Subclasses of Multivalent Starlike and Convex Functions

Rosihan M. Ali V. Ravichandran See Keong Lee*

Abstract

Subclasses of p-valent starlike and convex functions in the unit disk in the complex plane are investigated. Every p-valent convex function in a subclass is shown to belong to its corresponding subclass of starlike functions. A necessary and sufficient condition for functions to belong to these classes is obtained. Subordination properties, and sharp distortion, growth, covering and rotation estimates are obtained for these classes. Convolution results with prestarlike functions are also derived.

1 Preliminaries

Let $\mathcal{A}_p$ be the class of all p-valent analytic functions $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$ in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with $\mathcal{A} := \mathcal{A}_1$. Let f and g be analytic in Δ. Then f is subordinate to g, written $f(z) \prec g(z)$ $(z \in \Delta)$, if there is an analytic function w, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In the event that g is univalent in Δ, then f is subordinate to g when $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$.

Let ϕ be an analytic univalent function with positive real part in Δ and normalized by the conditions $\phi(0) = 1$ and $\phi'(0) > 0$. Further let ϕ maps the unit disk Δ onto a region starlike with respect to 1 and symmetric with respect to the

*The authors acknowledge support from the FRGS and Science Fund research grants. This work was completed during the second author's visit to USM, and the University's support is gratefully acknowledged.

Received by the editors February 2008.

Communicated by F. Brackx.

2000 *Mathematics Subject Classification* : 30C45, 30C55, 30C80.

Key words and phrases : Starlike functions, convex functions, p-valent analytic functions, subordination, convolution.

real axis. Several subclasses of starlike and convex univalent functions are respectively characterized by the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ lying in a region in the right-half plane, or equivalently, each function is subordinate to ϕ in Δ. Ma and Minda [7] gave a unified presentation of these various subclasses. They studied the class $S^*(\phi)$ consisting of functions $f \in \mathcal{A}$ for which $zf'(z)/f(z) \prec \phi(z)$, and the corresponding class $C(\phi)$ of functions $f \in \mathcal{A}$ satisfying $1 + zf''(z)/f'(z) \prec \phi(z) \quad (z \in \Delta)$. We extend these classes to the case of p-valent starlike and convex functions under a much more general setting.

Definition 1.1. *Let ϕ be an analytic univalent function with positive real part in Δ with $\phi(0) = 1$. The class $C_p(\phi)$ consists of functions $f \in \mathcal{A}_p$ satisfying*

$$\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z) \quad (z \in \Delta),$$

and the class $S_p^*(\phi)$ consists of all analytic functions $f \in \mathcal{A}_p$ satisfying

$$\frac{1}{p}\frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \Delta).$$

In the above definition, we do not make the assumption that the function ϕ maps the unit disk Δ onto a region starlike with respect to 1 nor is $\phi(\Delta)$ symmetric with respect to the real axis. Since ϕ is a function with positive real part, functions $f \in C_p(\phi)$ satisfy

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,$$

and hence they are p-valent convex functions. In particular, observe that $f(z)/z^p \neq 0$ and $f'(z)/z^{p-1} \neq 0$ in Δ. Similarly the functions in $S_p^*(\phi)$ are p-valent starlike functions.

Define the functions $k_{\phi,p}$ and $h_{\phi,p}$ respectively by

$$\frac{1}{p}\left(1 + \frac{zk''_{\phi,p}(z)}{k'_{\phi,p}(z)}\right) = \phi(z) \quad (z \in \Delta; k_{\phi,p} \in \mathcal{A}_p),$$

$$\frac{1}{p}\frac{zh'_{\phi,p}(z)}{h_{\phi,p}(z)} = \phi(z) \quad (z \in \Delta; h_{\phi,p} \in \mathcal{A}_p).$$

The function $k_{\phi,p}$ is in $C_p(\phi)$ while the function $h_{\phi,p}$ belongs to $S_p^*(\phi)$, and they play the role of extremal functions for these classes. We denote the functions $k_{\phi,1}$ and $h_{\phi,1}$ respectively by k_ϕ and h_ϕ. For $p = 1$, the classes $S_p^*(\phi)$ and $C_p(\phi)$ are precisely the classes $S^*(\phi)$ and $C(\phi)$ introduced by Ma and Minda [7]. More generally, the classes $S_p^*(\phi)$ and $C_p(\phi)$ include other well-known classes of p-valent functions, for instance, the class of uniformly convex p-valent functions. For these functions, Al-Kharsani and Al-Hajiry [3] recently obtained distortion inequalities and coefficients estimates.

In this paper, the classes $C_p(\phi)$ and $S_p^*(\phi)$ are studied. In Section 2, an Alexander-type connection between functions in $C_p(\phi)$ and $S_p^*(\phi)$, as well as relationships between $S_p^*(\phi)$ and $S^*(\phi)$, and also between $C_p(\phi)$ and $C(\phi)$ are obtained.

Using a result on the Briot-Bouquet differential subordination, we show that $C_p(\phi) \subset S_p^*(\phi)$. A necessary and sufficient condition for functions to belong to these classes is derived. Section 3 is devoted to subordination properties. In Section 4, we show that sharp distortion, growth, rotation and covering estimates for the classes $C(\phi)$ and $S^*(\phi)$ obtained by Ma and Minda [7] extend naturally to the p-valent cases. Finally in Section 5, we derive interesting and useful convolution results with prestarlike functions for functions in these classes.

Analogous inclusion and convolution properties for multivalent analytic and meromorphic functions have recently been investigated in [2,9,13].

2 The Alexander Theorem relating $C_p(\phi)$ and $S_p^*(\phi)$

A well-known result of Alexander states that a normalized analytic function f is convex if and only if the function $zf'(z)$ is starlike. There exists a similar close analytic connection between the classes $C_p(\phi)$ and $S_p^*(\phi)$.

Theorem 2.1. *The function f belongs to $C_p(\phi)$ if and only if $F(z) = zf'(z)/p \in S_p^*(\phi)$.*

Proof. The theorem follows directly from the identity

$$\frac{1}{p}\frac{zF'(z)}{F(z)} = \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right). \qquad \blacksquare$$

Next let $f \in \mathcal{A}_p$ and $g \in \mathcal{A}$ be related by

$$g(z) = z\left[\frac{f(z)}{z^p}\right]^{1/p},$$

where the power is chosen so that the function $\left[\frac{f(z)}{z^p}\right]^{1/p}$ takes the value 1 at $z = 0$. Since

$$\frac{zg'(z)}{g(z)} = \frac{1}{p}\frac{zf'(z)}{f(z)},$$

the function f belongs to $S_p^*(\phi)$ if and only if $g \in S^*(\phi)$. In particular,

$$h_\phi(z) = z\left[\frac{h_{\phi,p}(z)}{z^p}\right]^{1/p}.$$

Similarly if

$$g'(z) = \left(\frac{f'(z)}{pz^{p-1}}\right)^{1/p}, \tag{2.1}$$

then a computation yields

$$1 + \frac{zg''(z)}{g'(z)} = \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right),$$

and therefore $f \in C_p(\phi)$ if and only if $g \in C(\phi)$. In particular,

$$k'_\phi(z) = \left[\frac{k'_{\phi,p}(z)}{pz^{p-1}}\right]^{1/p}. \tag{2.2}$$

The following result is needed to show that $C_p(\phi) \subset S^*_p(\phi)$.

Lemma 2.1. *[4, 8] Let β, γ be any complex numbers and let h be convex in the unit disk Δ such that $\Re[\beta h(z) + \gamma] > 0$ in Δ. If ψ is analytic in Δ, $\psi(0) = h(0)$, and*

$$\psi(z) + \frac{z\psi'(z)}{\beta\psi(z) + \gamma} \prec h(z),$$

then $\psi(z) \prec h(z)$.

Theorem 2.2. *Let ϕ be convex in Δ with $\phi(0) = 1$ and $\Re\phi(z) > 0$ in Δ. Then $C_p(\phi) \subset S^*_p(\phi)$.*

Proof. For $f \in C_p(\phi)$, define the function ψ by

$$\psi(z) = \frac{1}{p}\frac{zf'(z)}{f(z)}.$$

Then a computation shows that

$$\psi(z) + \frac{1}{p}\frac{z\psi'(z)}{\psi(z)} = \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right), \tag{2.3}$$

and hence by (2.3),

$$\psi(z) + \frac{1}{p}\frac{z\psi'(z)}{\psi(z)} \prec \phi(z).$$

The result now follows by an application of Lemma 2.1. ∎

3 Subordination in $C_p(\phi)$ and $S^*_p(\phi)$

The following lemma is a restatement of a result of Ruscheweyh [11, Theorem 1, p. 275] for functions in the class $S^*(\phi)$ (also see Ruscheweyh [12, Theorem 2.37, pp. 86–88]).

Lemma 3.1. *Let ϕ be a convex function defined in Δ with $\phi(0) = 1$. Define Q by*

$$Q(z) = z \exp\left(\int_0^z \frac{\phi(x) - 1}{x} dx\right).$$

Let $q(z) = 1 + c_1 z + \cdots$ be analytic in Δ. Then

$$1 + \frac{zq'(z)}{q(z)} \prec \phi(z)$$

if and only if for all $|s| \leq 1$ and $|t| \leq 1$,

$$\frac{q(tz)}{q(sz)} \prec \frac{sQ(tz)}{tQ(sz)}.$$

Lemma 3.1 is used to establish the following result:

Theorem 3.1. *Let ϕ be a convex function defined in Δ with $\phi(0) = 1$. Define F by*

$$F(z) = z \exp\left(p \int_0^z \frac{\phi(x) - 1}{x} dx\right).$$

The function $f \in S_p^*(\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$,

$$\frac{s^p f(tz)}{t^p f(sz)} \prec \frac{sF(tz)}{tF(sz)}.$$

Proof. Define the function q by

$$q(z) := \frac{f(z)}{z^p}.$$

Then a computation shows that

$$1 + \frac{zq'(z)}{q(z)} = 1 + \left(\frac{zf'(z)}{f(z)} - p\right) \prec 1 + p(\phi(z) - 1).$$

and the result now follows from Lemma 3.1. ∎

Remark 3.1. *The case $p = 1$ in Theorem 3.1 was obtained by Ruscheweyh [11, Theorem 1, p. 275].*

The relation $f \in C_p(\phi)$ if and only if $zf'(z)/p \in S_p^*(\phi)$ yields the following corollary.

Corollary 3.1. *Let ϕ and F be as in Theorem 3.1. The function $f \in C_p(\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$,*

$$\frac{s^p f'(tz)}{t^p f'(sz)} \prec \frac{s^2 F(tz)}{t^2 F(sz)}.$$

By taking $t = 1$ and $s = 0$, the following corollary is obtained as an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let ϕ and F be as in Theorem 3.1. If $f \in S_p^*(\phi)$, then*

$$\frac{f(z)}{z^p} \prec \frac{F(z)}{z}.$$

Remark 3.2. *Since the function $F(z)/z$ is $h_{\phi,p}(z)/z^p$, the above subordination can be written in the form*

$$\frac{f(z)}{z^p} \prec \frac{h_{\phi,p}(z)}{z^p}.$$

This conclusion can also be drawn from a much weaker condition on $\phi(z)$ (see Theorem 3.2 below.)

For the proof of the next theorem, we need the following lemma:

Lemma 3.2. *[8, Corollary 3.4h.1, p.135] Let Q be univalent in Δ and let ψ be analytic in a domain containing $Q(\Delta)$. If $zQ'(z)/\psi(Q(z))$ is starlike, and*

$$zq'(z)\psi(q(z)) \prec zQ'(z)\psi(Q(z)),$$

then $q(z) \prec Q(z)$, and Q is the best dominant.

Theorem 3.2. *Let ϕ be starlike with respect to 1. If $f \in S_p^*(\phi)$, then*

$$\frac{f(z)}{z^p} \prec \frac{h_{\phi,p}(z)}{z^p}. \tag{3.1}$$

Proof. Define the functions q and Q respectively by

$$q(z) := \frac{f(z)}{z^p}, \quad Q(z) := \frac{h_{\phi,p}(z)}{z^p}.$$

Then a computation yields

$$\frac{zq'(z)}{q(z)} = \frac{zf'(z)}{f(z)} - p$$

and

$$\frac{zQ'(z)}{Q(z)} = \frac{zh'_{\phi,p}(z)}{h_{\phi,p}(z)} - p = p(\phi(z) - 1).$$

Since $f \in S_p^*(\phi)$,

$$\frac{zq'(z)}{q(z)} = \frac{zf'(z)}{f(z)} - p \prec p(\phi(z) - 1) = \frac{zQ'(z)}{Q(z)}.$$

By an application of Lemma 3.2, $q(z) \prec Q(z)$ and therefore the subordination (3.1) follows. ∎

Corollary 3.3. *Let ϕ be starlike with respect to 1. If $f \in C_p(\phi)$, then*

$$\frac{f'(z)}{z^{p-1}} \prec \frac{k'_{\phi,p}(z)}{z^{p-1}}.$$

4 Growth, Distortion, Covering and Rotation Theorems

As application of Corollary 3.3, we obtain growth and distortion estimates, as well as covering and rotation theorems for functions in $C_p(\phi)$. However these results are established under the assumptions that ϕ is an analytic function with positive real part in Δ, $\phi(\Delta)$ is symmetric with respect to the real axis, starlike with respect to $\phi(0) = 1$, and $\phi'(0) > 0$.

The result for p-valent functions follows from the corresponding results of Ma and Minda [7, Corollaries 1–4, pp. 159–161] for univalent functions.

Theorem 4.1. *If $f \in C_p(\phi)$, then for $|z| = r < 1$,*

$$\begin{aligned} k'_{\phi,p}(-r) &\leq |f'(z)| \leq k'_{\phi,p}(r), \\ -k_{\phi,p}(-r) &\leq |f(z)| \leq k_{\phi,p}(r). \end{aligned}$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of $k_{\phi,p}$. Also either f is a rotation of $k_{\phi,p}$ or $f(\Delta)$ contains the disk $|w| \leq -k_{\phi,p}(-1)$ where

$$-k_{\phi,p}(-1) = \lim_{r\to 1-}[-k_{\phi,p}(-r)].$$

Further, for $|z_0| = r < 1$,

$$|\operatorname{Arg}(f'(z_0))| \leq \max_{|z|=r} |\operatorname{Arg} k'_{\phi,p}(z)|.$$

Proof. Let $f \in C_p(\phi)$ and g be defined by (2.1). Then $g \in C(\phi)$ and from [7, Corollary 1, p. 159], yields

$$k'_{\phi}(-r) \leq |g'(z)| \leq k'_{\phi}(r).$$

Using (2.1) and (2.2) in the above inequality and rewriting the result, we get

$$k'_{\phi,p}(-r) \leq |f'(z)| \leq k'_{\phi,p}(r).$$

The other assertions in the theorem follow easily. ∎

The following corollary provides the corresponding results for functions in $S^*_p(\phi)$.

Corollary 4.1. *If $f \in S^*_p(\phi)$, then for $|z| = r < 1$,*

$$\begin{aligned} h'_{\phi,p}(-r) &\leq |f'(z)| \leq h'_{\phi,p}(r), \\ -h_{\phi,p}(-r) &\leq |f(z)| \leq h_{\phi,p}(r). \end{aligned}$$

The distortion inequality for $|f'(z)|$ holds only under the additional assumption that $\phi(r) = \max_{|z|=r<1}|\phi(z)|$ and $\phi(-r) = \min_{|z|=r<1}|\phi(z)|$. Equality holds for some $z \neq 0$ if and only if f is a rotation of $h_{\phi,p}$. Also either f is a rotation of $h_{\phi,p}$ or $f(\Delta)$ contains the disk $|w| \leq -h_{\phi,p}(-1)$ where

$$-h_{\phi,p}(-1) = \lim_{r\to 1-}[-h_{\phi,p}(-r)].$$

Further, for $|z_0| = r < 1$,

$$|\operatorname{Arg}(f'(z_0))| \leq \max_{|z|=r} |\operatorname{Arg} h'_{\phi,p}(z)|.$$

Remark 4.1. *Estimates for the first few coefficients of functions in $S^*_p(\phi)$, $C_p(\phi)$, and various other classes were obtained in Ali et. al [1].*

5 Convolution Properties

Let $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ be two analytic functions defined in Δ. Then their Hadamard product (or convolution) is the function $(f * g)(z)$ defined by

$$(f * g)(z) = \sum a_n b_n z^n.$$

The classes of starlike and convex functions are closed under convolution with convex functions. In this section, we consider convolutions with prestarlike functions: the class R_α of prestarlike functions of order $\alpha \leq 1$ consists of functions $f \in \mathcal{A}$ satisfying

$$\begin{cases} f * \frac{z}{(1-z)^{2-2\alpha}} \in S^*(\alpha) & (\alpha < 1) \\ \Re \frac{f(z)}{z} > 1/2 & (\alpha = 1). \end{cases}$$

The following lemma is used to prove our convolution theorem.

Lemma 5.1. *[11] If $f \in R_\alpha$ and $g \in S^*(\alpha)$, then for any analytic function $H(z)$ in Δ,*

$$\frac{f * (Hg)}{f * g}(\Delta) \subset \overline{co}(H(\Delta)),$$

where $\overline{co}(H(\Delta))$ denotes the closed convex hull of $H(\Delta)$.

Theorem 5.1. *Let $0 \leq \alpha < 1$ and ϕ be a convex function in Δ satisfying*

$$\Re \phi(z) > 1 - \frac{1-\alpha}{p}. \tag{5.1}$$

If $f \in S^*_p(\phi)$ and $g \in \mathcal{A}_p$ with $g(z)/z^{p-1} \in R_\alpha$, then $f * g \in S^*_p(\phi)$.

Proof. Define the functions H and ψ by

$$H(z) = \frac{1}{p} \frac{zf'(z)}{f(z)}, \quad \text{and} \quad \psi(z) = \frac{f(z)}{z^{p-1}}.$$

Then H and ψ are analytic in Δ. Since

$$\frac{z\psi'(z)}{\psi(z)} = \frac{zf'(z)}{f(z)} - p + 1,$$

using (5.1), we see that

$$\Re \frac{z\psi'(z)}{\psi(z)} \geq p \Re \phi(z) - p + 1 > \alpha,$$

and hence $\psi \in S^*(\alpha)$.

It is easy to see that, for f and $g \in \mathcal{A}_p$,

$$z(g * f)'(z) = (g * zf')(z)$$

and

$$(g * f)(z) = z^{p-1} \left(\frac{g}{z^{p-1}} * \frac{f}{z^{p-1}} \right)(z),$$

so that

$$\begin{aligned}\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)} &= \frac{1}{p}\frac{(g*zf')(z)}{(g*f)(z)}\\ &= \frac{(g*(Hf))(z)}{(g*f)(z)}\\ &= \frac{(G*(H\psi))(z)}{(G*\psi)(z)},\end{aligned} \tag{5.2}$$

where

$$G(z) := \frac{g(z)}{z^{p-1}} \in R_\alpha.$$

Using Lemma 5.1, we see that

$$\frac{(G*(H\psi))(z)}{(G*\psi)(z)} \prec \phi(z)$$

or, by using (5.2),

$$\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)} \prec \phi(z).$$

This proves that $f*g \in S_p^*(\phi)$. ■

Corollary 5.1. *Let* $0 \le \alpha < 1$ *and* ϕ *be a convex function in* Δ *satisfying*

$$\Re\phi(z) > 1 - \frac{1-\alpha}{p}.$$

If $f \in C_p(\phi)$ *and* $g \in \mathcal{A}_p$ *with* $g(z)/z^{p-1} \in R_\alpha$, *then* $f*g \in C_p(\phi)$.

Proof. The result follows from Theorem 5.1 and Theorem 2.1 ■

References

[1] R. M. Ali, V. Ravichandran and N. Seenivasagan, Coefficient bounds for p-valent functions, *Appl. Math. Comput.* **187** (2007), no. 1, 35–46.

[2] R.M. Ali, V. Ravichandran and N. Seenivasagan, Subordination and superordination of the Liu-Srivastava linear operator on meromorphic functions, *Bull. Malays. Math. Sci. Soc.* (2) **31**(2) (2008), 193-207.

[3] H. A. Al-Kharsani and S. S. Al-Hajiry, Subordination results for the family of uniformly convex p-valent functions, *J. Inequal. Pure Appl. Math.* **7**(1) (2006), Article 20, 11 pp. (electronic).

[4] P. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, On a Briot-Bouquet differential subordination, *Rev. Roumaine Math. Pures Appl.*, **29** (1984), no. 7, 567–573.

[5] A. Gangadharan, V. Ravichandran and T. N. Shanmugam, Radii of convexity and strong starlikeness for some classes of analytic functions, *J. Math. Anal. Appl.* **211**(1) (1997), 301–313.

[6] A. W. Goodman, *Univalent functions. Vol. I, II*, Mariner, Tampa, FL, 1983.

[7] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Internat. Press, Cambridge, MA.

[8] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications,* vol. 225 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 2000.

[9] M. H. Mohd, R.M. Ali, S. K. Lee and V. Ravichandran, Subclasses of meromorphic functions associated with convolution, *J. Inequal. Appl.* (2009) Article ID 190291, 9 pages. doi:10.1155/2009/190291

[10] Ch. R. Pommerenke, *Univalent functions*, Vandenhoeck, ruprecht in Göttingen, 1975.

[11] St. Ruscheweyh, A subordination theorem for Φ-like functions, *J. London Math. Soc.* **13** (1976), 275–280.

[12] S. Ruscheweyh, *Convolutions in geometric function theory*, Presses Univ. Montréal, Montreal, Que., 1982.

[13] S. Supramaniam, R.M. Ali, S.K. Lee and V. Ravichandran, Convolution and differential subordination for multivalent functions, *Bull. Malays. Math. Sci. Soc.* (2) **32**(3) (2009), to appear.

School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 USM, Penang, Malaysia
email:rosihan@cs.usm.my

Department of Mathematics
University of Delhi
Delhi 110 007, India
email:vravi@maths.du.ac.in

School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 USM, Penang, Malaysia
email:sklee@cs.usm.my