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Differential Subordination for Meromorphic Functions Defined by a Linear Operator

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Abstract. By making use of a linear operator that is defined by means of the Hadamard product (or convolution), we investigate the properties of a certain family of meromorphically multivalent functions.

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1. Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in \mathbf{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* and *p-valent* in the punctured unit disk

$$\Delta^* := \{z \in \mathbf{C} : 0 < |z| < 1\}.$$

The function $f \in \Sigma_p$ is *meromorphic p-valent starlike of order* α if

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > p\alpha \quad (\alpha < 1;\ z \in \Delta := \{z \in \mathbf{C} : |\mathbf{z}| < \mathbf{1}\}).$$

We denote by $\Sigma_p^*(\alpha)$ the class of all such meromorphic p-valent starlike functions in Δ^*.

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The function $f \in \Sigma_p$ is *meromorphic p-valent convex of order* α if $f'(z) \neq 0$ and

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > p\alpha, \quad (\alpha < 1; z \in \Delta).$$

The class of all meromorphic p-valent convex functions of order α is denoted by $\Sigma_p^c(\alpha)$.

The class $\Sigma_p^\gamma(\alpha)$ consists of functions $f \in \Sigma_p$ with $f(z)f'(z) \neq 0$ satisfying

$$-\Re\left[(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > p\alpha, \quad (\alpha < 1; z \in \Delta).$$

The function $f \in \Sigma_p^\gamma(\alpha)$ is called a *meromorphic p-valent* γ*-convex function of order* α. Let $\Sigma := \Sigma_1$, $\Sigma^*(\alpha) := \Sigma_1^0(\alpha)$, $\Sigma^c(\alpha) := \Sigma_1^1(\alpha)$ and $\Sigma^\gamma(\alpha) := \Sigma_1^\gamma(\alpha)$.

Recently Cho and Owa [2] proved the following:

Theorem 1.1. If $f \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in Δ^* and

$$\Re\left\{\alpha\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right\} < 2(2-\alpha)-\beta \quad (z \in \Delta), \tag{1.2}$$

then

$$-\Re\left\{\frac{z^{2-\alpha}f'(z)}{f^\alpha(z)}\right\} > \frac{1}{1+2(2-\alpha)-2\beta} \quad (z \in \Delta), \tag{1.3}$$

where $\alpha \leq 2$ and $[2(2-\alpha)-1]/2 \leq \beta < 2-\alpha$.

Also Nunokawa and Ahuja [5] have proved the following:

Theorem 1.2. *Let* $\alpha < 0$. *If*

$$f \in \Sigma^c\left(\frac{\alpha(3-2\alpha)}{2(1-\alpha)}\right),$$

then $f \in \Sigma^*(\alpha)$.

Theorem 1.3. *Let* $\alpha < 0$ *and* $\gamma \geq 0$. *If*

$$f \in \Sigma^\gamma\left(\frac{2\alpha-2\alpha^2+\gamma\alpha}{2(1-\alpha)}\right),$$

then $f \in \Sigma^*(\alpha)$.

For two functions f and g analytic in Δ, we say that the function $f(z)$ is *subordinate* to $g(z)$ in Δ, and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta),$$

if there exists a Schwarz function $w(z)$, analytic in Δ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta),$$

such that

$$f(z) = g\big(w(z)\big) \quad (z \in \Delta).$$

In particular, if the function g is *univalent* in Δ, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Ravichandran *et al.* [6] proved the following generalization of Theorem 1.2 and Theorem 1.3:

Theorem 1.4. *Let $q(z)$ be univalent and $q(z) \neq 0$ in Δ. Further assume that*

(1) $zq'(z)/q(z)$ *is starlike univalent in* Δ, *and*

(2) $\Re\left[1 + \frac{zq''(z)}{q(z)} - \frac{zq'(z)}{q(z)} - \frac{q(z)}{\gamma}\right] > 0$ for $z \in \Delta, \gamma \neq 0$.

If $f(z) \in \Sigma$ and

$$-\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec q(z) - \gamma\frac{zq'(z)}{q(z)},$$

then

$$-\frac{zf'(z)}{f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

In the present investigation of the above defined classes, we prove extensions of Theorem 1.1 and Theorem 1.4 for a certain class of meromorphic functions that is defined by a linear operator introduced by Liu and Srivastava [3].

For two functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_k z^{k-p},$$

we define the *Hadamard product* (or *convolution*) of f and g by

$$(f * g)(z) := \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p}.$$

Define the function $\phi_p(a, c; z)$ by

$$\phi_p(a, c; z) := \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p}$$

$$(z \in \Delta; a \in \mathbf{R};\ c \in \mathbf{R} \setminus \{0, -1, -2, \ldots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \begin{cases} 1, & n = 0; \\ a(a+1)(a+2)\ldots(a+n-1), & n \in \mathbf{N}. \end{cases}$$

Corresponding to the function $\phi_p(a, c; z)$, Liu and Srivastava [3] defined a *linear operator* $\mathcal{L}_p(a, c)$ on Σ_p by

$$\mathcal{L}_p(a, c) f(z) := \phi_p(a, c; z) * f(z) \quad (f \in \Sigma_p)$$

or, equivalently, by

$$\mathcal{L}_p(a, c) f(z) := \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_k z^{k-p} \qquad (z \in \Delta).$$

The definition of the linear operator $\mathcal{L}_p(a, c)$ is motivated essentially by the familiar Carlson-Shaffer operator which has been used widely on the space of analytic and univalent functions in Δ. See [1, 8, 9] for details.

To prove our main results, we need the following Lemma due to Miller and Mocanu:

Lemma 1.5. [4, Corollary 3.4h.1, p.135] *Let $q(z)$ be univalent in Δ and let $\varphi(z)$ be analytic in a domain containing $q(\Delta)$. If $zq'(z)\varphi(q(z))$ is starlike and*

$$z\psi'(z)\varphi(\psi(z)) \prec zq'(z)\varphi(q(z)) \quad (z \in \Delta),$$

then $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 1.6. [7, Theorem 2, p.195] *Let α, β be any complex numbers, $\beta \neq 0$. Let $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ be univalent in Δ, $q(z) \neq 0$. Let $Q(z) = \beta z q'(z)/q(z)$ be starlike and*

$$\Re \left\{ \frac{\alpha}{\beta} q(z) + \frac{zQ'(z)}{Q(z)} \right\} > 0, \quad (z \in \Delta).$$

If $\psi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in Δ and satisfies

$$\alpha\psi(z) + \beta\frac{z\psi'(z)}{\psi(z)} \prec \alpha q(z) + \beta\frac{zq'(z)}{q(z)} \quad (z \in \Delta),$$

then $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant.

If $q(z)$ is a convex function that maps Δ onto a region in the right half-plane and $zq'(z)/q(z)$ is starlike, $\alpha\beta > 0$, then the conditions of Lemma 1.6 are satisfied.

2. Main Results

By appealing to Lemma 1.5, we first prove the following generalization of Theorem 1.1:

Theorem 2.1. *Let $q(z) \neq 0$ be univalent in Δ and $zq'(z)/q(z)$ be starlike in Δ. If $f \in \Sigma_p$ satisfies*

$$\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \prec 1 + (1-\alpha)p - \frac{zq'(z)}{q(z)}, \tag{2.1}$$

then

$$-\frac{z^{1+(1-\alpha)p}f'(z)}{pf^{\alpha}(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define the function $\psi(z)$ by

$$\psi(z) := -\frac{z^{1+(1-\alpha)p}f'(z)}{pf^{\alpha}(z)} \quad (z \in \Delta). \tag{2.2}$$

Then a computation using (2.2) shows that

$$\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} = 1 + (1-\alpha)p - \frac{z\psi'(z)}{\psi(z)}. \tag{2.3}$$

In view of (2.3), the subordination (2.1) becomes

$$\frac{z\psi'(z)}{\psi(z)} \prec \frac{zq'(z)}{q(z)}.$$

By an application of Lemma 1.5, with $\phi(w) := 1/w$, it follows that $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant.

Corollary 2.2. *Let $-1 \leq B < A \leq 1$. If $f \in \Sigma$ satisfies*

$$\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \prec 2 - \alpha - \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$-\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)} \prec \frac{1+Az}{1+Bz}.$$

Proof. Define $q(z)$ by

$$q(z) := \frac{1+Az}{1+Bz}. \tag{2.4}$$

Using (2.4), we obtain that

$$h(z) := \frac{zq'(z)}{q(z)} = \frac{(A-B)z}{(1+Az)(1+Bz)}. \tag{2.5}$$

We prove that the function $h(z)$ is starlike. A computation using (2.5) shows that

$$Q(z) := \frac{zh'(z)}{h(z)} = \frac{1-ABz^2}{(1+Az)(1+Bz)}.$$

Now

$$\begin{aligned}\Re Q(e^{i\theta}) &= \Re\frac{1-ABr^2e^{2i\theta}}{(1+Are^{i\theta})(1+Bre^{i\theta})}\\ &= \frac{(1-ABr^2)(1+ABr^2+r(A+B)\cos\theta)}{|(1+Are^{i\theta})(1+Bre^{i\theta})|^2} > 0\end{aligned}$$

provided $1+ABr^2+r(A+B)\cos\theta > 0$. Since $-1 \leq B \leq A \leq 1$, $1+ABr^2+r(A+B)\cos\theta > (1-Ar)(1-Br) > 0$ when $(A+B) \geq 0$. Also $1+ABr^2+r(A+B)\cos\theta > (1+Ar)(1+Br) > 0$ when $(A+B) \leq 0$. The result now follows from Theorem 2.1. □

Remark 1. By taking $A = 1-2\delta$ and $B = -1$ where

$$\delta := \frac{1}{1+2(2-\alpha)-2\beta} \quad (\alpha \leq 2,\ [2(2-\alpha)-1]/2 \leq \beta < 2-\alpha)$$

in the above Corollary 2.2, we see that (1.3) follows if

$$\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \prec 2-\alpha-\frac{2(1-\delta)z}{(1+(1-2\delta)z)(1-z)}. \tag{2.6}$$

Since the image of Δ under the function

$$w(z) := (2-\alpha) - \frac{2(1-\delta)z}{(1-z)(1+(1-2\delta)z)}$$

contains the half-plane

$$\Re w \leq (2-\alpha) - \frac{\delta - 1}{2\delta} = 2(2-\alpha) - \beta,$$

we see that (1.2) implies (2.6) and hence Theorem 1.1 follows from our Corollary 2.2.

By appealing again to Lemma 1.5, we now prove the following extension of Theorem 1.1 involving the linear operator $\mathcal{L}_p(a,c)$:

Theorem 2.3. *Let $q(z) \neq 0$ be univalent in Δ and $zq'(z)/q(z)$ be starlike. If $f \in \Sigma_p$ satisfies*

$$(a+1)\frac{\mathcal{L}_p(a+2,c)f(z)}{\mathcal{L}_p(a+1,c)f(z)} - \alpha a \frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} \prec 1 - (a+p)(\alpha - 1) + \frac{zq'(z)}{q(z)}, \quad (\alpha \in \mathbf{C}) \quad (2.7)$$

then

$$\frac{\mathcal{L}_p(a+1,c)f(z)}{[\mathcal{L}_p(a,c)f(z)]^\alpha} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define the function $\psi(z)$ by

$$\psi(z) := \frac{\mathcal{L}_p(a+1,c)f(z)}{[\mathcal{L}_p(a,c)f(z)]^\alpha} \quad (z \in \Delta). \quad (2.8)$$

In view of (2.8), we get

$$\frac{z\psi'(z)}{\psi(z)} = \frac{z[\mathcal{L}_p(a+1,c)f(z)]'}{\mathcal{L}_p(a+1,c)f(z)} - \alpha \frac{z[\mathcal{L}_p(a,c)f(z)]'}{\mathcal{L}_p(a,c)f(z)} \quad (2.9)$$

and by making use of the identity

$$z[\mathcal{L}_p(a,c)f(z)]' = a\mathcal{L}_p(a+1,c)f(z) - (a+p)\mathcal{L}_p(a,c)f(z) \quad (2.10)$$

we have, from (2.9),

$$\begin{aligned} \frac{z\psi'(z)}{\psi(z)} = \; & (a+1)\frac{\mathcal{L}_p(a+2,c)f(z)}{\mathcal{L}_p(a+1,c)f(z)} \\ & -\alpha a \frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} \\ & +(a+p)(\alpha-1) - 1, \end{aligned}$$

or

$$\begin{aligned}\frac{z\psi'(z)}{\psi(z)} - (a+p)(\alpha-1)+1 \quad &= \quad (a+1)\frac{\mathcal{L}_p(a+2,c)f(z)}{\mathcal{L}_p(a+1,c)f(z)} \\ &\quad -\alpha a\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}. \qquad (2.11)\end{aligned}$$

Therefore, in view of (2.11), the subordination (2.7) becomes

$$\frac{z\psi'(z)}{\psi(z)} \prec \frac{zq'(z)}{q(z)}.$$

By an application of Lemma 1.5, with $\phi(w) := 1/w$, it follows that $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant. □

By taking

$$q(z) = \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1)$$

in Theorem 2.3, we have the following:

Corollary 2.4. *Let $-1 \le B < A \le 1$. If $f \in \Sigma_p$ satisfies*

$$\begin{aligned}&(a+1)\frac{\mathcal{L}_p(a+2,c)f(z)}{\mathcal{L}_p(a+1,c)f(z)} - \alpha a\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} \\ &\prec 1-(a+p)(\alpha-1)+\frac{(A-B)z}{(1+Az)(1+Bz)}, \qquad (\alpha \in \mathbf{C})\end{aligned}$$

then

$$\frac{\mathcal{L}_p(a+1,c)f(z)}{[\mathcal{L}_p(a,c)f(z)]^\alpha} \prec \frac{1+Az}{1+Bz}.$$

By appealing to Lemma 1.6, we now prove the following extension of Theorem 1.4:

Theorem 2.5. *Let $a \neq -1$ and $\gamma \neq 0$. Let $q(z)$ satisfies the conditions of Lemma 1.6 with $\beta := \gamma$ and $\alpha := 1+a-\gamma$. If $f \in \Sigma_p$ satisfies*

$$\begin{aligned}&(1-\gamma)\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} + \gamma\frac{\mathcal{L}_p(a+2,c)f(z)}{\mathcal{L}_p(a+1,c)f(z)} \\ &\prec \frac{1}{a+1}\left[\gamma+(1+a-\gamma)q(z)+\gamma\frac{zq'(z)}{q(z)}\right], \qquad (2.12)\end{aligned}$$

then

$$\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Define the function $\psi(z)$ by

$$\psi(z) := \frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} \quad (z \in \Delta). \tag{2.13}$$

In view of (2.13), a computation using (2.10) shows that

$$\frac{\mathcal{L}_p(a+2,c)f(z)}{\mathcal{L}_p(a+1,c)f(z)} = \frac{1}{a+1}\left[1 + a\psi(z) + \frac{z\psi'(z)}{\psi(z)}\right]$$

and therefore we have

$$\begin{aligned} & (1-\gamma)\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} + \gamma\frac{\mathcal{L}_p(a+2,c)f(z)}{\mathcal{L}_p(a+1,c)f(z)} \\ = & \frac{1}{a+1}\left[\gamma + (1+a-\gamma)\psi(z) + \gamma\frac{z\psi'(z)}{\psi(z)}\right] \end{aligned}$$

and the subordination (2.12) becomes

$$(1+a-\gamma)\psi(z) + \gamma\frac{z\psi'(z)}{\psi(z)} \prec (1+a-\gamma)q(z) + \gamma\frac{zq'(z)}{q(z)}.$$

By an application of Lemma 1.6, it follows that $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant. □

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References

[1] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., 15 (1984), 737–745.

[2] N. E. Cho and S. Owa, *Sufficient conditions for meromorphic starlikeness and close-to-convexity of order α*, Internat. J. Math. Math. Sci., 26 (2001), 317–319.

[3 J.-L. Liu and H.M. Srivastava, *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl., 259 (2001), 566–581.

[4] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics (No. 225), Marcel Dekker, New York and Basel, 2000.

[5] M. Nunokawa and O. P. Ahuja, *On meromorphic starlike and convex functions*, Indian J. Pure Appl. Math., 32 (2001), 1027–1032.

[6] V. Ravichandran, S. Sivaprasad Kumar and M. Darus, *On a subordination theorem for a class of meromorphic functions*, J. Inequal. Pure Appl. Math., 5 (2004), Article No. 8 (Electronic). (http://jipam.vu.edu.au)

[7] V. Ravichandran and M. Jayamala, *On sufficient conditions for Caratheodory functions*, Far East J. Math. Sci.(FJMS), 12 (2004), 191–201.

[8] H. Saitoh, *A linear operator and its applications of first order differential subordinations*, Math. Japon., 44 (1996), 31–38.

[9] H. M. Srivastava and S. Owa (Eds.), *Current Topics in Analytic Function Theory*, World Scientific, Singapore, 1992.

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