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CLASSES OF MEROMORPHIC p-VALENT PARABOLIC STARLIKE FUNCTIONS WITH POSITIVE COEFFICIENTS

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ABSTRACT. In the present paper, we consider two general subclasses of meromorphic p-valent starlike functions with positive coefficients and obtain a necessary and sufficient condition for functions to be in these classes. Also we obtain certain other related results as a consequences of our main results.

Key words and phrases: Analytic functions, Univalent functions, Uniformly k-convex function, Meromorphic p-valent starlike functions.

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1. INTRODUCTION

A normalized analytic function f is uniformly convex (starlike) if for every circular arc γ contained in $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ with center $\zeta \in \Delta$ the image arc $f(\gamma)$ is convex (starlike with respect to $f(\zeta)$). The class of all uniformly convex (starlike) functions is denoted by $UCV(UST)$. These classes were introduced and studied by Goodman [4, 3] and he has shown that

$$f \in UCV \iff \Re\left\{1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right\} \geq 0, \quad (z, \zeta \in \Delta), \tag{1.1}$$

$$f \in UST \iff \Re\left\{\frac{(z-\zeta)f'(z)}{f(z) - f(\zeta)}\right\} \geq\ 0, \quad (z, \zeta \in \Delta). \tag{1.2}$$

Rønning [9] and Ma and Minda [6, 7] have given the following one variable characterization of the class of uniformly convex functions.

Theorem 1.1. *Let $f \in \mathcal{A}$. Then $f \in UCV$ if and only if*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in \Delta). \tag{1.3}$$

Since the Alexander type result $f \in UCV$ if and only if $zf' \in UST$ failed [11], the class

$$S_p = \{f :\ f = zF',\ F \in UCV\} \tag{1.4}$$

was introduced by Rønning [9]. Also Rønning [12] generalized this class to $SP(\alpha, \beta)$. Subramanian *et al.* [14], Bharathi *et al.* [1] and Kanas [5] have studied the class of functions called uniformly k-convex functions.

In this paper, we define two subclasses of meromorphic starlike functions with positive coefficients which are similar to the classes $SP(\alpha, \beta)$ and the uniformly k-convex functions and obtain a necessary and sufficient condition for functions to be in these classes and obtained certain other related results as a consequences of our main results.

Let Σ_p be the class of *meromorphic* functions

$$f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n \quad \left(p \in \mathbb{N} := \{1, 2, 3, \ldots\}\right), \tag{1.5}$$

which are *analytic* in the punctured disk

$$\mathbb{E} = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\}.$$

Let $\Sigma_p^*(\alpha)$ be the subclass of Σ_p consisting of functions $f(z)$ satisfying

$$\Re\left(-\frac{zf'(z)}{f(z)}\right) > p\alpha \qquad (z \in \Delta). \tag{1.6}$$

The class $\widetilde{\Sigma}_p$ consists of functions $f(z) \in \Sigma_p$ of the form (1.5) where $a_n \geq 0$ and

$$\widetilde{\Sigma}_p^*(\alpha) = \Sigma_p^*(\alpha) \cap \widetilde{\Sigma}_p.$$

Definition 1.1. Let $\alpha \geq 0;\ 0 \leq \beta < 1$. Define the class $\Sigma_p(\alpha, \beta)$ by

$$\Sigma_p(\alpha, \beta) = \left\{f \in \Sigma_p :\ \Re\left\{-\frac{1}{p}\frac{zf'(z)}{f(z)}\right\} \geq \alpha\left|\frac{1}{p}\frac{zf'(z)}{f(z)} + 1\right| + \beta, \quad (z \in \Delta)\right\}. \tag{1.7}$$

The class $\widetilde{\Sigma}_p(\alpha, \beta)$ is the subclass of $\Sigma_p(\alpha, \beta)$ consisting of functions $f(z) \in \Sigma_p$ of the form (1.5) where $a_n \geq 0$:

$$\widetilde{\Sigma}_p(\alpha, \beta) = \widetilde{\Sigma}_p \cap \Sigma_p(\alpha, \beta). \tag{1.8}$$

Definition 1.2. Let $\alpha \geq 0;\ 0 \leq \beta < 1$. Define the class $\Sigma_p^*(\alpha, \beta)$ by

$$\Sigma_p^*(\alpha,\beta) = \left\{ f \in \Sigma_p : \left| \frac{1}{p}\frac{zf'(z)}{f(z)} + \alpha + \beta \right| \leq \Re\left\{ -\frac{1}{p}\frac{zf'(z)}{f(z)} \right\} + \alpha - \beta, \quad (z \in \Delta) \right\}. \tag{1.9}$$

The class $\widetilde{\Sigma}_p^*(\alpha, \beta)$ is defined by

$$\widetilde{\Sigma}_p^*(\alpha,\beta) = \widetilde{\Sigma}_p \cap \Sigma_p^*(\alpha,\beta). \tag{1.10}$$

2. THE CLASSES $\Sigma_p(\alpha, \beta)$ AND $\widetilde{\Sigma}_p(\alpha, \beta)$

We now prove the following:

Theorem 2.1. *If $f(z) \in \Sigma_p(\alpha, \beta)$, then*

$$f(z) \in \Sigma_p^*\left(\frac{\alpha+\beta}{1+\alpha}\right).$$

In particular if $f(z) \in \widetilde{\Sigma}_p(\alpha, \beta)$, then

$$f(z) \in \widetilde{\Sigma}_p^*\left(\frac{\alpha+\beta}{1+\alpha}\right).$$

Proof. If $f \in \Sigma_p(\alpha, \beta)$, then by (1.7) we have

$$\begin{aligned} -\Re\left\{\frac{zf'(z)}{pf(z)}\right\} &\geq \alpha\left|\frac{zf'(z)}{pf(z)} + 1\right| + \beta \\ &\geq \alpha\Re\left\{\frac{zf'(z)}{pf(z)} + 1\right\} + \beta \end{aligned}$$

or

$$-\Re\left\{\frac{zf'(z)}{pf(z)}\right\}(1+\alpha) \geq (\alpha+\beta)$$

or

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq p\left(\frac{\alpha+\beta}{1+\alpha}\right).$$

Hence by (1.6), we have

$$f(z) \in \Sigma_p^*\left(\frac{\alpha+\beta}{1+\alpha}\right),$$

as desired. ∎

To obtain a converse to Theorem 2.1, we first determine the largest radius R_a such that the disk $|w-a| \leq R_a$ is contained in $\Re(w) \geq \alpha|w-1|+\beta$. Note that this inequality can be written as

$$\left(\frac{u-\beta}{\alpha}\right)^2 = (u-1)^2 + v^2, \quad w = u + iv.$$

The square of the distance from $(a, 0)$ to a point in $\Re(w) \geq \alpha|w-1|+\beta$ is therefore given by

$$D = (u-a)^2 + \left(\frac{u-\beta}{\alpha}\right)^2 - (u-1)^2.$$

Since

$$\frac{dD}{du} = 2[\frac{u-\beta}{\alpha^2} - a + 1],$$

we see that the minimum of d is given by

$$u = \alpha^2(a-1)+\beta \quad \text{or} \quad u = \frac{\alpha+\beta}{1+\alpha}.$$

Also, if $\alpha^2(a-1)+\beta \leq \frac{\alpha+\beta}{1+\alpha}$, the minimum is at $u = \frac{\alpha+\beta}{1+\alpha}$. Otherwise it is the minimum of the two extremum values.

Note that

$$\alpha^2(a-1)+\beta \leq \frac{\alpha+\beta}{1+\alpha} \Leftrightarrow a \leq \frac{1-\beta}{\alpha(1+\alpha)}+1.$$

Case (i). Let $a \leq 1+\frac{1-\beta}{\alpha(1+\alpha)}$. In this case,

$$\begin{aligned} R_a &= \sqrt{\left(\frac{\alpha+\beta}{1+\alpha}-a\right)^2+\frac{1}{\alpha^2}\left(\frac{\alpha+\beta}{1+\alpha}-\beta\right)^2-\left(\frac{\alpha+\beta}{1+\alpha}-1\right)^2} \\ &= \sqrt{\left(\frac{\alpha+\beta}{1+\alpha}-a\right)^2+\left(\frac{1-\beta}{1+\alpha}\right)^2-\left(\frac{\beta-1}{1+\alpha}\right)^2} \\ &= \left|a-\frac{\alpha+\beta}{1+\alpha}\right| \\ &= a-\frac{\alpha+\beta}{1+\alpha}. \end{aligned}$$

Case (ii). Let $a \geq 1+\frac{1-\beta}{\alpha(1+\alpha)}$. In this case, $R_a = \min\left\{a-\frac{\alpha+\beta}{1+\alpha}, \sqrt{X}\right\}$ where

$$\begin{aligned} X &= (1-a)^2-2\alpha^2(a-1)^2+2(\beta-1)(1-a)+\alpha^2(a-1)^2 \\ &= (1-a)^2(1-\alpha^2)-2(1-\beta)(1-a) \\ &= (1-a)[(1-a)(1-\alpha^2)-2(1-\beta)]. \end{aligned}$$

For $\beta=0, \alpha=1, X=(1-a)(-2)=2a-2$. Then

$$\begin{aligned} R_a &= \min\left\{a-\frac{\alpha+\beta}{1+\alpha}, \quad \sqrt{(1-a)[(1-a)(1-\alpha^2)-2(1-\beta)]}\right\} \\ &= \min\left\{a-\frac{\alpha+\beta}{1+\alpha}, \quad \sqrt{((1-a)^2(1-\alpha^2)-2(1-\beta)(1-a)}\right\}. \end{aligned}$$

We show that

$$\left(-a+\frac{\alpha+\beta}{1+\alpha}\right)^2 > (1-a)^2(1-\alpha^2)-2(1-\beta)(1-a). \tag{2.1}$$

Note that the inequality (2.1) is equivalent to

$$\left(1-a+\frac{\alpha+\beta}{1+\alpha}-1\right)^2 > (1-a)^2(1-\alpha^2)-2(1-\beta)(1-a). \tag{2.2}$$

Setting $1-a=x$ in (2.2) yields

$$\left(x+\frac{\beta-1}{1+\alpha}\right)^2 > x^2(1-\alpha^2)-2(1-\beta)x$$

or

$$\alpha^2x^2+2x\left(\frac{\beta-1}{1+\alpha}+1-\beta\right)+\left(\frac{\beta-1}{1+\alpha}\right)^2 = \left(\alpha x+\frac{1-\beta}{1+\alpha}\right)^2 \geq 0. \tag{2.3}$$

Thus we have proved the following:

Lemma 2.2. *Let*

$$R_a = \begin{cases} a - \frac{\alpha+\beta}{1+\alpha} & \text{for } a \le 1 + \frac{1-\beta}{\alpha(1+\alpha)} \\ \sqrt{((1-a)^2(1-\alpha^2) - 2(1-\beta)(1-a)} & \text{for } a \ge 1 + \frac{1-\beta}{\alpha(1+\alpha)} \end{cases}.$$

Then

$$\{w : |w - a| \le R_a\} \subseteq \{w : \Re(w) \ge \alpha|w-1| + \beta\}.$$

Note that

$$R_a < a - \frac{\alpha+\beta}{1+\alpha} \quad \text{for all} \quad a \ge 1 + \frac{1-\beta}{\alpha(1+\alpha)}$$

and

$$R_a = a - \frac{\alpha+\beta}{1+\alpha} \quad \text{for all} \quad a \le 1 + \frac{1-\beta}{\alpha(1+\alpha)}.$$

Hence if

$$1 \le 1 + \frac{1-\beta}{\alpha(1+\alpha)} \quad \text{or} \quad 0 \le \frac{1-\beta}{\alpha(1+\alpha)},$$

we have

$$R_1 = 1 - \frac{\alpha+\beta}{1+\alpha}.$$

Therefore

$$R_1 \le 1 - \frac{\alpha+\beta}{1+\alpha}.$$

Hence we have the following:

Theorem 2.3. *If* $f(z) \in \widetilde{\Sigma}_p^*\left(\frac{\alpha+\beta}{1+\alpha}\right)$*, then* $f(z) \in \widetilde{\Sigma}_p(\alpha, \beta)$.

Corollary 2.4. *The two classes* $\widetilde{\Sigma}_p^*\left(\frac{\alpha+\beta}{1+\alpha}\right)$ *and* $\widetilde{\Sigma}_p(\alpha,\beta)$ *are equal:*

$$\widetilde{\Sigma}_p^*\left(\frac{\alpha+\beta}{1+\alpha}\right) = \widetilde{\Sigma}_p(\alpha,\beta).$$

Theorem 2.5. [15] *Let* $f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n$*. Then*

$$f(z) \in \widetilde{\Sigma}_p^*(\alpha) \quad \textit{if and only if} \quad \sum_{n=p}^{\infty} (n + p\alpha)a_n \le p(1-\alpha).$$

Corollary 2.6. *Let* $f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n$*. Then*

$$f(z) \in \widetilde{\Sigma}_p(\alpha,\beta) \quad \textit{if and only if} \quad \sum_{n=p}^{\infty} [n(1+\alpha) + p(\alpha+\beta)]a_n \le p(1-\beta). \tag{2.4}$$

Corollary 2.7. *If* $f(z) \in \widetilde{\Sigma}_p(\alpha,\beta)$*, then*

$$a_n \le \frac{p(1-\beta)}{(1+\alpha)n + p(\alpha+\beta)}.$$

The result is sharp for $f(z) = \frac{1}{z^p} + \frac{p(1-\beta)}{(1+\alpha)n+p(\alpha+\beta)} z^n$.

Corollary 2.8. *If* $f(z) \in \widetilde{\Sigma}_p(\alpha,\beta)$*, then*

$$\frac{1}{r^p} - \frac{1-\beta}{1+2\alpha+\beta} r^p \le |f(z)| \le \frac{1}{r^p} + \frac{1-\beta}{1+2\alpha+\beta} r^p \quad (z \in \mathbb{E} \quad \textit{and} \quad |z| = r).$$

The result is sharp for $f(z) = \frac{1}{z^p} + \frac{1-\beta}{1+2\alpha+\beta} z^p$.

3. THE CLASSES $\Sigma_p^*(\alpha,\beta)$ AND $\widetilde{\Sigma}_p^*(\alpha,\beta)$

Theorem 3.1. *If $f(z) \in \Sigma_p^*(\alpha,\beta)$, then $f(z) \in \Sigma_p^*(\beta)$. In particular if $f(z) \in \widetilde{\Sigma}_p^*(\alpha,\beta)$, then $f(z) \in \widetilde{\Sigma}_p^*(\beta)$.*

Proof. Let $f \in \Sigma_p^*(\alpha,\beta)$. Then by (1.9) we have

$$\begin{aligned} -\Re\left\{\frac{zf'(z)}{pf(z)}\right\} + \alpha - \beta &\geq \left|\frac{zf'(z)}{pf(z)} + \alpha + \beta\right| \\ &\geq \Re\left\{\frac{zf'(z)}{pf(z)} + \alpha + \beta\right\} \end{aligned}$$

or

$$-\Re\left\{\frac{zf'(z)}{pf(z)}\right\} \geq \beta.$$

Hence by(1.6), we have $f \in \Sigma_p^*(\beta)$. ∎

To obtain a converse to Theorem 3.1, we note that $f \in \Sigma_p^*(\beta)$ if

$$\left|-\frac{zf'}{pf} - 1\right| \leq 1 - \beta.$$

This is a circular disk $|w-1| \leq 1-\beta$. This disk should be contained in the parabolic region $|w-(\alpha+\beta)| \leq \Re(w+\alpha-\beta)$ or equivalently in $v^2 \leq 4\alpha(u-\beta)$ where $w = -zf'(z)/pf(z) = u+iv$. Note that when $\alpha \to 0$, the parabolic region reduced to the line segment $v = 0$. Therefore, the class $\Sigma_p^*(\beta) \not\subseteq \Sigma_p^*(\alpha,\beta)$ for small $\alpha \geq 0$. Hence we determine conditions on α, β such that the disk $|w-1| \leq 1-\beta$ is inside $|w-(\alpha+\beta)| \leq \Re(w+\alpha-\beta)$. To find such condition, we first find the radius of largest disk $|w-a| \leq R_a$ contained in the parabolic region. Then the required condition on α, β will be given by $R_1 \geq 1-\beta$. The expression for R_a is given in the following:

Lemma 3.2. *Let $a > \beta$. Then the disk $|w-a| < R_a$ is contained in*

$$|w-(\alpha+\beta)| \leq \Re\{w+\alpha-\beta\}$$

where

$$R_a = \begin{cases} a-\beta & \text{for } a \leq 2\alpha+\beta \\ 2\sqrt{\alpha(a-\alpha-\beta)} & \text{for } a \geq 2\alpha+\beta \end{cases}.$$

Note that

$$R_1 = \begin{cases} 1-\beta & \text{for } 1 \leq 2\alpha+\beta \\ 2\sqrt{\alpha(1-\alpha-\beta)} & \text{for } 1 \geq 2\alpha+\beta \end{cases}.$$

If $2\alpha+\beta \geq 1$, then $R_1 = 1-\beta \geq 1-\beta$. If $2\alpha+\beta \leq 1$, then $R_1 \geq 1-\beta$ provided $2\sqrt{\alpha(1-\alpha-\beta)} \geq 1-\beta$ and this is not the case since R_1 obtained by taking minimum of $1-\beta,\ 2\sqrt{\alpha(1-\alpha-\beta)}$. Therefore we have the following:

Corollary 3.3. *Let $\alpha \geq \frac{1-\beta}{2}$. Then*

$$\widetilde{\Sigma}_p^*(\beta) \subseteq \widetilde{\Sigma}_p^*(\alpha,\beta).$$

Corollary 3.4. *Let $\alpha \geq \frac{1-\beta}{2}$. Then*

$$\widetilde{\Sigma}_p^*(\beta) = \widetilde{\Sigma}_p^*(\alpha,\beta).$$

Corollary 3.5. *Let* $\alpha \geq \frac{1-\beta}{2}$. *Then*

$$f \in \widetilde{\Sigma}_p^*(\alpha, \beta) \quad \textit{if and only if} \quad \sum_{n=p}^{\infty}(n + p\beta)a_n \leq p(1 - \beta).$$

Corollary 3.6. *If* $f(z) \in \widetilde{\Sigma}_p^*(\alpha, \beta)$, *then*

$$a_n \leq \frac{p(1-\beta)}{n+p\beta}.$$

The result is sharp for

$$f(z) = \frac{1}{z^p} + \frac{p(1-\beta)}{n+p\beta}z^n.$$

Corollary 3.7. *If* $f(z) \in \widetilde{\Sigma}_p^*(\alpha, \beta)$, *then*

$$\frac{1}{r^p} - \frac{1-\beta}{1+\beta}r^p \leq |f(z)| \leq \frac{1}{r^p} + \frac{1-\beta}{1+\beta}r^p \quad (z \in \mathbb{E} \quad \textit{and} \quad |z| = r).$$

The result is sharp for

$$f(z) = \frac{1}{z^p} + \frac{1-\beta}{1+\beta}z^p.$$

4. PARTIAL SUMS

Silverman [13] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. Since to a certain extend the work on meromorphic case is analogous to analytic case, one is interested to search results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of Silverman [13] and Cho and Owa [2] we will investigate the ratio of a function of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} \quad (p \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{4.1}$$

to its sequence of partial sums

$$f_1(z) = z^{-p} \quad \text{and} \quad f_k(z) = z^{-p} + \sum_{n=1}^{k-1} a_n z^{n-p} \quad (k \in \mathbb{N} \setminus \{1\}) \tag{4.2}$$

when the coefficients are sufficiently small to satisfy the condition 4.3 below. More precisely we will determine sharp lower bounds for $\Re\{f(z)/f_k(z)\}$ and $\Re\{f_k(z)/f(z)\}$.

Theorem 4.1. *Let* $f(z) \in \widetilde{\Sigma}_p(\alpha, \beta)$ *be given by (4.1) and define the partial sums* $f_1(z)$ *and* $f_k(z)$ *given by (4.2). Suppose also that*

$$\sum_{n=1}^{\infty} d_n |a_n| \leq 1, \tag{4.3}$$

where

$$d_n := \frac{(n+p)(1+\alpha) + p(\alpha+\beta)}{p(1-\beta)}. \tag{4.4}$$

Then $f \in \Sigma_p(\alpha, \beta)$. *Furthermore,*

$$\Re\left\{\frac{f(z)}{f_k(z)}\right\} > 1 - \frac{1}{d_k} \quad z \in E, k \in N \tag{4.5}$$

and

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\} > \frac{d_k}{1+d_k}. \tag{4.6}$$

Proof. For the coefficients d_n given by (4.4) it is not difficult to verify that

$$d_{n+1} > d_n > 1.$$

Therefore we have

$$\sum_{n=1}^{k-1}|a_n| + d_k\sum_{n=k}^{\infty}|a_n| \leq \sum_{n=1}^{\infty} d_n|a_n| \leq 1 \tag{4.7}$$

by using the hypothesis (4.3). By setting

$$\begin{aligned} g_1(z) &= d_k\left\{\frac{f(z)}{f_k(z)} - \left(1-\frac{1}{d_k}\right)\right\} \\ &= 1 + \frac{d_k \sum_{n=k}^{\infty} a_n z^{n-1}}{1+\sum_{n=1}^{k} a_n z^{n-1}} \end{aligned}$$

and applying (4.7), we find that

$$\begin{aligned} \left|\frac{g_1(z)-1}{g_1(z)+1}\right| &\leq \frac{d_k \sum_{n=k}^{\infty}|a_n|}{2-2\sum_{n=1}^{k-1}|a_n| - d_k\sum_{n=k}^{\infty}|a_n|} \\ &\leq 1, \quad z \in E, \end{aligned}$$

which readily yields the assertion (4.5) of Theorem 4.1. If we take

$$f(z) = z^{-p} - \frac{z^{k-p}}{d_k} \tag{4.8}$$

then

$$\frac{f(z)}{f_k(z)} = 1 - \frac{z^k}{d_k} \to 1 - \frac{1}{d_k} \quad \text{as} \quad z \to 1-,$$

which shows the bound (4.5) is the best possible for each $k \in \mathbb{N}$.

Similarly, if we take

$$\begin{aligned} g_2(z) &= (1+d_k)\left\{\frac{f_k(z)}{f(z)} - \frac{d_k}{1+d_k}\right\} \\ &= 1 - \frac{(1+d_k)\sum_{n=k}^{\infty} a_n z^n}{1+\sum_{n=1}^{\infty} a_n z^n} \end{aligned}$$

and making use of (4.7), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_k)\sum_{n=k}^{\infty}|a_n|}{2-2\sum_{n=1}^{k-1}|a_n| + (1-d_k)\sum_{n=k}^{\infty}|a_n|}$$

which leads us immediately to the assertion (4.6) of Theorem 4.1.

The bound in (4.6) is sharp for each $k \in N$ with the extremal function $f(z)$ given by (4.8). The proof of the Theorem 4.1 is thus completed. ∎

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