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A CLASS OF MULTIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY CONVOLUTION

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ABSTRACT. For a given p-valent analytic function g with positive coefficients in the open unit disk Δ, we study a class of functions $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n$, $a_n \geq 0$ satisfying

$$\frac{1}{p}\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) < \alpha \quad \left(z \in \Delta; 1 < \alpha < \frac{m+p}{2p}\right).$$

Coefficient inequalities, distortion and covering theorems, as well as closure theorems are determined. The results obtained extend several known results as special cases.

Key words and phrases: Starlike function, Ruscheweyh derivative, Convolution, Positive coefficients, Coefficient inequalities, Growth and distortion theorems.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all analytic functions $f(z)$ in the unit disk $\Delta := \{z \in \mathcal{C} : |z| < 1\}$ with $f(0) = 0 = f'(0) - 1$. The class $M(\alpha)$ defined by

$$M(\alpha) := \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha \quad \left(1 < \alpha < \frac{3}{2};\ z \in \Delta\right)\right\}$$

was investigated by Uralegaddi *et al.* [6]. A subclass of $M(\alpha)$ was recently investigated by Owa and Srivastava [3]. Motivated by $M(\alpha)$, we introduce a more general class $PM_g(p, m, \alpha)$ of analytic functions with positive coefficients. For two analytic functions

$$f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z^p + \sum_{n=m}^{\infty} b_n z^n,$$

the convolution (or Hadamard product) of f and g, denoted by $f * g$ or $(f * g)(z)$, is defined by

$$(f * g)(z) := z^p + \sum_{n=m}^{\infty} a_n b_n z^n.$$

Let $T(p, m)$ be the class of all analytic p-valent functions $f(z) = z^p - \sum_{n=m}^{\infty} a_n z^n$ $(a_n \geq 0)$, defined on the unit disk Δ and let $T := T(1, 2)$. A function $f(z) \in T(p, m)$ is called a function with negative coefficients. The subclass of T consisting of starlike functions of order α, denoted by $TS^*(\alpha)$, was studied by Silverman [5]. Several other classes of starlike functions with negative coefficients were studied; for e.g. see [2].

Let $P(p, m)$ be the class of all analytic functions

$$f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n \quad (a_n \geq 0) \tag{1.1}$$

and $P := P(1, 2)$.

Definition 1.1. Let

$$g(z) = z^p + \sum_{n=m}^{\infty} b_n z^n \quad (b_n > 0) \tag{1.2}$$

be a fixed analytic function in Δ. Define the class $PM_g(p, m, \alpha)$ by

$$PM_g(p, m, \alpha) := \left\{ f \in P(p, m) : \frac{1}{p}\Re\left(\frac{z(f * g)'(z)}{(f * g)(z)}\right) < \alpha, \quad \left(1 < \alpha < \frac{m + p}{2p}; z \in \Delta\right)\right\}.$$

When $g(z) = z/(1 - z)$, $p = 1$ and $m = 2$, the class $PM_g(p, m, \alpha)$ reduces to the subclass $PM(\alpha) := P \cap M(\alpha)$. When $g(z) = z/(1 - z)^{\lambda+1}$, $p = 1$ and $m = 2$, the class $PM_g(p, m, \alpha)$ reduces to the class:

$$P_\lambda(\alpha) = \left\{ f \in P : \Re\left(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}\right) < \alpha, \quad \left(\lambda > -1, 1 < \alpha < \frac{3}{2}; z \in \Delta\right)\right\},$$

where D^λ denotes the Ruscheweyh derivative of order λ. When

$$g(z) = z + \sum_{n=2}^{\infty} n^l z^n,$$

the class of functions $PM_g(1, 2, \alpha)$ reduces to the class $PM_l(\alpha)$ where

$$PM_l(\alpha) = \left\{ f \in P : \Re\left(\frac{z(\mathcal{D}^l f(z))'}{\mathcal{D}^l f(z)}\right) < \alpha, \quad \left(1 < \alpha < \frac{3}{2}; l \geq 0;\ \ z \in \Delta\right)\right\},$$

where $\mathcal{D}^l$ denotes the Salagean derivative of order l. Also we have

$$PM(\alpha) \equiv P_0(\alpha) \equiv PM_0(\alpha).$$

A function $f \in \mathcal{A}(p,m)$ is in $PPC(p,m,\alpha,\beta)$ if

$$\frac{1}{p}\Re\left(\frac{(1-\beta)zf'(z)+\frac{\beta}{p}z(zf')'(z)}{(1-\beta)f(z)+\frac{\beta}{p}zf'(z)}\right) < \alpha \quad \left(\beta \geq 0;\ 0 \leq \alpha < \frac{m+p}{2p}\right)$$

This class is similar to the class of β-Pascu convex functions of order α and it unifies the class of $PM(\alpha)$ and the corresponding convex class.

For the newly defined class $PM_g(p,m,\alpha)$, we obtain coefficient inequalities, distortion and covering theorems, as well as closure theorems. As special cases, we obtain results for the classes $P_\lambda(\alpha)$, and $PM_l(\alpha)$. Similar results for the class $PPC(p,m,\alpha,\beta)$ also follow from our results, the details of which are omitted here.

2. COEFFICIENT INEQUALITIES

Throughout the paper, we assume that the function $f(z)$ is given by the equation (1.1) and $g(z)$ is given by by (1.2). We first prove a necessary and sufficient condition for functions to be in the class $PM_g(p,m,\alpha)$ in the following:

Theorem 2.1. *A function $f \in PM_g(p,m,\alpha)$ if and only if*

$$\sum_{n=m}^{\infty}(n-p\alpha)a_nb_n \leq p(\alpha-1) \quad \left(1 < \alpha < \frac{m+p}{2p}\right). \tag{2.1}$$

Proof. If $f \in PM_g(p,m,\alpha)$, then (2.1) follows from

$$\frac{1}{p}\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) < \alpha$$

by letting $z \to 1-$ through real values. To prove the converse, assume that (2.1) holds. Then by making use of (2.1), we obtain

$$\left|\frac{z(f*g)'(z)-p(f*g)(z)}{z(f*g)'(z)-(2\alpha-1)p(f*g)(z)}\right| \leq \frac{\sum_{n=m}^{\infty}(n-p)a_nb_n}{2(\alpha-1)p-\sum_{n=m}^{\infty}[n-(2\alpha-1)p]a_nb_n} \leq 1$$

or equivalently $f \in PM_g(p,m,\alpha)$. □

Corollary 2.2. *A function $f \in P_\lambda(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_nB_n(\lambda) \leq \alpha-1 \quad \left(1 < \alpha < \frac{3}{2}\right),$$

where

$$B_n(\lambda) = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}{(n-1)!}. \tag{2.2}$$

Corollary 2.3. *A function $f \in PM_m(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_nn^m \leq \alpha-1 \quad \left(1 < \alpha < \frac{3}{2}\right).$$

Our next theorem gives an estimate for the coefficient of functions in the class $PM_g(p,m,\alpha)$.

Theorem 2.4. *If $f \in PM_g(p,m,\alpha)$, then*

$$a_n \leq \frac{p(\alpha-1)}{(n-p\alpha)b_n}$$

with equality only for functions of the form

$$f_n(z) = z^p + \frac{p(\alpha-1)}{(n-p\alpha)b_n} z^n.$$

Proof. Let $f \in PM_g(p,m,\alpha)$. By making use of the inequality (2.1), we have

$$(n-p\alpha)a_n b_n \leq \sum_{n=m}^{\infty}(n-p\alpha)a_n b_n \leq p(\alpha-1)$$

or

$$a_n \leq \frac{p(\alpha-1)}{(n-p\alpha)b_n}.$$

Clearly for

$$f_n(z) = z^p + \frac{p(\alpha-1)}{(n-p\alpha)b_n} z^n \in PM_g(p,m,\alpha),$$

we have

$$a_n = \frac{p(\alpha-1)}{(n-p\alpha)b_n}.$$

□

Corollary 2.5. *If $f \in P_\lambda(\alpha)$, then*

$$a_n \leq \frac{\alpha-1}{(n-\alpha)B_n(\lambda)}$$

with equality only for functions of the form

$$f_n(z) = z + \frac{\alpha-1}{(n-\alpha)B_n(\lambda)} z^n,$$

where $B_n(\lambda)$ is given by (2.2).

Corollary 2.6. *If $f \in PM_m(\alpha)$, then*

$$a_n \leq \frac{\alpha-1}{(n-\alpha)n^m}$$

with equality only for functions of the form

$$f_n(z) = z + \frac{\alpha-1}{(n-\alpha)n^m} z^n.$$

3. GROWTH AND DISTORTION THEOREMS

We now prove the growth theorem for the functions in the class $PM_g(p,m,\alpha)$.

Theorem 3.1. *If $f \in PM_g(p,m,\alpha)$, then*

$$r^p - \frac{p(\alpha-1)}{(m-p\alpha)b_m} r^m \leq |f(z)| \leq r^p + \frac{p(\alpha-1)}{(m-p\alpha)b_m} r^m, \quad |z| = r < 1,$$

provided $b_n \geq b_m \geq 1$. The result is sharp for

$$f(z) = z^p + \frac{p(\alpha-1)}{(m-p\alpha)b_m} z^m. \tag{3.1}$$

Proof. By making use of the inequality (2.1) for $f \in PM_g(p,m,\alpha)$ together with

$$(m-p\alpha)b_m \leq (n-p\alpha)b_n,$$

we obtain

$$b_m(m-p\alpha)\sum_{n=m}^{\infty} a_n \leq \sum_{n=m}^{\infty}(n-p\alpha)a_n b_n \leq p(\alpha-1)$$

or

$$\sum_{n=m}^{\infty} a_n \leq \frac{p(\alpha-1)}{(m-p\alpha)b_m}. \tag{3.2}$$

By using (3.2) for the function $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n \in PM_g(p,m,\alpha)$, we have for $|z| = r$,

$$\begin{aligned}|f(z)| &\leq r^p + \sum_{n=m}^{\infty} a_n r^n \\ &\leq r^p + r^m \sum_{n=m}^{\infty} a_n \\ &\leq r^p + \frac{p(\alpha-1)}{(m-p\alpha)b_m} r^m,\end{aligned}$$

and similarly,

$$|f(z)| \geq r^p - \frac{p(\alpha-1)}{(m-p\alpha)b_m} r^m.$$

□

Theorem 3.1 also shows that $f(\Delta)$ for every $f \in PM_g(p,m,\alpha)$ contains the disk of radius $1 - \frac{p(\alpha-1)}{(m-p\alpha)b_m}$.

Corollary 3.2. *If $f \in P_\lambda(\alpha)$, then*

$$r - \frac{\alpha-1}{(2-\alpha)(\lambda+1)}r^2 \leq |f(z)| \leq r + \frac{\alpha-1}{(2-\alpha)(\lambda+1)}r^2 \quad (|z|=r).$$

The result is sharp for

$$f(z) = z + \frac{\alpha-1}{(2-\alpha)(\lambda+1)}z^2. \tag{3.3}$$

Corollary 3.3. *If $f \in PM_m(\alpha)$, then*

$$r - \frac{\alpha-1}{(2-\alpha)2^m}r^2 \leq |f(z)| \leq r + \frac{\alpha-1}{(2-\alpha)2^m}r^2 \quad (|z|=r).$$

The result is sharp for

$$f(z) = z + \frac{\alpha-1}{(2-\alpha)2^m}z^2. \tag{3.4}$$

The distortion estimates for the functions in the class $PM_g(p,m,\alpha)$ is given in the following:

Theorem 3.4. *If $f \in PM_g(p,m,\alpha)$, then*

$$pr^{p-1} - \frac{mp(\alpha-1)}{(m-p\alpha)b_m}r^{m-1} \leq |f'(z)| \leq pr^{p-1} + \frac{mp(\alpha-1)}{(m-p\alpha)b_m}r^{m-1}, \quad |z| = r < 1,$$

provided $b_n \geq b_m$. The result is sharp for the function given by (3.1).

Proof. By making use of the inequality (2.1) for $f \in PM_g(p, m, \alpha)$, we obtain

$$\sum_{n=m}^{\infty} a_n b_n \leq \frac{p(\alpha - 1)}{(m - p\alpha)}$$

and therefore, again using the inequality (2.1), we get

$$\sum_{n=m}^{\infty} n a_n \leq \frac{mp(\alpha - 1)}{(m - p\alpha) b_m}.$$

For the function $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n \in PM_g(p, m, \alpha)$, we now have

$$\begin{aligned} |f'(z)| &\leq pr^{p-1} + \sum_{n=m}^{\infty} n a_n r^{n-1} \quad (|z| = r) \\ &\leq pr^{p-1} + r^{m-1} \sum_{n=m}^{\infty} n a_n \\ &\leq pr^{p-1} + \frac{mp(\alpha - 1)}{(m - p\alpha) b_m} r^{m-1} \end{aligned}$$

and similarly we have

$$|f'(z)| \geq pr^{p-1} - \frac{mp(\alpha - 1)}{(m - p\alpha) b_m} r^{m-1}.$$

□

Corollary 3.5. *If $f \in P_\lambda(\alpha)$, then*

$$1 - \frac{2(\alpha - 1)}{(2 - \alpha)(\lambda + 1)} r \leq |f'(z)| \leq 1 + \frac{2(\alpha - 1)}{(2 - \alpha)(\lambda + 1)} r \quad (|z| = r).$$

The result is sharp for the function given by (3.3)

Corollary 3.6. *If $f \in PM_m(\alpha)$, then*

$$1 - \frac{2(\alpha - 1)}{(2 - \alpha) 2^m} r \leq |f(z)| \leq 1 + \frac{2(\alpha - 1)}{(2 - \alpha) 2^m} r \quad (|z| = r).$$

The result is sharp for the function given by (3.4)

4. CLOSURE THEOREMS

We shall now prove the following closure theorems for the class $PM_g(p, m, \alpha)$. Let the functions $F_k(z)$ be given by

$$F_k(z) = z^p + \sum_{n=m}^{\infty} f_{n,k} z^n, \quad (k = 1, 2, \ldots, M). \tag{4.1}$$

Theorem 4.1. *Let $\lambda_k \geq 0$ for $k = 1, 2, \ldots, M$ and $\sum_{k=1}^{M} \lambda_k \leq 1$. Let the function $F_k(z)$ defined by (4.1) be in the class $PM_g(p, m, \alpha)$ for every $k = 1, 2, \ldots, M$. Then the function $f(z)$ defined by*

$$f(z) = z^p + \sum_{n=m}^{\infty} \left(\sum_{k=1}^{M} \lambda_k f_{n,k} \right) z^n$$

belongs to the class $PM_g(p, m, \alpha)$.

Proof. Since $F_k(z) \in PM_g(p,m,\alpha)$, it follows from Theorem 2.1 that

$$\sum_{n=m}^{\infty}(n-p\alpha)b_n f_{n,k} \leq p(\alpha-1) \tag{4.2}$$

for every $k = 1, 2, \dots, M$. Hence

$$\begin{aligned}\sum_{n=m}^{\infty}(n-p\alpha)b_n\left(\sum_{k=1}^{M}\lambda_k f_{n,k}\right) &= \sum_{k=1}^{M}\lambda_k\left(\sum_{n=m}^{\infty}(n-p\alpha)b_n f_{n,k}\right)\\ &\leq \sum_{k=1}^{M}\lambda_k p(\alpha-1)\\ &\leq p(\alpha-1).\end{aligned}$$

By Theorem 2.1, it follows that $f(z) \in PM_g(p,m,\alpha)$. □

Corollary 4.2. *The class $PM_g(p,m,\alpha)$ is closed under convex linear combinations.*

Theorem 4.3. *Let*

$$F_p(z) = z^p \text{ and } F_n(z) = z^p + \frac{p(\alpha-1)}{(n-p\alpha)b_n}z^n$$

for $n = m, m+1, \dots$. Then $f(z) \in PM_g(p,m,\alpha)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = \lambda_p z^p + \sum_{n=m}^{\infty}\lambda_n F_n(z), \tag{4.3}$$

where each $\lambda_j \geq 0$ and $\lambda_p + \sum_{n=m}^{\infty}\lambda_n = 1$.

Proof. Let $f(z)$ be of the form (4.3). Then

$$f(z) = z^p + \sum_{n=m}^{\infty}\frac{\lambda_n p(\alpha-1)}{(n-p\alpha)b_n}z^n$$

and therefore

$$\sum_{n=m}^{\infty}\frac{\lambda_n p(\alpha-1)}{(n-p\alpha)b_n}\frac{(n-p\alpha)b_n}{p(\alpha-1)} = \sum_{n=m}^{\infty}\lambda_n = 1-\lambda_p \leq 1.$$

By Theorem 2.1, we have $f(z) \in PM_g(p,m,\alpha)$.

Conversely, let $f(z) \in PM_g(p,m,\alpha)$. From Theorem 2.4, we have

$$a_n \leq \frac{p(\alpha-1)}{(n-p\alpha)b_n} \quad \text{for} \quad n = m, m+1, \dots.$$

Therefore we may take

$$\lambda_n = \frac{(n-p\alpha)b_n a_n}{p(\alpha-1)} \quad \text{for} \quad n = m, m+1, \dots$$

and

$$\lambda_p = 1 - \sum_{n=m}^{\infty}\lambda_n.$$

Then

$$f(z) = \lambda_p z^p + \sum_{n=m}^{\infty}\lambda_n F_n(z).$$

□

We now prove that the class $PM_g(p,m,\alpha)$ is closed under convolution with certain functions and give an application of this result to show that the class $PM_g(p,m,\alpha)$ is closed under the familiar Bernardi integral operator.

Theorem 4.4. *Let* $h(z) = z^p + \sum_{n=m}^{\infty} h_n z^n$ *be analytic in* Δ *with* $0 \le h_n \le 1$. *If* $f(z) \in PM_g(p,m,\alpha)$*, then* $(f * h)(z) \in PM_g(p,m,\alpha)$.

Proof. The result follows directly from Theorem 2.1. □

The generalized Bernardi integral operator is defined by the following integral:

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt \quad (c > -1;\ z \in \Delta). \tag{4.4}$$

Since

$$F(z) = f(z) * \left(z^p + \sum_{n=m}^{\infty} \frac{c+p}{c+n} z^n \right),$$

we have the following:

Corollary 4.5. *If* $f(z) \in PM_g(p,m,\alpha)$*, then* $F(z)$ *given by (4.4) is also in* $PM_g(p,m,\alpha)$.

5. ORDER AND RADIUS RESULTS

Let $PS_h^*(p,m,\beta)$ be the subclass of $P(m,p)$ consisting of functions f for which $f * h$ is starlike of order β.

Theorem 5.1. *Let* $h(z) = z^p + \sum_{n=m}^{\infty} h_n z^n$ *with* $h_n > 0$. *Let* $(\alpha-1)nh_n \le (n-p\alpha)b_n$. *If* $f \in PM_g(p,m,\alpha)$*, then* $f \in PS_h^*(p,m,\beta)$*, where*

$$\beta := \inf_{n \ge m}\left[\frac{(n-p\alpha)b_n - (\alpha-1)nh_n}{(n-p\alpha)b_n - (\alpha-1)ph_n}\right].$$

Proof. Let us first note that the condition $(\alpha-1)nh_n \le (n-p\alpha)b_n$ implies $f \in PS_h^*(p,m,0)$. From the definition of β, it follows that

$$\beta \le \frac{(n-p\alpha)b_n - (\alpha-1)nh_n}{(n-p\alpha)b_n - (\alpha-1)ph_n}$$

or

$$\frac{(n-p\beta)h_n}{1-\beta} \le \frac{(n-p\alpha)b_n}{\alpha-1}$$

and therefore, in view of (2.1),

$$\sum_{n=m}^{\infty}\frac{(n-p\beta)}{p(1-\beta)}a_n h_n \le \sum_{n=m}^{\infty}\frac{(n-p\alpha)}{p(\alpha-1)}a_n b_n \le 1.$$

Thus

$$\left|\frac{1}{p}\cdot\frac{z(f*h)'(z)}{(f*h)(z)} - 1\right| \le \frac{\sum_{n=m}^{\infty}(n/p-1)a_n h_n}{1-\sum_{n=m}^{\infty} a_n h_n} \le 1-\beta$$

and therefore $f \in PS_h^*(p,m,\beta)$. □

Similarly we can prove the following:

Theorem 5.2. *If* $f \in PM_g(p,m,\alpha)$*, then* $f \in PM_h(p,m,\beta)$ *in* $|z| < r(\alpha,\beta)$ *where*

$$r(\alpha,\beta) := \min\left\{1; \inf_{n\ge m}\left[\frac{(n-p\alpha)}{(n-p\alpha)}\frac{(\beta-1)}{(\alpha-1)}\frac{b_n}{h_n}\right]^{\frac{1}{n-p}}\right\}.$$

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