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DIFFERENTIAL SANDWICH THEOREMS FOR CERTAIN ANALYTIC FUNCTIONS

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Abstract

Let q_1, q_2 be univalent in $\Delta := \{z : |z| < 1\}$. We give some applications of first order differential superordinations to obtain sufficient conditions for normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec zf'(z)/f(z) \prec q_2(z).$$

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\Delta := \{z : |z| < 1\}$ and $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots$. Let $\mathcal{A}$ be the class of all analytic functions $f(z) = z + a_2 z^2 + \ldots$ $(z \in \Delta)$. Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

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$$h(z) \prec \phi(p(z),\ zp'(z),\ z^2p''(z);\ z), \tag{1.1}$$

then p is a solution of the differential superordination (1.1). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be *best subordinant*. Recently Miller and Mocanu [3] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z),\ zp'(z),\ z^2p''(z);\ z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu [3], Bulboaca [2] have considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [1]. In the present paper, we give some applications of first order differential superordinations for functions in $\mathcal{A}$.

In our present investigation, we shall need the following:

Definition 1.1 [3, Definition 2, p. 817]. Denote by Q, the set of all functions $f(z)$ that are analytic and injective on $\overline{\Delta} - E(f)$, where

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Lemma 1.2 [2]. *Let $q(z)$ be univalent in the unit disk Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*

(1) $\Re[\vartheta'(q(z))/\varphi(q(z))] \geq 0$ *for* $z \in \Delta$,

(2) $zq'(z)\varphi(q(z))$ *is starlike univalent in* Δ.

If $p(z) \in \mathcal{H}(q(0), 1) \cap Q$, *with* $p(\Delta) \subseteq D$, *and* $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ *is univalent in* Δ, *then*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)) \tag{1.2}$$

implies $q(z) \prec p(z)$ *and* $q(z)$ *is the best subordinant.*

2. Sandwich Theorems

By making use of Lemma 1.2, we obtain the following results.

Lemma 2.1. *Let* $q(z)$ *be convex univalent in* Δ *and* $\alpha, \beta, \gamma \in \mathbb{C}$. *Further assume that*

$$\Re\left[\frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z)\right] \geq 0.$$

If $p(z) \in \mathcal{H}(q(0), 1) \cap Q$, $\alpha p(z) + \beta p^2(z) + \gamma z p'(z)$ *is univalent in* Δ, *then*

$$\alpha q(z) + \beta q^2(z) + \gamma z q'(z) \prec \alpha p(z) + \beta p^2(z) + \gamma z p'(z)$$

implies $q(z) \prec p(z)$ *and* $q(z)$ *is the best subordinant.*

Proof. Define the functions ϑ and φ by

$$\vartheta(w) := \alpha w + \beta w^2 \text{ and } \varphi(w) := \gamma.$$

Clearly, $\vartheta(w)$ and $\varphi(w)$ are analytic in $\mathbb{C}$. Also

$$\Re \frac{\vartheta'(q(z))}{\varphi(q(z))} = \Re\left[\frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z)\right] \geq 0$$

and the function $\gamma z q'(z)$ is starlike univalent in Δ. Lemma 2.1 now follows by an application of Lemma 1.2.

Remark 1. When $\alpha = 1$ and $\beta = 0$, Lemma 2.1 reduces to [3, Theorem 8, p. 822]. When $\alpha = \beta = 0$ and $\gamma = 1$ Lemma 2.1 reduces to [3, Theorem 9, p. 823].

By making use of Lemma 2.1, we now prove the following:

Theorem 2.2. *Let* $\alpha \in \mathbb{C}$. *Let* $q(z)$ *be convex univalent in* Δ *and* $\Re q(z) \geq \Re \frac{\alpha - 1}{2\alpha}$. *If* $f \in \mathcal{A}$, $zf'(z)/f(z) \in \mathcal{H}(1, 1) \cap Q$, $\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}$ *is univalent in* Δ, *then*

$$(1 - \alpha) q(z) + \alpha q^2(z) + \alpha z q'(z) \prec \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}$$

implies

$$q(z) \prec \frac{zf'(z)}{f(z)}$$

and $q(z)$ is the best subordinant.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{zf'(z)}{f(z)}.$$

Then a computation shows that

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} = (1-\alpha)p(z) + \alpha p^2(z) + \alpha z p'(z).$$

By using Lemma 2.1, we have the result.

Together with the corresponding result for differential subordination (see Ravichandran [4]), we obtain the following "sandwich result":

Corollary 2.3. *Let $q_1(z)$ and $q_2(z)$ be convex univalent in Δ. Let $\alpha \in \mathbb{C}$. Assume that $\Re q_i(z) \geq \Re \frac{\alpha - 1}{2\alpha}$ for $i = 1, 2$. If $f \in \mathcal{A}$, $zf'(z)/f(z) \in \mathcal{H}(1, 1) \cap Q$, $\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}$ is univalent in Δ, then*

$$(1-\alpha)q_1(z) + \alpha q_1^2(z) + \alpha z q_1'(z) \prec \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec (1-\alpha)q_2(z) + \alpha q_2^2(z) + \alpha z q_2'(z)$$

implies

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.

Lemma 2.4. *Let $q(z) \neq 0$ be univalent in Δ and $\alpha, \beta \in \mathbb{C}$. Further assume that $\Re[\alpha \bar{\beta} q(z)] \geq 0$ and $zq'(z)/q(z)$ is starlike univalent in Δ. If $p(z) \in \mathcal{H}(q(0), 1) \cap Q$, $p(z) \neq 0$, $\alpha p(z) + \beta \frac{zp'(z)}{p(z)}$ is univalent in Δ, then*

$$\alpha q(z) + \beta \frac{zq'(z)}{q(z)} \prec \alpha p(z) + \beta \frac{zp'(z)}{p(z)}$$

implies $q(z) \prec p(z)$ *and* $q(z)$ *is the best subordinant.*

Proof. The Lemma 2.4 follows from Lemma 1.2 when the functions ϑ and φ are given by $\vartheta(w) := \alpha w$ and $\varphi(w) := \beta/w$.

By making use of Lemma 2.4, we now prove the following:

Theorem 2.5. *Let* $\alpha \in \mathbb{C}$. *Let* $q(z) \neq 0$ *be univalent in* Δ. *Further assume that* $\Re[\bar{\alpha} q(z)] \geq 0$ *and* $zq'(z)/q(z)$ *is starlike univalent in* Δ. *If* $f \in \mathcal{A}$, $0 \neq zf'(z)/f(z) \in \mathcal{H}(1, 1) \cap Q$, $(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$ *is univalent in* Δ, *then*

$$q(z) + \alpha \frac{zq'(z)}{q(z)} \prec (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

implies

$$q(z) \prec \frac{zf'(z)}{f(z)}$$

and $q(z)$ *is the best subordinant.*

Proof. Theorem 2.5 follows from Lemma 2.4 by taking $p(z)$ to be the function given by $p(z) := zf'(z)/f(z)$.

Together with the corresponding result for differential subordination (see Ravichandran and Darus [6]), we obtain the following:

Corollary 2.6. *Let* $\alpha \in \mathbb{C}$. *Let* $q_i(z) \neq 0$ $(i = 1, 2)$ *be univalent in* Δ. *Further assume that* $\Re[\bar{\alpha} q_i(z)] \geq 0$ *for* $i = 1, 2$ *and* $zq_i'(z)/q_i(z)$ $(i = 1, 2)$ *is starlike univalent in* Δ. *If* $f \in \mathcal{A}$, $0 \neq zf'(z)/f(z) \in \mathcal{H}(1, 1) \cap Q$, $(1-\alpha)\frac{zf'(z)}{f(z)}$ $+ \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$ *is univalent in* Δ, *then*

$$q_1(z) + \alpha \frac{zq_1'(z)}{q_1(z)} \prec (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec q_2(z) + \alpha \frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and the best dominant.*

By making use of Lemma 2.4, we obtain the following:

Theorem 2.7. *Let* $q(z) \neq 0$ *be univalent in* Δ *and* $zq'(z)/q(z)$ *be starlike univalent in* Δ. *If* $f \in \mathcal{A}$, $0 \neq z^2 f'(z)/f^2(z) \in \mathcal{H}(1, 1) \cap Q$, $\frac{(zf)''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}$ *is univalent in* Δ, *then*

$$\frac{zq'(z)}{q(z)} \prec \frac{(zf)''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}$$

implies

$$q(z) \prec \frac{z^2 f'(z)}{f^2(z)}$$

and $q(z)$ *is the best subordinant.*

Together with the corresponding result for differential subordination (see Ravichandran [4]), we obtain the following:

Corollary 2.8. *Let* $q_i(z) \neq 0$ *be univalent in* Δ *and* $zq_i'(z)/q_i(z)$ *be starlike univalent in* Δ *for* $i = 1, 2$. *If* $f \in \mathcal{A}$, $0 \neq z^2 f'(z)/f^2(z) \in \mathcal{H}(1, 1) \cap Q$, $\frac{(zf)''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}$ *is univalent in* Δ, *then*

$$\frac{zq_1'(z)}{q_1(z)} \prec \frac{(zf)''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)} \prec \frac{zq_2'(z)}{q_2(z)},$$

implies

$$q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and best dominant.*

Lemma 2.9. *Let* $q(z) \neq 0$ *be univalent in* Δ *and* $zq'(z)/q^2(z)$ *be starlike univalent in* Δ. *If* $p(z) \in \mathcal{H}(q(0), 1) \cap Q$, $p(z) \neq 0$, $zp'(z)/p^2(z)$ *is univalent in* Δ, *then*

$$\frac{zq'(z)}{q^2(z)} \prec \frac{zp'(z)}{p^2(z)}$$

implies $q(z) \prec p(z)$ *and* $q(z)$ *is the best subordinant.*

Proof. Lemma 2.9 follows from Lemma 1.2 when $\vartheta(w) := 0$ and $\varphi(w) := 1/w^2$.

Theorem 2.10. *Let* $q(z) \neq 0$ *be univalent in* Δ *and* $zq'(z)/q^2(z)$ *be starlike univalent in* Δ. *If* $f \in \mathcal{A}$, $0 \neq zf'(z)/f(z) \in \mathcal{H}(1, 1) \cap Q$, $\dfrac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)}$ *is univalent in* Δ, *then*

$$1 + \frac{zq'(z)}{q^2(z)} \prec \frac{1 + z''f(z)/f'(z)}{zf'(z)/f(z)}$$

implies $q(z) \prec zf'(z)/f(z)$ *and* $q(z)$ *is the best subordinant.*

Proof. The result follows from Lemma 2.9 by taking $p(z) = zf'(z)/f(z)$.

Together with the corresponding result for differential subordination (see Ravichandran and Darus [5]), we obtain the following:

Theorem 2.11. *Let* $q_i(z) \neq 0$ *be univalent in* Δ *and* $zq_i'(z)/q_i^2(z)$ *be starlike univalent in* Δ *for* $i = 1, 2$. *If* $f \in \mathcal{A}$, $0 \neq zf'(z)/f(z) \in \mathcal{H}(1, 1) \cap Q$, $\dfrac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)}$ *is univalent in* Δ, *then*

$$1 + \frac{zq_1'(z)}{q_1^2(z)} \prec \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{zq_2'(z)}{q_2^2(z)}$$

implies $q_1(z) \prec zf'(z)/f(z) \prec q_2(z)$ *and* $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and the best dominant.*

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