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ON CERTAIN APPLICATIONS OF DIFFERENTIAL SUBORDINATIONS FOR Φ-LIKE FUNCTIONS

V. RAVICHANDRAN, N. MAGESH AND R. RAJALAKSHMI

Abstract. Let $f(z)$ be a normalized analytic function in $\Delta = \{z|z \in \mathbb{C}$ and $|z| < 1\}$ satisfying $f(0) = 0$ and $f'(0) = 1$. Let Φ be an analytic function in a domain containing $f(\Delta)$, with $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(\omega) \neq 0$ for $\omega \in f(\Delta) - \{0\}$. Let $q(z)$ be a fixed analytic function in Δ, $q(0) = 1$. The function f is called Φ-like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z) \quad (z \in \Delta).$$

In this paper, we obtain some sufficient conditions for functions to be Φ-like with respect to $q(z)$.

1. Introduction and Definitions

Let $\mathcal{A}$ be the class of all normalized analytic functions $f(z)$ in the open unit disk $\Delta = \{z|z \in \mathbb{C}$ and $|z| < 1\}$ satisfying $f(0) = 0$ and $f'(0) = 1$. Let Φ be an analytic function in a domain containing $f(\Delta)$, $\Phi(0) = 0$ and $\Phi'(0) > 0$. The function $f \in \mathcal{A}$ is called Φ-*like* if

$$\Re\frac{zf'(z)}{\Phi(f(z))} > 0 \quad (z \in \Delta).$$

This concept was introduced by Brickman [1] and it was shown that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is Φ-like for some Φ. When $\Phi(w) = w$ and $\Phi(w) = \lambda w$, the function f is starlike and spirallike of type $\arg\lambda$ respectively. Later, Ruscheweyh [13] introduced and studied the following general class of Φ-like functions:

Definition 1.1. Let Φ be analytic in a domain containing $f(\Delta)$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(\omega) \neq 0$ for $\omega \in f(\Delta) - \{0\}$. Let $q(z)$ be a fixed analytic function in Δ, $q(0) = 1$. The function $f \in \mathcal{A}$ is called Φ-like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z) \quad (z \in \Delta).$$

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When $\Phi(w) = w$, we denote the class of all Φ-like functions with respect to q by $S^*(q)$.

Li and Owa [4], Lewandowski *et al.* [3] and Ramesha *et al.* [10], Nunokawa *et al.* [6], Ravichandran *et al.* [11], and Padamanbhan [9] have found out sufficient conditions for functions to be starlike or starlike of order α in terms of

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}.$$

Ravichandran [12, Theorem 3, p.44] have obtained a sufficient condition for functions to be in the class $S^*(q)$ which generalize or improve the earlier results. Our first aim in this paper is to obtain a sufficient conditions for a function $f(z)$ to be a Φ-like functions with respect to $q(z)$.

Kim [2] have found out estimates on $C(\alpha)$ for which

$$p(z)\left(1 + \frac{zp'(z)}{p(z)}\right)^{\alpha} \prec C(\alpha)h(z) \quad (0 \le \alpha \le 1)$$

implies $p(z) \prec h(z)$. In fact, they have shown that $\left(\frac{3}{2}\right)^{\alpha} \le C(\alpha) \le 2^{\alpha}$ for $\alpha \in [0,1]$ for certain $p(z)$ and $h(z)$. Parvatham [8] considered a similar problem. In this paper, our second aim is to determine the function $h(z)$ such that

$$p(z)\left(1 + \frac{zp'(z)}{p(z)}\right)^{\alpha} \prec h(z)$$

implies $p(z) \prec q(z)$ and apply this result to derive a sufficient condition for a function to Φ-like with respect to $q(z)$.

2. Subordination Results for Φ-like Functions

In our present investigation, we need the following result of Miller and Mocanu [5]:

Lemma 2.1.(cf. Miller and Mocanu [5, p.132, Theorem 3.4h]) *Let $q(z)$ be univalent in the unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set*

$$Q(z) := zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) := \theta(q(z)) + Q(z).$$

Suppose that

1. *$Q(z)$ is starlike univalent in Δ and*
2. *$\Re\frac{zh'(z)}{Q(z)} > 0$ for $z \in \Delta$.*

If $p(z)$ is analytic with $p(0) = q(0)$, $p(\Delta) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.2.1}$$

then

$$p(z) \prec q(z)$$

and $q(z)$ is the best dominant.

By appealing to Lemma 2.1, we first prove the following:

Theorem 2.2. *Let $\alpha \neq 0$ be a complex number and $q(z)$ be convex univalent in Δ. Define $h(z)$ by*

$$h(z) := \alpha q^2(z) + (1-\alpha)q(z) + \alpha z q'(z). \tag{2.2.2}$$

Further assume that

$$\Re\left\{\frac{1-\alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \quad (z \in \Delta). \tag{2.2.3}$$

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\Phi(f(z))}\left[1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha z[f'(z) - \{\Phi(f(z))\}']}{\Phi(f(z))}\right] \prec h(z), \tag{2.2.4}$$

then

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{zf'(z)}{\Phi(f(z))} \quad (z \in \Delta). \tag{2.2.5}$$

Then the function $p(z)$ is analytic in Δ and $p(0) = 1$. Therefore, by making use of (2.2.5), we obtain

$$\begin{aligned} &\frac{zf'(z)}{\Phi(f(z))}\left[1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha z[f'(z) - (\Phi(f(z)))']}{\Phi(f(z))}\right] \\ &\qquad = p(z)\left[1 + \alpha\left(\frac{zp'(z)}{p(z)} - 1\right) + \alpha p(z)\right] \\ &\qquad = \alpha p^2(z) + (1-\alpha)p(z) + \alpha z p'(z). \end{aligned} \tag{2.2.6}$$

By using equation (2.2.6) in subordination (2.2.4), we have

$$\alpha p^2(z) + (1-\alpha)p(z) + \alpha z p'(z) \prec \alpha q^2(z) + (1-\alpha)q(z) + \alpha z q'(z). \tag{2.2.7}$$

The subordination (2.2.7) is same as (2.2.1) with

$$\phi(\omega) := \alpha\omega^2 + (1-\alpha)\omega \quad \text{and} \quad \theta(\omega) := \alpha.$$

The result now follows by an application of Lemma 2.1.

Remark 2.1. By taking $\Phi(w) = w$, we obtain [12, Theorem 3, p.44] as a special case of Theorem 2.2.

Remark 2.2. The condition

$$\Re\left\{\frac{1-\alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \quad (z \in \Delta)$$

is satisfied by any convex function which maps Δ onto a convex region in the right-half plane when $0 < \alpha \leq 1$.

Remark 2.3. By taking $\Phi(w) := w$ and $q(z) := (1+Az)/(1+Bz)$, $-1 \leq B < A \leq 1$, in the above Theorem 2.1, we get a sufficient condition for function to be Janowski starlike.

3. Another Sufficient Condition

We need the following Lemma to our main result of this section.

Lemma 3.1. *Let $\alpha \neq 0$ be any complex number and $\beta := \max\{0. -\Re\frac{1}{\alpha}\}$. Let $q(z) \neq 0$ be analytic in Δ and*

$$Q(z) := zq'(z)q^{\frac{1}{\alpha}-1}(z)$$

be starlike of order β in Δ. If $p(z)$ is analytic in Δ and

$$p(z)\left(1 + \frac{zp'(z)}{p(z)}\right)^{\alpha} \prec q(z)\left(1 + \frac{zq'(z)}{q(z)}\right)^{\alpha}, \tag{3.3.1}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Since the subordination (3.3.1) is equivalent to

$$p^{\frac{1}{\alpha}}(z) + zp'(z)p^{\frac{1}{\alpha}-1}(z) \prec q^{\frac{1}{\alpha}}(z) + zq'(z)q^{\frac{1}{\alpha}-1}(z), \tag{3.3.2}$$

the result follows by an application of Lemma 2.1.

Remark 3.1. Let $0 < \alpha \leq 1$. If $q(z)$ is a convex function that maps Δ onto a region in the right half-plane, then we see that $Q(z)$ is starlike and the hypothesis of Lemma 3.1 is satisfied.

By making use of the above Lemma 3.1, we obtain the following:

Theorem 3.2. *Let $\alpha \neq 0$ be any complex number and $\beta = \max\{0, -\Re\frac{1}{\alpha}\}$. Let $q(z) \neq 0$ be analytic in Δ and*

$$Q(z) = zq'(z)q^{\frac{1}{\alpha}-1}(z)$$

be starlike of order β in Δ. If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\Phi(f(z))}\left(1+\frac{zf''(z)}{f'(z)}-\frac{\Phi'(f(z))}{\Phi(f(z))}zf'(z)\right)^{\alpha} \prec q(z)\left(1+\frac{zq'(z)}{q(z)}\right)^{\alpha}, \tag{3.3.3}$$

then

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. By taking

$$p(z) := \frac{zf'(z)}{\Phi(f(z))} \quad (z \in \Delta), \tag{3.3.4}$$

we see that Theorem 3.2 follows from Lemma 3.1.

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Department of Computer Applications, Sri Venkateswara College of Engineerig, Pennalur, Sripermubudur 602 105, India.

E-mail: vravi@svce.ac.in

Department of Mathematics, Adhiyamaan College of Engineering, Hosur 635 109, India.

Department of Applied Mathematics, Faculty of Natural Sciences, Debub University, Awassa, Ethiopia.