mathbf{i}\delta_2}{2}\right) & \cdots & \alpha_n f_2\bar{e}(n-1)\,T_{n-1}\left(\frac{\mathbf{i}\delta_2}{2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 f_n\bar{e}(0)\,T_0\left(\frac{\mathbf{i}\delta_n}{2}\right) & \alpha_2 f_n\bar{e}(1)\,T_1\left(\frac{\mathbf{i}\delta_n}{2}\right) & \cdots & \alpha_n f_n\bar{e}(n-1)\,T_{n-1}\left(\frac{\mathbf{i}\delta_n}{2}\right) \end{bmatrix}$$

By combining (2.9), (2.11) and (2.13) and using the equalities $B^r=NJ^rN^{-1}$and $J^r=(\lambda_1^r,\lambda_2^r,\ldots,\lambda_n^r)$, we compute the rth powers of the matrix B of order n. (i,j)th entry of the matrix $B^r=[s_{ij}]$ can be given as:

$$s_{ij}=\alpha_j\sum_{k=1}^{n} f_k(\lambda_k)^r e(i-1)\bar{e}(j-1)T_{i-1}\left(\frac{\mathbf{i}\delta_k}{2}\right)T_{j-1}\left(\frac{\mathbf{i}\delta_k}{2}\right)\ \text{for } i,j=1,2,...,n,$$

or, by substituting $\delta_k=(\lambda_k-a)/b$,
(2.14)

$$s_{ij}=\alpha_j\sum_{k=1}^{n} f_k(\lambda_k)^r e(i-1)\bar{e}(j-1)T_{i-1}\left(\frac{\mathbf{i}(\lambda_k-a)}{2b}\right)T_{j-1}\left(\frac{\mathbf{i}(\lambda_k-a)}{2b}\right)\ \text{for } i,j=1,2,...,n.$$

3. Numerical considerations

Taking into account the derived expressions, we can compute arbitrary positive integer powers of the matrix given by (1.1).

Example 3.1. Let us consider the matrix B for $n=4,\ a=2$ and $b=3$, and recall the matrix as B_1

$$B_1=\begin{bmatrix} 2 & 6 & 0 & 0 \\ -3 & 2 & 3 & 0 \\ 0 & -3 & 2 & 3 \\ 0 & 0 & -6 & 2 \end{bmatrix}.$$

3th power of B_1 is computed as in the following.

From (2.5), eigenvalues of the matrix B_1 can be written for $k=1,2,3,4$ as:

$$\lambda_k=2-6\mathbf{i}\cos\frac{(k-1)\pi}{3},$$

namely, $\lambda_1=2-6\mathbf{i}$, $\lambda_2=2-3\mathbf{i}$, $\lambda_3=2+3\mathbf{i}$ and $\lambda_4=2+6\mathbf{i}$. We also write the transforming matrix N_1, whose columns consist of eigenvectors of B_1, and its inverse as:

$$N_1=\begin{bmatrix} e(0)\,T_0\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) & e(0)\,T_0\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) & e(0)\,T_0\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) & e(0)\,T_0\left(\frac{\mathbf{i}(\lambda_4-2)}{6}\right) \\ e(1)\,T_1\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) & e(1)\,T_1\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) & e(1)\,T_1\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) & e(1)\,T_1\left(\frac{\mathbf{i}(\lambda_4-2)}{6}\right) \\ e(2)\,T_2\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) & e(2)\,T_2\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) & e(2)\,T_2\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) & e(2)\,T_2\left(\frac{\mathbf{i}(\lambda_4-2)}{6}\right) \\ e(3)\,T_3\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) & e(3)\,T_3\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) & e(3)\,T_3\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) & e(3)\,T_3\left(\frac{\mathbf{i}(\lambda_4-2)}{6}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\mathbf{i} & -\frac{\mathbf{i}}{2} & \frac{\mathbf{i}}{2} & \mathbf{i} \\ -1 & \frac{1}{2} & \frac{1}{2} & -1 \\ \mathbf{i} & -\mathbf{i} & \mathbf{i} & -\mathbf{i} \end{bmatrix}$$

and

$$N_1^{-1} = \begin{bmatrix} \alpha_1 f_1 \bar{e}(0) T_0\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) & \alpha_2 f_1 \bar{e}(1) T_1\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) & \alpha_3 f_1 \bar{e}(2) T_2\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) & \alpha_4 f_1 \bar{e}(3) T_3\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) \\ \alpha_1 f_2 \bar{e}(0) T_0\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) & \alpha_2 f_2 \bar{e}(1) T_1\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) & \alpha_3 f_2 \bar{e}(2) T_2\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) & \alpha_4 f_2 \bar{e}(3) T_3\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) \\ \alpha_1 f_3 \bar{e}(0) T_0\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) & \alpha_2 f_3 \bar{e}(1) T_1\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) & \alpha_3 f_3 \bar{e}(2) T_2\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) & \alpha_4 f_3 \bar{e}(3) T_3\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) \\ \alpha_1 f_4 \bar{e}(0) T_0\left(\frac{\mathbf{i}(\lambda_4-2)}{6}\right) & \alpha_2 f_4 \bar{e}(1) T_1\left(\frac{\mathbf{i}(\lambda_4-2)}{6}\right) & \alpha_3 f_4 \bar{e}(2) T_2\left(\frac{\mathbf{i}(\lambda_4-2)}{6}\right) & \alpha_4 f_4 \bar{e}(3) T_3\left(\frac{\mathbf{i}(\lambda_4-2)}{6}\right) \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 2\mathbf{i} & -2 & -\mathbf{i} \\ 2 & 2\mathbf{i} & 2 & 2\mathbf{i} \\ 2 & -2\mathbf{i} & 2 & -2\mathbf{i} \\ 1 & -2\mathbf{i} & -2 & \mathbf{i} \end{bmatrix}.$$

Then we get

$$B_1^3 = N_1 J_1^3 N_1^{-1} = \begin{bmatrix} -100 & -90 & 108 & 54 \\ 45 & -154 & -99 & 54 \\ 54 & 99 & -154 & -45 \\ -54 & 108 & 90 & -100 \end{bmatrix}.$$

(i, j)th entry of the B_1^3, can be verified by formula given in (2.14).

Example 3.2. Let us consider the matrix B for $n = 3,\ a = 1 - \mathbf{i}$ and $b = 5 + 2\mathbf{i}$, and recall the matrix as B_2

$$B_2 = \begin{bmatrix} 1-\mathbf{i} & 10+4\mathbf{i} & 0 \\ -5-2\mathbf{i} & 1-\mathbf{i} & 5+2\mathbf{i} \\ 0 & -10-4\mathbf{i} & 1-\mathbf{i} \end{bmatrix}.$$

2th power of B_2 is computed as in the following.

From (2.5), eigenvalues of the matrix B_2 can be written for $k = 1, 2, 3$ as:

$$\lambda_k = (1-\mathbf{i}) + (4-10\mathbf{i})\cos\frac{(k-1)\pi}{2},$$

namely, $\lambda_1 = 5 - 11\mathbf{i},\ \lambda_2 = 1 - \mathbf{i}$ and $\lambda_3 = -3 + 9\mathbf{i}$. We also write the transforming matrix N_2, whose columns consist of eigenvectors of the B_2, and its inverse as:

$$N_2 = \begin{bmatrix} e(0) T_0\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) & e(0) T_0\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) & e(0) T_0\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) \\ e(1) T_1\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) & e(1) T_1\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) & e(1) T_1\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) \\ e(2) T_2\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) & e(2) T_2\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) & e(2) T_2\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -\mathbf{i} & 0 & \mathbf{i} \\ -1 & 1 & -1 \end{bmatrix}$$

and

$$N_2^{-1} = \begin{bmatrix} \alpha_1 f_1 \bar{e}(0) T_0\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) & \alpha_2 f_1 \bar{e}(1) T_1\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) & \alpha_3 f_1 \bar{e}(2) T_2\left(\frac{\mathbf{i}(\lambda_1-2)}{6}\right) \\ \alpha_1 f_2 \bar{e}(0) T_0\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) & \alpha_2 f_2 \bar{e}(1) T_1\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) & \alpha_3 f_2 \bar{e}(2) T_2\left(\frac{\mathbf{i}(\lambda_2-2)}{6}\right) \\ \alpha_1 f_3 \bar{e}(0) T_0\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) & \alpha_2 f_3 \bar{e}(1) T_1\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) & \alpha_3 f_3 \bar{e}(2) T_2\left(\frac{\mathbf{i}(\lambda_3-2)}{6}\right) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 2\mathbf{i} & -1 \\ 2 & 0 & 2 \\ 1 & -2\mathbf{i} & -1 \end{bmatrix}.$$

Then we get

$$B_2^2 = N_2 J_2^2 N_2^{-1} = \begin{bmatrix} -42-42\mathbf{i} & 28-12\mathbf{i} & 42+40\mathbf{i} \\ -14+6\mathbf{i} & -84-82\mathbf{i} & 14-6\mathbf{i} \\ 42+40\mathbf{i} & -28+12\mathbf{i} & -42-42\mathbf{i} \end{bmatrix}.$$

(i,j)th entry of the B_2^2, can be verified by formula given in (2.14).

4. Complex factorization of Fibonacci and Pell numbers

In this section we find two complex factorization formulas for the Fibonacci and Pell numbers in terms of determinant of the matrix B given in (1.1). Calculations given in this section can be verified by using Maple 13 procedures given in Appendix B.

In [1, Section 2], authors obtained that

$$|tridiag_n(\mathbf{i},1,\mathbf{i})| = F_{n+1}. \tag{4.1}$$

In [12], authors also obtained that

$$m\,|tridiag_n(1,2\mathbf{i},1)| = P_{n+1} \tag{4.2}$$

where

$$m = \begin{cases} 1 & n \equiv 0 \pmod 4, \\ -\mathbf{i} & n \equiv 1 \pmod 4, \\ -1 & n \equiv 2 \pmod 4, \\ \mathbf{i} & n \equiv 3 \pmod 4. \end{cases}$$

Equality (4.2) can be written as

$$|tridiag_n(\mathbf{i},2,\mathbf{i})| = P_{n+1}. \tag{4.3}$$

Theorem 4.1. *Let B be n-square matrix as in (1.1). Then*

$$\det(B) = \begin{cases} (1+2\mathbf{i})F_n & \text{if } a=1 \text{ and } b=\mathbf{i}, \\ (2+2\mathbf{i})P_n & \text{if } a=2 \text{ and } b=\mathbf{i}, \end{cases} \tag{4.4}$$

where F_n and P_n denote nth Fibonacci and Pell numbers.

Proof. By applying the Laplace expansion according to the first and last rows of B, we get

$$\begin{aligned} |B| &= (a+b)^2\,|tridiag_{n-2}(b,a,b)| \\ &\quad - 2b^2(a+b)\,|tridiag_{n-3}(b,a,b)| + b^4\,|tridiag_{n-4}(b,a,b)|. \end{aligned} \tag{4.5}$$

If we choose $a=1$ and $b=\mathbf{i}$ in (4.5), and take (4.1) into account, we write

$$\begin{aligned} \det(B) &= (1+\mathbf{i})^2\,|tridiag_{n-2}(\mathbf{i},1,\mathbf{i})| + 2(1+\mathbf{i})\,|tridiag_{n-3}(\mathbf{i},1,\mathbf{i})| + |tridiag_{n-4}(\mathbf{i},1,\mathbf{i})| \\ &= (1+\mathbf{i})^2 F_{n-1} + 2(1+\mathbf{i})F_{n-2} + F_{n-3} \\ &= (1+2\mathbf{i})F_n. \end{aligned}$$

If we also choose $a=2$ and $b=\mathbf{i}$ in (4.5), and take (4.3) into account, we write

$$\begin{aligned} \det(B) &= (2+\mathbf{i})^2\,|tridiag_{n-2}(\mathbf{i},2,\mathbf{i})| + 2(2+\mathbf{i})\,|tridiag_{n-3}(\mathbf{i},2,\mathbf{i})| + |tridiag_{n-4}(\mathbf{i},2,\mathbf{i})| \\ &= (2+\mathbf{i})^2 P_{n-1} + 2(2+\mathbf{i})P_{n-2} + P_{n-3} \end{aligned}$$

$$= (2+2\mathbf{i})\,P_n.$$

Thus, the proof completes. ∎

Conclusion. Let n-square matrix B be as in (1.1). Then, complex factorization of the Fibonacci and Pell numbers are

$$F_n = \prod_{k=1}^{n-1} \left[1 - 2\mathbf{i}\cos\frac{k\pi}{n}\right]$$

and

$$P_n = \prod_{k=1}^{n-1} \left[2 - 2\mathbf{i}\cos\frac{k\pi}{n}\right].$$

Proof. Since eigenvalues of B for $k = 1,2,\ldots,n$ are

$$\lambda_k = a - 2b\cos\frac{k\pi}{n}$$

from (2.5), determinant of B can be written as

$$\det(B) = \prod_{k=1}^{n} \left[a - 2b\cos\frac{k\pi}{n}\right].$$

By using (4.4), we write

$$(1+2\mathbf{i})\,F_n = \prod_{k=1}^{n} \left[1 - 2\mathbf{i}\cos\frac{k\pi}{n}\right]$$

and

$$(2+2\mathbf{i})\,P_n = \prod_{k=1}^{n} \left[2 - 2\mathbf{i}\cos\frac{k\pi}{n}\right].$$

So, we obtain

$$\begin{aligned} F_n &= \frac{1}{1+2\mathbf{i}} \prod_{k=1}^{n} \left[1 - 2\mathbf{i}\cos\frac{k\pi}{n}\right] \\ &= \prod_{k=1}^{n-1} \left[1 - 2\mathbf{i}\cos\frac{k\pi}{n}\right] \end{aligned}$$

as in [1] and

$$\begin{aligned} P_n &= \frac{1}{2+2\mathbf{i}} \prod_{k=1}^{n} \left[2 - 2\mathbf{i}\cos\frac{k\pi}{n}\right] \\ &= \prod_{k=1}^{n-1} \left[2 - 2\mathbf{i}\cos\frac{k\pi}{n}\right]. \end{aligned}$$

Thus, proof is completed. ∎

Appendix A. Following Maple 13 procedure calculates the rth power of n-square complex tridiagonal matrix given in (1.1) and (i,j)th entry of B^r for $b \neq 0$.

```
restart:
with(LinearAlgebra):
power:=proc(n,r,a,b,i,j)
local c,s,B,e,f, lambda, delta, T, alpha, beta, M;
c:=(i,j)->piecewise(i=1 and j=2,2*b,i=n and
j=n-1,-2*b,i=j,a,j-i=1,b,j-i=-1,-b,0);
B:=Matrix(n,n,c):
lambda:=(k)->evalf(a-2*I*b*cos((k-1)*Pi/(n-1)));
delta:=(j)->((lambda(j)-a)/b);
T:=(k,x)->evalf(cos(k*arccos(x)));
e:=(k)->exp(-I*(k*Pi)/2);
alpha:=(j)->piecewise(j=1,1,j=n,1,1>j and j>n,2,0);
beta:=(k)->piecewise(k=1,1,k=n,1,1>k and k>n,2,0);
f:=(k)->(beta(k)/(2*n-2));
s:=(i,j)->alpha(j)*sum(f(k)*lambda(k)\symbol{94}%
r*e(i-1)*conjugate(e(j-1))*T(i-1,1/2*I*delta(k))*T(j-1,1/2*I*delta(k)),k = 1
.. n);
M:=Matrix(n,n,s);
print(M);
print(s(i,j));
end proc:
power( , , , , , );
```

Appendix B. **(i)** Following Maple 13 procedure calculates n-square matrix B given in (1.1) for $a = 1$ and $b = \mathbf{i}$, determinant of B and complex factorization formula for Fibonacci numbers given in Conclusion.

```
restart:
with(LinearAlgebra):
F:=proc(n)
local c,B,Factorization;
c:=(i,j)->piecewise(i=1 and j=1,1+I,i=n and
j=n,1+I,i=j,1,abs(i-j)=1,I,0);
B:=Matrix(n,n,c):
Factorization:=(1/(1+2*I))*product(1-2*I*cos(k*Pi/n),k=1..n);
print(B);
print(Determinant(B));
print(evalf(Factorization));
end proc:
F( );
```

(ii) Following Maple 13 procedure calculates n-square matrix B given in (1.1) for $a = 2$ and $b = \mathbf{i}$, determinant of B and complex factorization formula for Pell numbers given in Conclusion.

```
restart:
with(LinearAlgebra):
P:=proc(n)
```

```
local c,B,Factorization;
c:=(i,j)->piecewise(i=1 and j=1,2+I,i=n and
j=n,2+I,i=j,2,abs(i-j)=1,I,0);
B:=Matrix(n,n,c):
Factorization:=(1/(2+2*I))*product(2-2*I*cos(k*Pi/n),k=1..n);
print(B);
print(Determinant(B));
print(evalf(Factorization));
end proc:
P( );
```

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