DIFFERENTIAL SANDWICH THEOREMS FOR CERTAIN ANALYTIC FUNCTIONS

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Abstract

Let \( q_1, q_2 \) be univalent in \( \Delta := \{ z : |z| < 1 \} \). We give some applications of first order differential superordinations to obtain sufficient conditions for normalized analytic functions \( f(z) \) to satisfy

\[ q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z). \]

1. Introduction

Let \( \mathcal{H} \) be the class of analytic functions in \( \Delta := \{ z : |z| < 1 \} \) and \( \mathcal{H}(a, n) \) be the subclass of \( \mathcal{H} \) consisting of functions of the form \( f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \ldots \). Let \( \mathcal{A} \) be the class of all analytic functions \( f(z) = z + a_2z^2 + \ldots (z \in \Delta) \). Let \( p, h \in \mathcal{H} \) and let \( \phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \rightarrow \mathbb{C} \). If \( p \) and \( \phi(p(z), zp'(z), z^2p''(z); z) \) are univalent and if \( p \) satisfies the second order superordination

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then $p$ is a solution of the differential superordination (1.1). (If $f$ is subordinate to $F$, then $F$ is superordinate to $f$.) An analytic function $q$ is called a subordinant if $q < p$ for all $p$ satisfying (1.1). A univalent subordinant $\tilde{q}$ that satisfies $q < \tilde{q}$ for all subordinants $q$ of (1.1) is said to be best subordinant. Recently Miller and Mocanu [3] obtained conditions on $h$, $q$ and $\phi$ for which the following implication holds:

$$h(z) < \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) < p(z).$$

Using the results of Miller and Mocanu [3], Bulboaca [2] have considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [1]. In the present paper, we give some applications of first order differential superordinations for functions in $A$.

In our present investigation, we shall need the following:

**Definition 1.1** [3, Definition 2, p. 817]. Denote by $Q$, the set of all functions $f(z)$ that are analytic and injective on $\Delta - E(f)$, where

$$E(f) = \{\zeta \in \partial \Delta : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial \Delta - E(f)$.

**Lemma 1.2** [2]. Let $q(z)$ be univalent in the unit disk $\Delta$ and $\phi$ and $\psi$ be analytic in a domain $D$ containing $q(\Delta)$. Suppose that

1. $\Re[\phi'(q(z))/\psi(q(z))] \geq 0$ for $z \in \Delta$,

2. $zq'(z)\psi(q(z))$ is starlike univalent in $\Delta$.

If $p(z) \in \mathcal{H}(q(0), 1) \cap Q$, with $p(\Delta) \subset D$, and $q(p(z)) + zp'(z)\psi(p(z))$ is univalent in $\Delta$, then

$$q(q(z)) + zq'(z)\psi(q(z)) < q(p(z)) + zp'(z)\psi(p(z)) \quad (1.2)$$

implies $q(z) < p(z)$ and $q(z)$ is the best subordinant.
2. Sandwich Theorems

By making use of Lemma 1.2, we obtain the following results.

**Lemma 2.1.** Let \( q(z) \) be convex univalent in \( \Delta \) and \( \alpha, \beta, \gamma \in \mathbb{C} \). Further assume that

\[
\Re \left[ \frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z) \right] \geq 0.
\]

If \( p(z) \in \mathcal{H}(q(0), 1) \cap Q \), \( \alpha p(z) + \beta p^2(z) + \gamma p'(z) \) is univalent in \( \Delta \), then

\[
\alpha q(z) + \beta q^2(z) + \gamma q'(z) < \alpha p(z) + \beta p^2(z) + \gamma p'(z)
\]

implies \( q(z) < p(z) \) and \( q(z) \) is the best subordinant.

**Proof.** Define the functions \( \beta \) and \( \varphi \) by

\[
\beta(w) := \alpha w + \beta w^2 \quad \text{and} \quad \varphi(w) := \gamma.
\]

Clearly, \( \beta(w) \) and \( \varphi(w) \) are analytic in \( \mathbb{C} \). Also

\[
\Re \left[ \frac{\beta(q(z))}{\varphi(q(z))} \right] = \Re \left[ \frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z) \right] \geq 0
\]

and the function \( \gamma q'(z) \) is starlike univalent in \( \Delta \). Lemma 2.1 now follows by an application of Lemma 1.2.

**Remark 1.** When \( \alpha = 1 \) and \( \beta = 0 \), Lemma 2.1 reduces to [3, Theorem 8, p. 822]. When \( \alpha = \beta = 0 \) and \( \gamma = 1 \) Lemma 2.1 reduces to [3, Theorem 9, p. 823].

By making use of Lemma 2.1, we now prove the following:

**Theorem 2.2.** Let \( \alpha \in \mathbb{C} \). Let \( q(z) \) be convex univalent in \( \Delta \) and \( \Re q(z) \)

\[
\geq \Re \frac{\alpha - 1}{2\alpha}.
\]

If \( f \in \mathcal{A}, zf'(z)/f(z) \in \mathcal{H}(1, 1) \cap Q \), then \( z \frac{f'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \) is univalent in \( \Delta \), then

\[
(1 - \alpha) q(z) + \alpha q^2(z) + \alpha q'(z) < \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}
\]
implies

\[ q(z) < \frac{zf'(z)}{f(z)} \]

and \( q(z) \) is the best subordinant.

**Proof.** Define the function \( p(z) \) by

\[ p(z) := \frac{zf'(z)}{f(z)}. \]

Then a computation shows that

\[ \frac{zf'(z)}{f(z)} + \alpha \frac{z^2f''(z)}{f(z)} = (1 - \alpha)p(z) + \alpha p^2(z) + \alpha zp'(z). \]

By using Lemma 2.1, we have the result.

Together with the corresponding result for differential subordination (see Ravichandran [4]), we obtain the following "sandwich result":

**Corollary 2.3.** Let \( q_1(z) \) and \( q_2(z) \) be convex univalent in \( \Delta \). Let \( \alpha \in \mathbb{C} \). Assume that \( \Re q_i(z) \geq \Re \frac{\alpha - 1}{2\alpha} \) for \( i = 1, 2 \). If \( f \in A, zf'(z)/f(z) \in \mathcal{H}(1, 1) \cap Q, \frac{zf'(z)}{f(z)} + \alpha \frac{z^2f''(z)}{f(z)} \) is univalent in \( \Delta \), then

\[
(1 - \alpha)q_1(z) + \alpha q_1^2(z) + \alpha zq_1'(z) < \frac{zf'(z)}{f(z)} + \alpha \frac{z^2f''(z)}{f(z)} < (1 - \alpha)q_2(z) + \alpha q_2^2(z) + \alpha zq_2'(z)
\]

implies

\[ q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z) \]

and \( q_1(z) \) and \( q_2(z) \) are respectively the best subordinant and best dominant.

**Lemma 2.4.** Let \( q(z) \neq 0 \) be univalent in \( \Delta \) and \( \alpha, \beta \in \mathbb{C} \). Further assume that \( \Re [\alpha \beta q(z)] \geq 0 \) and \( zq'(z)/q(z) \) is starlike univalent in \( \Delta \). If \( p(z) \in \mathcal{H}(q(0), 1) \cap Q, p(z) \neq 0, \alpha p(z) + \beta zp'(z)/p(z) \) is univalent in \( \Delta \), then
\[ a q(z) + \beta \frac{z q'(z)}{q(z)} < a p(z) + \beta \frac{z p'(z)}{p(z)} \]

implies \( q(z) < p(z) \) and \( q(z) \) is the best subordinant.

**Proof.** The Lemma 2.4 follows from Lemma 1.2 when the functions \( \varphi \) and \( \varphi \) are given by \( \varphi(w) := \alpha w \) and \( \varphi(w) := \beta/w \).

By making use of Lemma 2.4, we now prove the following:

**Theorem 2.5.** Let \( \alpha \in \mathbb{C} \). Let \( q(z) \neq 0 \) be univalent in \( \Delta \). Further assume that \( \Re[\alpha q(z)] \geq 0 \) and \( z q'(z)/q(z) \) is starlike univalent in \( \Delta \). If \( f \in A, \) \( 0 \neq z f''(z)/f(z) \in H(1,1) \cap Q, \) \( (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f(z)} \right) \) is univalent in \( \Delta \), then

\[ q(z) + \alpha \frac{z q'(z)}{q(z)} < (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f(z)} \right) \]

implies

\[ q(z) < \frac{zf'(z)}{f(z)} \]

and \( q(z) \) is the best subordinant.

**Proof.** Theorem 2.5 follows from Lemma 2.4 by taking \( p(z) \) to be the function given by \( p(z) := zf'(z)/f(z) \).

Together with the corresponding result for differential subordination (see Ravichandran and Darus [6]), we obtain the following:

**Corollary 2.6.** Let \( \alpha \in \mathbb{C} \). Let \( q_i(z) \neq 0 \) \( (i = 1, 2) \) be univalent in \( \Delta \). Further assume that \( \Re[\alpha q_i(z)] \geq 0 \) for \( i = 1, 2 \) and \( z q_i'(z)/q_i(z) \) \( (i = 1, 2) \) is starlike univalent in \( \Delta \). If \( f \in A, \) \( 0 \neq z f''(z)/f(z) \in H(1,1) \cap Q, \) \( (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f(z)} \right) \)

\[ + \alpha \left(1 + \frac{zf''(z)}{f(z)} \right) \] is univalent in \( \Delta \), then

\[ q_1(z) + \alpha \frac{z q_1'(z)}{q_1(z)} < (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f(z)} \right) < q_2(z) + \alpha \frac{z q_2'(z)}{q_2(z)} \]
implies

\[ q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z) \]

and \( q_1(z) \) and \( q_2(z) \) are respectively the best subordinant and the best dominant.

By making use of Lemma 2.4, we obtain the following:

**Theorem 2.7.** Let \( q(z) \neq 0 \) be univalent in \( \Delta \) and \( zq'(z)/q(z) \) be starlike univalent in \( \Delta \). If \( f \in \mathcal{A}, 0 < z^2f'(z)/f(z) \in \mathcal{H}(1,1) \cap Q, \frac{(zf')''(z)}{f'(z)} - 2 \frac{zf'(z)}{f(z)} \) is univalent in \( \Delta \), then

\[ \frac{zq'(z)}{q(z)} < \frac{(zf')'(z)}{f'(z)} - 2 \frac{zf'(z)}{f(z)} \]

implies

\[ q(z) < \frac{z^2f'(z)}{f^2(z)} \]

and \( q(z) \) is the best subordinant.

Together with the corresponding result for differential subordination (see Ravichandran [4]), we obtain the following:

**Corollary 2.8.** Let \( q_i(z) \neq 0 \) be univalent in \( \Delta \) and \( zq_i'(z)/q_i(z) \) be starlike univalent in \( \Delta \) for \( i = 1, 2 \). If \( f \in \mathcal{A}, 0 < z^2f'(z)/f^2(z) \in \mathcal{H}(1,1) \cap Q, \frac{(zf')''(z)}{f'(z)} - 2 \frac{zf'(z)}{f(z)} \) is univalent in \( \Delta \), then

\[ \frac{zq_1(z)}{q_1(z)} < \frac{(zf')'(z)}{f'(z)} - 2 \frac{zf'(z)}{f(z)} < \frac{zq_2(z)}{q_2(z)} \]

implies

\[ q_1(z) < \frac{z^2f'(z)}{f^2(z)} < q_2(z) \]
and \( q_1(z) \) and \( q_2(z) \) are respectively the best subordinant and best dominant.

**Lemma 2.9.** Let \( q(z) \neq 0 \) be univalent in \( \Delta \) and \( zq'(z)/q^2(z) \) be starlike univalent in \( \Delta \). If \( p(z) \in \mathcal{H}(0,1) \cap Q, p(z) \neq 0, \) \( zp'(z)/p^2(z) \) is univalent in \( \Delta \), then

\[
\frac{zq'(z)}{q^2(z)} < \frac{zp'(z)}{p^2(z)}
\]

implies \( q(z) < p(z) \) and \( q(z) \) is the best subordinant.

**Proof.** Lemma 2.9 follows from Lemma 1.2 when \( g(w) := 0 \) and \( \varphi(w) := 1/w^2 \).

**Theorem 2.10.** Let \( q(z) \neq 0 \) be univalent in \( \Delta \) and \( zq'(z)/q^2(z) \) be starlike univalent in \( \Delta \). If \( f \in \mathcal{A}, 0 \neq zf''(z)/f'(z) \in \mathcal{H}(1,1) \cap Q, \) \( 1 + zf''(z)/f'(z) \) is univalent in \( \Delta \), then

\[
1 + \frac{zq'(z)}{q^2(z)} < \frac{1 + zf''(z)/f'(z)}{zf''(z)/f'(z)}
\]

implies \( q(z) < zf''(z)/f'(z) \) and \( q(z) \) is the best subordinant.

**Proof.** The result follows from Lemma 2.9 by taking \( p(z) = zf'(z)/f(z) \).

Together with the corresponding result for differential subordination (see Ravichandran and Darus [5]), we obtain the following:

**Theorem 2.11.** Let \( q_i(z) \neq 0 \) be univalent in \( \Delta \) and \( zq^i(z)/q^2(z) \) be starlike univalent in \( \Delta \) for \( i = 1, 2 \). If \( f \in \mathcal{A}, 0 \neq zf''(z)/f'(z) \in \mathcal{H}(1,1) \cap Q, \) \( 1 + zf''(z)/f'(z) \) is univalent in \( \Delta \), then

\[
1 + \frac{zq_1(z)}{q_1^2(z)} < \frac{1 + zf''(z)/f'(z)}{zf''(z)/f'(z)} < 1 + \frac{zq_2(z)}{q_2^2(z)}
\]

implies \( q_1(z) < zf''(z)/f'(z) < q_2(z) \) and \( q_1(z) \) and \( q_2(z) \) are respectively the best subordinant and the best dominant.
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References


