DIFFERENTIAL SUBORDINATION ASSOCIATED WITH LINEAR OPERATORS DEFINED FOR MULTIVALENT FUNCTIONS

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Abstract. In this paper we give certain sufficient conditions for functions defined through the Dziok-Srivastava linear operator and the multiplier transformation.

1. Introduction

Let $A_p$ denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$$  \hspace{1cm} (1.1)

$(z \in \Delta := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}; \ p < k; \ p, k \in \mathbb{N} := \{1, 2, 3, \ldots \})$

and $A := A_1$. Recently several authors [8, 12, 13, 16, 18, 19, 25] obtained sufficient conditions associated with starlikeness in terms of the expression

$$zf'(z) f(z) + \alpha z^2 f''(z) f(z).$$

In fact, Ravichandran [19] obtained the following more general result:

**Theorem 1.1.** [19, Theorem 3, p.44] Let $q(z)$ be convex univalent and $0 < \alpha \leq 1$,

$$\text{Re} \left\{ \frac{1 - \alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right) \right\} > 0.$$  \hspace{1cm} \text{If } f \in A \text{ satisfies}

$$zf'(z) f(z) + \alpha z^2 f''(z) f(z) \prec (1 - \alpha)q(z) + \alpha q^2(z) + \alpha q'(z),$$

then $zf'(z) f(z) \prec q(z)$ and $q(z)$ is the best dominant.

Also the following extension of a result of Darus and Frasin [6] was obtained:

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Theorem 1.2. [19, Theorem 4, p.48] Let \( q(z) \) be analytic in \( \Delta \), \( q(0) = 1 \) and \( h(z) = zq'(z)/q(z) \) be starlike univalent in \( \Delta \). If \( f \in \mathcal{A} \) satisfies
\[
\frac{(zf(z))''}{f'(z)} - 2\frac{zf'(z)}{f(z)} < h(z),
\]
then
\[
\frac{z^2f'(z)}{f^2(z)} < q(z).
\]
The dominant \( q(z) \) is the best dominant.

In the present paper, the authors present extension of the above two theorems for functions defined through Dziok-Srivastava linear operator and the multiplier transformation on the space of multivalent functions \( \mathcal{A}_p \).

2. Preliminaries

For two analytic functions \( f(z) \) given by (1.1) and
\[
g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k,
\]
their Hadamard product (or convolution) is the function \((f \ast g)(z)\) defined by
\[
(f \ast g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k.
\]

For \( \alpha_j \in \mathbb{C} \ (j = 1, 2, \ldots, l) \) and \( \beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, \ j = 1, 2, \ldots, m \), the generalized hypergeometric function \( \left(_l \!F_m\right)(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) \) is defined by the infinite series
\[
\left(_l \!F_m\right)(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!}
\]
\((l \leq m + 1; \ l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})\)
where \((a)_n\) is the Pochhammer symbol defined by
\[
(a)_n := \frac{\Gamma(a + n)}{\Gamma(a)} = \begin{cases} 1 & \text{if } n = 0; \\ a(a + 1)(a + 2) \cdots (a + n - 1) & \text{if } n \in \mathbb{N}. \end{cases}
\]

Corresponding to the function
\[
h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) := z^p \left(_l \!F_m\right)(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m),
\]
the Dziok-Srivastava operator [5] (see also [23]) \( H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) \) is defined by the Hadamard product
\[
H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) := h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) \ast f(z)
\]
\[
= z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p} \cdots (\alpha_l)_{n-p} a_n z^n}{(\beta_1)_{n-p} \cdots (\beta_m)_{n-p} (n-p)!}.
\]
It is well known [5] that
\[
\alpha_1 H_p^{(l,m)}(\alpha_1 + 1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) = z[H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z)]' + (\alpha_1 - p) H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z).
\]

The Dziok-Srivastava linear operator includes special cases the Hohlov linear operator [7], the Carlson-Shaffer linear operator [2], the Ruscheweyh derivative operator [20], the generalized Bernardi-Libera-Livingston linear integral operator (cf. [1], [9], [10]) and the Srivastava-Owa fractional derivative operators (cf. [15], [14]).

Motivated by the multiplier transformation on \( A \), we define the operator \( I_p(n, \lambda) \) on \( A_p \) by the following infinite series
\[
I_p(n, \lambda) f(z) := z^p + \sum_{k=p+1}^{\infty} \left( \frac{k + \lambda}{p + \lambda} \right)^n a_k z^k.
\]

A straight forward calculation shows that
\[
(p + \lambda) I_p(n + 1, \lambda) f(z) = z [I_p(n, \lambda) f(z)]' + \Lambda_p(n, \lambda) f(z).
\]

The operator \( I_p(n, \lambda) \) is closely related to the Sălăgean derivative operators [21].

The operator \( I_n := I_1(n, 1) \) was recently studied by Cho and Srivastava [3] and Cho and Kim [4]. The operator \( I_n := I_1(n, 1) \) was studied by Uralegaddi and Somanatha [24].

We shall need the following lemma due to Miller and Mocanu

**Lemma 2.1.** [11, Theorem 3.4h, p.132] Let \( q(z) \) be univalent in the unit disk \( \Delta \).
Let \( \vartheta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(\Delta) \) with \( \varphi(w) \neq 0 \) when \( w \in q(\Delta) \). Set
\[
Q(z) := z q'(z) \varphi(q(z)), \quad h(z) := \vartheta(q(z)) + Q(z).
\]

Suppose that either
(1) \( h(z) \) is convex, or
(2) \( Q(z) \) is starlike univalent in \( \Delta \).

In addition, assume that
\[
\Re \frac{z h'(z)}{Q(z)} > 0 \text{ for } z \in \Delta.
\]

If \( p(z) \) is analytic with \( p(0) = q(0) \), \( p(\Delta) \subseteq D \) and
\[
\vartheta(p(z)) + z p'(z) \varphi(p(z)) \prec \vartheta(q(z)) + z q'(z) \varphi(q(z)),
\]
then
\[
p(z) \prec q(z)
\]
and \( q(z) \) is the best dominant.
The following result which is a special case of Lemma 2.1 is also useful.

**Lemma 2.2.** [11, Corollary 3.4h.1, p.135] Let \( q(z) \) be univalent in \( \Delta \) and let \( \varphi(z) \) be analytic in a domain containing \( q(\Delta) \). If \( zq'(z)/\varphi(z) \) is starlike, then

\[
z\varphi'(z)\varphi(\psi(z)) < zq'(z)\varphi(q(z)) \quad (z \in \Delta),
\]
then \( \psi(z) \prec q(z) \) and \( q(z) \) is the best dominant.

By making use of Lemma 2.1, we prove the following

**Lemma 2.3.** If \( p(z) \) and \( q(z) \) are analytic in \( \Delta \), \( q(z) \) is convex univalent, \( \alpha, \beta \) and \( \gamma \) are complex and \( \gamma \neq 0 \). Further assume that

\[
\Re \left\{ \frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z) + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0.
\]

If \( p(z) = 1 + cz + \cdots \) is analytic in \( \Delta \) and satisfies

\[
\alpha p(z) + \beta p^2(z) + \gamma zp'(z) < \alpha q(z) + \beta q^2(z) + \gamma zq'(z),
\]
then \( p(z) \prec q(z) \) and \( q(z) \) is the best dominant.

**Proof.** Let \( \vartheta(w) := \alpha w + \beta w^2 \) and \( \varphi(w) := \gamma \). Then clearly \( \vartheta(w) \) and \( \varphi(w) \) are analytic in \( \mathbb{C} \) and \( \varphi(w) \neq 0 \). Also let

\[
Q(z) = zq'(z)\varphi(q(z)) = \gamma zq'(z)
\]
and

\[
h(z) = \vartheta(q(z)) + Q(z) = \alpha q(z) + \beta q^2(z) + \gamma zq'(z).
\]

Since \( q(z) \) is convex univalent, \( zq'(z) \) is starlike univalent. Therefore \( Q(z) \) is starlike univalent in \( \Delta \), and

\[
\Re \frac{zh'(z)}{Q(z)} = \Re \left\{ \frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z) + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0
\]
for \( z \in \Delta \). Hence the result follows from Lemma 2.1.

**Remark.** Note that the condition \( \Re \left\{ \frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z) + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0 \) is satisfied by any convex function that maps \( \Delta \) onto a convex region in the right-half plane when \( \alpha > 0, \beta > 0 \) and \( \gamma > 0 \).

3. **SUFFICIENT CONDITIONS INVOLVING DZIOK-SRIVASTAVA LINEAR OPERATOR**

To make the notation simple, we write

\[
H_{p}^{l,m}(\alpha_1)f(z) := H_{p}^{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z).
\]

By making use of Lemma 2.3 we first prove the following

**Theorem 3.1.** Let \( q(z) \) be convex univalent, \( \alpha \neq 0 \). Further assume that

\[
\Re \left\{ \frac{1 + \alpha_1(1 - \alpha)}{\alpha} + \frac{2(1 + \alpha_1(1 - \alpha))}{\alpha\alpha_1} q(z) + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0.
\]
If \( f(z) \in A_p \) satisfies
\[
\frac{H_p^{l,m}(\alpha_1 + 1)f(z)}{H_p^{l,m}(\alpha_1)f(z)} \left\{ 1 - \alpha + \frac{H_p^{l,m}(\alpha_1 + 2)f(z)}{H_p^{l,m}(\alpha_1)f(z)} \right\} < \frac{1 + \alpha_1(1 - \alpha)}{1 + \alpha_1} q(z) + \frac{\alpha_1}{1 + \alpha_1} q^2(z) + \frac{\alpha}{1 + \alpha_1} z q'(z)
\]
(3.1) then
\[
\frac{H_p^{l,m}(\alpha_1 + 1)f(z)}{H_p^{l,m}(\alpha_1)f(z)} < q(z)
\]
(3.2) and \( q(z) \) is the best dominant.

**Proof.** Define the function \( \psi(z) \) by
\[
\psi(z) := \frac{H_p^{l,m}(\alpha_1 + 1)f(z)}{H_p^{l,m}(\alpha_1)f(z)}.
\]
(3.3) By a simple computation from (3.3) we get
\[
\frac{z \psi'(z)}{\psi(z)} = \frac{z[H_p^{l,m}(\alpha_1 + 1)f(z)]'}{H_p^{l,m}(\alpha_1 + 1)f(z)} - \frac{z[H_p^{l,m}(\alpha_1)f(z)]'}{H_p^{l,m}(\alpha_1)f(z)}.
\]
(3.4) By making use of (2.2) in the equation (3.4), we obtain
\[
\frac{z \psi'(z)}{\psi(z)} = (\alpha_1 + 1) \frac{H_p^{l,m}(\alpha_1 + 2)f(z)}{H_p^{l,m}(\alpha_1 + 1)f(z)} - \alpha_1 - \frac{H_p^{l,m}(\alpha_1)f(z)}{H_p^{l,m}(\alpha_1)f(z)} - 1.
\]
(3.5) Using (3.3) in (3.5), we get
\[
\frac{H_p^{l,m}(\alpha_1 + 2)f(z)}{H_p^{l,m}(\alpha_1 + 1)f(z)} = \frac{1}{1 + \alpha_1} \left[ \frac{z \psi'(z)}{\psi(z)} + \alpha_1 \psi(z) + 1 \right].
\]
(3.6) Therefore we have from (3.6),
\[
\frac{H_p^{l,m}(\alpha_1 + 1)f(z)}{H_p^{l,m}(\alpha_1)f(z)} \left\{ 1 - \alpha + \frac{H_p^{l,m}(\alpha_1 + 2)f(z)}{H_p^{l,m}(\alpha_1)f(z)} \right\} = \frac{1 + \alpha_1(1 - \alpha)}{1 + \alpha_1} \psi(z) + \frac{\alpha_1}{1 + \alpha_1} \psi^2(z) + \frac{\alpha}{1 + \alpha_1} z \psi'(z).
\]
(3.7) In view of the equation (3.7), the subordination (3.1) becomes
\[
[1 + \alpha_1(1 - \alpha)] \psi(z) + \alpha_1 \psi^2(z) + \alpha z \psi'(z) < [1 + \alpha_1(1 - \alpha)] q(z) + \alpha_1 q^2(z) + \alpha z q'(z)
\]
and the result now follows by an application of Lemma 2.3.

By making use of Lemma 2.2, we now prove the following
Theorem 3.2. Let \( q(z) \) be univalent in \( \Delta \), \( q(0) = 1 \). Let \( zq'(z)/q(z) \) be starlike univalent in \( \Delta \). If \( f(z) \in A_p \) satisfies
\[
\begin{align*}
(\alpha + 1) \frac{H_p^{l,m}(\alpha + 2)f(z)}{H_p^{l,m}(\alpha + 1)f(z)} - \alpha \alpha_1 \frac{H_p^{l,m}(\alpha + 1)f(z)}{H_p^{l,m}(\alpha_1)f(z)} < \frac{zq'(z)}{q(z)} + 1 - \alpha_1 - \alpha \alpha_1
\end{align*}
\]
(3.8)
then
\[
\begin{align*}
\frac{z^{p(\alpha - 1)}H_p^{l,m}(\alpha + 1)f(z)}{(H_p^{l,m}(\alpha_1)f(z))^\alpha} < q(z)
\end{align*}
\]
(3.9)
and \( q(z) \) is the best dominant.

Proof. Define the function \( \psi(z) \) by
\[
\begin{align*}
\psi(z) := \frac{z^{p(\alpha - 1)}H_p^{l,m}(\alpha + 1)f(z)}{(H_p^{l,m}(\alpha_1)f(z))^\alpha}.
\end{align*}
\]
(3.10)
By a simple computation from (3.10) we get
\[
\begin{align*}
\frac{z\psi'(z)}{\psi(z)} = \alpha - 1 + \frac{z[H_p^{l,m}(\alpha + 1)f(z)]'}{H_p^{l,m}(\alpha + 1)f(z)} - \alpha \frac{z[H_p^{l,m}(\alpha_1)f(z)]'}{H_p^{l,m}(\alpha_1)f(z)}.
\end{align*}
\]
(3.11)
By making use of (2.2) in the equation (3.11), we obtain
\[
\begin{align*}
(\alpha + 1) \frac{H_p^{l,m}(\alpha + 2)f(z)}{H_p^{l,m}(\alpha + 1)f(z)} - \alpha \alpha_1 \frac{H_p^{l,m}(\alpha + 1)f(z)}{H_p^{l,m}(\alpha_1)f(z)} = \frac{z\psi'(z)}{\psi(z)} + 1 - \alpha_1 - \alpha \alpha_1.
\end{align*}
\]
(3.12)
In view of the equation (3.12), the subordination (3.8) becomes
\[
\begin{align*}
\frac{z\psi'(z)}{\psi(z)} < \frac{zq'(z)}{q(z)}
\end{align*}
\]
and the result now follows by an application of Lemma 2.2.

4. SUFFICIENT CONDITIONS INVOLVING MULTIPLIER TRANSFORM

By making use of Lemma 2.3, we prove the following

Theorem 4.1. Let \( q(z) \) be convex univalent, \( \alpha \neq 0 \). Further assume that
\[
\Re \left\{ \frac{(1 - \alpha)(p + \lambda)}{\alpha} + 2(p + \lambda)q(z) + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0.
\]
If \( f(z) \in A_p \) satisfies
\[
\begin{align*}
\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \left\{ 1 - \alpha + \frac{I_p(n + 2, \lambda)f(z)}{I_p(n + 1, \lambda)f(z)} \right\} < (1 - \alpha)q(z) + \alpha q^2(z) + \frac{\alpha}{p + \lambda} zq'(z),
\end{align*}
\]
(4.1)
then
\[
\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} < q(z)
\]
and \(q(z)\) is the best dominant.

**Proof.** Define the function \(\psi(z)\) by
\[
\psi(z) := \frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)}.
\]
By a simple computation from (4.3) we get
\[
\frac{z\psi'(z)}{\psi(z)} = \frac{z[I_p(n+1, \lambda)f(z)']}{I_p(n+1, \lambda)f(z)} - \frac{z[I_p(n, \lambda)f(z)']}{I_p(n, \lambda)f(z)}.
\]
By making use of (2.4) in the equation (4.4), we obtain
\[
\frac{z\psi'(z)}{\psi(z)} = (p + \lambda) \left[ \frac{I_p(n+2, \lambda)f(z)}{I_p(n+1, \lambda)f(z)} - \frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \right].
\]
Using (4.3) in (4.5), we get
\[
\frac{I_p(n+2, \lambda)f(z)}{I_p(n+1, \lambda)f(z)} = \frac{1}{p + \lambda} \left[ \frac{z\psi'(z)}{\psi(z)} + (p + \lambda)\psi(z) \right]
\]
Therefore we have from (4.6),
\[
\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \left\{ 1 - \alpha + \frac{\alpha}{p + \lambda} z\psi'(z) \right\} = (1 - \alpha)\psi(z) + \alpha\psi^2(z) + \frac{\alpha}{p + \lambda} z\psi'(z).
\]
In view of the equation (4.7), the subordination (4.1) becomes
\[
(1 - \alpha)\psi(z) + \alpha\psi^2(z) + \frac{\alpha}{p + \lambda} z\psi'(z) < (1 - \alpha)q(z) + \alpha q^2(z) + \frac{\alpha}{p + \lambda} zq'(z)
\]
and the result now follows by an application of Lemma 2.3.

By making use of Lemma 2.2, we now prove the following

**Theorem 4.2.** Let \(q(z)\) be univalent in \(\Delta\), \(q(0) = 1\). Let \(\alpha q'(z)/q(z)\) be starlike univalent in \(\Delta\). If \(f(z) \in A_p\) satisfies
\[
\frac{I_p(n+2, \lambda)f(z)}{I_p(n+1, \lambda)f(z)} - \alpha \frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \leq \frac{1}{p + \lambda} \frac{zq'(z)}{q(z)} + 1 - \alpha,
\]
then
\[
\frac{z^{p(\alpha-1)}I_p(n+1, \lambda)f(z)}{(I_p(n, \lambda)f(z))^\alpha} < q(z)
\]
and \(q(z)\) is the best dominant.
Proof. Define the function $\psi(z)$ by

\begin{equation}
\psi(z) = \frac{z^{p(a-1)}I_p(n+1, \lambda)f(z)}{(I_p(n, \lambda)f(z))^{a}}.
\end{equation}

By a simple computation from (4.9) we get

\begin{equation}
\frac{z\psi'(z)}{\psi(z)} = \alpha - 1 + \frac{z[I_p(n+1, \lambda)f(z)]'}{I_p(n+1, \lambda)f(z)} - \alpha \frac{z[I_p(n, \lambda)f(z)]'}{I_p(n, \lambda)f(z)}.
\end{equation}

By making use of (2.4) in the equation (4.10), we obtain

\begin{equation}
\frac{I_p(n+2, \lambda)f(z)}{I_p(n+1, \lambda)f(z)} - \alpha \frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} = \frac{1}{p+\lambda} \frac{z\psi'(z)}{\psi(z)} + 1 - \alpha.
\end{equation}

In view of the equation (4.11), the subordination (4.2) becomes

\begin{equation}
\frac{z\psi'(z)}{\psi(z)} \prec \frac{zq'(z)}{q(z)}
\end{equation}

and the result now follows by an application of Lemma 2.2.

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