CLASSES OF MEROMORPHIC $\alpha$-CONVEX FUNCTIONS

Rosihan M. Ali and V. Ravichandran

Abstract. For a fixed analytic univalent function $\phi$, the class of meromorphic univalent $\alpha$-convex functions with respect to $\phi$ is introduced. A representation theorem for functions in the class, as well as a necessary and sufficient condition for functions to belong to the class are obtained. Also we obtain a sharp growth theorem and estimate on a certain coefficient functional for meromorphic starlike functions with respect to $\phi$. Differential subordination and superordination conditions are also obtained for the subclass of meromorphic starlike functions with respect to $\phi$.

1. INTRODUCTION

Let $\Sigma$ denote the class of meromorphic univalent functions $f$ defined on the punctured unit disk $\Delta^* := \{ z \in \mathbb{C} : 0 < |z| < 1 \}$ having the form $f(z) = 1/z + \sum_{k=0}^{\infty} a_k z^k$. A function $f \in \Sigma$ is said to be meromorphic starlike of order $\alpha$ $(0 \leq \alpha < 1)$ if $-\Re [zf'(z)/f(z)] > \alpha$ for all $z \in \Delta := \{ z \in \mathbb{C} : |z| < 1 \}$. We denote by $\Sigma^* (\alpha)$ the class of all such meromorphic starlike functions of order $\alpha$ in $\Delta^*$.

Several authors [2, 3, 7, 10, 11, 14, 16, 17] have studied various subclasses of $\Sigma^* (\alpha)$, as well as subclasses of meromorphic convex functions of order $\alpha$. The latter class is characterized by the property $-\Re [1 + zf''(z)/f'(z)] > \alpha$. We shall unify these functions in Definition 1.1.

First we recall the definition of subordination. For two functions $f$ and $g$ analytic in $\Delta$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\Delta$, and write $f \prec g$ or $f(z) \prec g(z)$ $(z \in \Delta)$, if there exists a Schwarz function $w(z)$, analytic in $\Delta$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \Delta)$, such that $f(z) = g(w(z))$ $(z \in \Delta)$. In particular, if the function $g$ is univalent in $\Delta$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.
Definition 1.1. Let \( \phi(z) \) be an analytic univalent function in \( \Delta \) with \( \phi(0) = 1 \). Let \( \Sigma_\alpha^*(\phi) \) be the class of functions \( f \in \Sigma \) satisfying \( f(z)f'(z) \neq 0 \) and

\[
(1.1) \quad -\left[ (1-\alpha) \left( \frac{zf'(z)}{f(z)} \right) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] \prec \phi(z) \quad (z \in \Delta).
\]

The function \( f \in \Sigma_\alpha^*(\phi) \) is called a meromorphic \( \alpha \)-convex function with respect to \( \phi \). (Here \( \prec \) denotes subordination between analytic functions.) We shall write \( \Sigma_0^*(\phi) \) by \( \Sigma^*(\phi) \).

With \( \phi(z) = 1 + (1-2\alpha)z \) \( (0 \leq \alpha < 1) \), it is obvious that \( \Sigma_0^*(\phi) \) is the class of meromorphic starlike functions of order \( \alpha \), while \( \Sigma_1^*(\phi) \) is the class of meromorphic convex functions of order \( \alpha \). The class \( \Sigma_\alpha^*(\phi) \) reduces to the class \( \Sigma(\alpha, \beta, \gamma) \) introduced by Kulkarni and Joshi [5] when

\[
(1.2) \quad \phi(z) = \frac{1 + \beta(1-2\alpha\gamma)z}{1 + \beta(1-2\gamma)z} \quad (0 \leq \alpha < 1; \ 0 < \beta \leq 1; \ 1/2 \leq \gamma \leq 1).
\]

Karunakaran [4] have considered a special case of the class \( \Sigma^*(\phi) \) consisting of functions \( f \in \Sigma \) for which

\[
-zf'(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (0 \leq B < 1; \ -B < A < B),
\]

where \( w(z) \) is an analytic function in \( \Delta \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \( (z \in \Delta) \). He denoted this class by \( K_1(A, B) \).

In this paper, a representation theorem as well as a necessary and sufficient condition for functions to belong to \( \Sigma_\alpha^*(\phi) \) is obtained. Also we obtain a sharp growth theorem and estimate for the coefficient functional \( |a_1 - \mu a_0| \) for functions in \( \Sigma_\alpha^*(\phi) \). Finally we investigate the subclass \( \Sigma_\alpha^*(\phi) \) from the perspective of first-order differential subordination and superordination [8, 9].

2. A REPRESENTATION THEOREM

We first prove a representation formula for functions in the class \( \Sigma_\alpha^*(\phi) \).

Theorem 2.1. A function \( f(z) \in \Sigma_\alpha^*(\phi) \) if and only if

\[
[zf(z)]^{1-\alpha}[-zf'(z)]^\alpha = \exp \left( \int_0^z \frac{1-\phi(w(\eta))}{\eta} \, d\eta \right),
\]

where \( w(z) \) is analytic in \( \Delta \) satisfying \( w(0) = 0 \) and \( |w(z)| \leq 1 \).
Proof. Let \( f(z) \in \Sigma_{\alpha}^*(\phi) \). Then (1.1) holds and therefore there is a function \( w(z) \) analytic in \( \Delta \) with \( w(0) = 0 \) and \( |w(z)| \leq 1 \) such that
\[
- \left[ (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] = \phi(w(z)), \quad (z \in \Delta).
\]
Rewriting the above equation in the form
\[
\left[ (1 - \alpha) \left( \frac{1}{z} + \frac{f'(z)}{f(z)} \right) + \alpha \left( \frac{2}{z} + \frac{f''(z)}{f'(z)} \right) \right] = \frac{1 - \phi(w(z))}{z}, \quad (z \in \Delta)
\]
and integrating from 0 to \( z \), we obtain the desired expression upon exponentiation. The converse follows directly by differentiation. \( \blacksquare \)

Example 2.1. For the function \( \phi(z) \) given by (1.2) and with \( \alpha = 0 \), we obtain [5, Theorem 1, p. 198]: Let \( f \in \Sigma \) and \( 0 \leq \alpha < 1, \; 0 < \beta \leq 1 \) and \( 1/2 \leq \gamma \leq 1 \). Then \( f \in \Sigma(\alpha, \beta, \gamma) \) if and only if
\[
zf(z) = \exp \left( - \int_0^z \frac{2\beta\gamma(1-\alpha)w(\eta)}{1+\beta(1-2\gamma)w(\eta)\eta} \right)
\]
where \( w(\zeta) \) is analytic in \( \Delta \) satisfying \( w(0) = 0 \) and \( |w(z)| \leq 1 \).

3. A Necessary and Sufficient Condition

We need the following subordination result.

Lemma 3.1. [13]. Let \( \phi \) be a convex univalent function defined on \( \Delta \) and \( \phi(0) = 1 \). Define \( F(z) \) by
\[
F(z) = z \exp \left( \int_0^z \frac{\phi(\eta) - 1}{\eta} d\eta \right).
\]
Let \( q(z) \) be analytic in \( \Delta \) and \( q(0) = 1 \). Then
\[
1 + \frac{zq'(z)}{q(z)} < \phi(z)
\]
if and only if for all \( |s| \leq 1 \) and \( |t| \leq 1 \),
\[
\frac{q(tz)}{q(sz)} < \frac{sF'(tz)}{tF'(sz)}.
\]

Using Lemma 3.1, we obtain the following necessary and sufficient conditions for functions to belong to \( \Sigma_{\alpha}^*(\phi) \).
Theorem 3.1. Let \( \phi(z) \) and \( F(z) \) be as in Lemma 3.1. A function \( f \) belongs to \( \Sigma^\alpha_\phi \) if and only if for all \( |s| \leq 1 \) and \( |t| \leq 1 \),
\[
\left( \frac{s f(sz)}{t f(tz)} \right)^{1-\alpha} \left( \frac{s^2 f'(sz)}{t^2 f'(tz)} \right)^\alpha \prec \frac{s F(tz)}{t F(sz)}.
\]

Proof. Define the function \( q(z) \) by
\[
\frac{1}{q(z)} := (zf(z))^{1-\alpha} (-z^2 f'(z))^\alpha.
\]
Then a computation shows that
\[
1 + zq'(z)q(z) = -\left[ (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right]
\]
and the result now follows from Lemma 3.1.

Example 3.2. Let \( \Sigma^\alpha_{A,B} \) be the class of all meromorphic \( \alpha \)-convex functions \( f \in \Sigma \) satisfying
\[
- \left[ (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] < \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1; z \in \Delta).
\]
The function \( f \in \Sigma^\alpha_{A,B} \) if and only if for all \( |s| \leq 1 \) and \( |t| \leq 1 \),
\[
\left( \frac{s f(sz)}{t f(tz)} \right)^{1-\alpha} \left( \frac{s^2 f'(sz)}{t^2 f'(tz)} \right)^\alpha \prec \begin{cases} \left( \frac{1+Az}{1+Bz} \right)^{(A-B)/B} & \text{if } B \neq 0 \\ e^{A(t-s)z} & \text{if } B = 0 \end{cases}
\]

4. Growth Theorem for Functions in \( \Sigma^\alpha(\phi) \)

We need the following Lemma in the proof of Theorem 4.1.

Lemma 4.1. [8, Corollary 3.4h.1, p.135]. Let \( q(z) \) be univalent in \( \Delta \) and let \( \psi(z) \) be analytic in a domain containing \( q(\Delta) \). If \( zq'(z)/\psi(q(z)) \) is starlike, and
\[
zp'(z)\psi(p(z)) < zq'(z)\psi(q(z)),
\]
then \( p(z) \prec q(z) \) and \( q(z) \) is the best dominant.

Theorem 4.1 below is a special case of Theorem 3.1 if \( \phi \) is a convex univalent function. However we prove Theorem 4.1 without the convexity assumption.
Theorem 4.1. Let $\phi(z)$ be an analytic function with positive real part on $\Delta$ with $\phi(0) = 1$, $\phi'(0) > 0$ and maps the unit disk $\Delta$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let the functions $h_{\phi_n}$ $(n = 2, 3, \ldots)$ be defined by

$$z h_{\phi}'(z) = \phi(z) \quad (h_{\phi}(0) = 0 = h_{\phi}'(0) - 1).$$

If $f(z) \in \Sigma^*(\phi)$, then

$$zf(z) \prec \frac{z}{h_{\phi}(z)}.$$

**Proof.** Define the function $p(z)$ by

$$p(z) := zf(z) \quad (z \in \Delta).$$

Then a computation shows that

$$\frac{zf'(z)}{f(z)} = 1 - \frac{zp'(z)}{p(z)}.$$

If $f(z) \in \Sigma^*(\phi)$, then

$$\frac{zp'(z)}{p(z)} < 1 - \phi(z).$$

Since $\phi(z)$ is starlike in $\Delta$, by an application of Lemma 4.1, we obtain $p(z) \prec q(z)$ where $q(z)$ is given by

$$\frac{zq'(z)}{q(z)} = 1 - \frac{zh_{\phi}'(z)}{h_{\phi}(z)}$$

or $q(z) = z/h_{\phi}(z)$.

As a consequence of Theorem 4.1, we immediately obtain

Theorem 4.2. (Growth Theorem). Let $\phi(z)$ be an analytic function with positive real part on $\Delta$ with $\phi(0) = 1$, $\phi'(0) > 0$ and maps the unit disk $\Delta$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. If $f(z) \in \Sigma^*(\phi)$, then

$$[h_{\phi}(r)]^{-1} \leq |f(z)| \leq [-h_{\phi}(-r)]^{-1} \quad (|z| = r < 1).$$

For the choice $p(z) = (1 - Az)/(1 - Bz)$, $0 \leq B \leq 1; -B < A < B$, we obtain the following result of Karanukaran:
Corollary 4.1. [4]. If \( f \in K_1(A, B) \), then
\[
r^{-1}(1 - Br)^{(B - A)/B} \leq |f(z)| \leq r^{-1}(1 + Br)^{(B - A)/B}
\]

5. COEFFICIENT PROBLEM FOR THE CLASS \( \Sigma^*(\phi) \)

Now we consider coefficient problems for the class \( \Sigma^*(\phi) \).

Theorem 5.1. Let \( \phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots \). If \( f(z) = 1/z + \sum_{k=0}^{\infty} a_kz^k \) belongs to \( \Sigma^*(\phi) \), then
\[
|a_1 - \mu a_0^2| \leq \begin{cases} 
\frac{1}{2}(B_1^2 - 2\mu B_1^2 - B_2) & \text{if } 2\mu B_1^2 \leq B_1^2 - B_1 - B_2 \\
\frac{1}{2}B_1 & \text{if } B_1^2 - B_1 - B_2 \leq 2\mu B_1^2 \leq B_1^2 + B_1 - B_2 \\
\frac{1}{2}(-B_1^2 + 2\mu B_1^2 + B_2) & \text{if } B_1^2 + B_1 - B_2 \leq 2\mu B_1^2
\end{cases}
\]
The result is sharp.

Proof. Our proof of Theorem 5.1 is essentially similar to the proof of Theorem 3 of Ma and Minda[6]. If \( f(z) \in \Sigma^*(\phi) \), then there is a Schwarz function \( w(z) \), analytic in \( \Delta \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( \Delta \) such that
\[
(5.1) \quad \frac{zf'(z)}{f(z)} = \phi(w(z)).
\]
Define the function \( p_1(z) \) by
\[
(5.2) \quad p_1(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \cdots.
\]
Since \( w(z) \) is a Schwarz function, we see that \( \Re p_1(z) > 0 \) and \( p_1(0) = 1 \). Define the function \( p(z) \) by
\[
(5.3) \quad p(z) := -\frac{zf''(z)}{f'(z)} = 1 + b_1z + b_2z^2 + \cdots.
\]
In view of the equations (5.1), (5.2), (5.3), we have
\[
(5.4) \quad p(z) = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right).
\]
and from this equation (5.4), we obtain
\[
b_1 = \frac{1}{2}B_1c_1
\]
and
\[ b_2 = \frac{1}{2} B_1 (c_2 - \frac{1}{2} c_1^2) + \frac{1}{4} B_2 c_1^2. \]

From the equation (5.3), we see that
\[ b_1 + a_0 = 0 \tag{5.5} \]
\[ b_2 + b_1 a_0 + 2a_1 = 0 \tag{5.6} \]
or equivalently
\[ a_0 = -b_1 = -\frac{B_1 c_1}{2} \tag{5.7} \]
and
\[ a_1 = \frac{1}{2} (b_1^2 - b_2) \]
\[ = \frac{1}{8} \left\{ B_1^2 c_1^2 - 2B_1 c_2 + B_1 c_1^2 - B_2 c_2^2 \right\}. \]

Therefore,
\[ a_1 - \mu a_0^2 = -\frac{B_1}{4} \left\{ c_2 - vc_1^2 \right\} \tag{5.8} \]
where
\[ v := \frac{1}{2} \left[ 1 + B_1 - \frac{B_2}{B_1} - 2\mu B_1 \right]. \]

Our result now follows by an application of Lemma 5.2 below. The sharpness is also an immediate consequence of Lemma 5.2.

Lemma 5.2. [6] If \( p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is a function with positive real part in \( \Delta \), then
\[ |c_2 - vc_1^2| \leq \begin{cases} 4v - 2 & \text{if } v \geq 1 \\ 2 & \text{if } 0 \leq v \leq 1 \\ -4v + 2 & \text{if } v \leq 0 \end{cases} \]

When \( v < 0 \) or \( v > 1 \), equality holds if and only if \( p_1(z) \) is \((1 + z) / (1 - z)\) or one of its rotations. If \( 0 < v < 1 \), then equality holds if and only if \( p_1(z) \) is \((1 + z^2) / (1 - z^2)\) or one of its rotations. If \( v = 0 \), equality holds if and only if
\[ p_1(z) = \left( \frac{1}{2} + \frac{1}{2} \lambda \right) \frac{1+z}{1-z} + \left( \frac{1}{2} - \frac{1}{2} \lambda \right) \frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1) \]
or one of its rotations. If \( v = 1 \), equality holds if and only if \( p_1 \) is the reciprocal of one of the functions such that the equality holds in the case of \( v = 0 \).
When $\mu$ is complex, we have the following:

**Theorem 5.2.** Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z) = 1/z + \sum_{k=0}^{\infty} a_k z^k$ belongs to $\Sigma^*(\phi)$, then for $\mu$ a complex number,

$$|a_1 - \mu a_0^2| \leq \frac{B_1}{2} \max\{1, |B_1 - 2\mu B_1 - \frac{B_2}{B_1}|\}.$$  

The result is sharp.

Theorem 5.2 follows from the following result. For a function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ with positive real part, we have

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1 + z^2}{1 - z^2}, \quad p(z) = \frac{1 + z}{1 - z}.$$  

6. **Differential Subordination and Superordination for $\Sigma^*(\phi)$**

In this section, we discuss differential implications for the subclass $\Sigma^*(\phi)$. We shall require the following definition and lemmas:

**Definition 6.1.** [9, Definition 2, p. 817]. Denote by $Q$, the set of all functions $f(z)$ that are analytic and injective on $\Delta - E(f)$, where

$$E(f) = \{\zeta \in \partial \Delta : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial \Delta - E(f)$.

**Lemma 6.1.** (cf. Miller and Mocanu [8, Theorem 3.4h, p. 132]). Let $q(z)$ be univalent in the unit disk $\Delta$ and $\vartheta$ and $\varphi$ be analytic in a domain $D$ containing $q(\Delta)$ with $\varphi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) := zq'(z)\varphi(q(z))$ and $h(z) := \vartheta(q(z)) + Q(z)$. Suppose that either $h(z)$ is convex, or $Q(z)$ is starlike univalent in $\Delta$. In addition, assume that $\Re[z h'(z)/Q(z)] > 0$ for $z \in \Delta$. If $p(z)$ is analytic in $\Delta$ with $p(0) = q(0)$, $p(\Delta) \subseteq D$ and

$$\vartheta(p(z)) + z p'(z) \varphi(p(z)) < \vartheta(q(z)) + z q'(z) \varphi(q(z)),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.
Lemma 6.2. [1]. Let $q(z)$ be univalent in the unit disk $\Delta$ and $\vartheta$ and $\varphi$ be analytic in a domain $D$ containing $q(\Delta)$. Suppose that $\Re\left[\vartheta'(q(z))/\varphi(q(z))\right] > 0$ for $z \in \Delta$ and $zq'(z)/\varphi(q(z))$ is starlike univalent in $\Delta$. If $p(z) \in \mathcal{H}(q(0), 1) \cap \mathcal{Q}$, with $p(\Delta) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $\Delta$, then

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) < \vartheta(p(z)) + zp'(z)\varphi(p(z)),$$

implies $q(z) \prec p(z)$ and $q(z)$ is the best subordinant. (Here $\mathcal{H}[a, n]$ denotes the class of all analytic functions $f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \cdots$ ($z \in \Delta$.)

First we prove a differential subordination result for the class $\Sigma^*(\phi)$.

Theorem 6.1. Let $\alpha$ be a nonzero complex number. Let $q(z)$ be univalent in $\Delta$, $q(0) = 1$. Assume that $q(z)$ or $(\alpha - 1)q(z) + \alpha q^2(z) - \alpha zq'(z)$ is convex univalent and

$$\Re\left\{\frac{1 - \alpha}{\alpha} - 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0.$$  

If $f \in \Sigma$ satisfies

$$\frac{zf''(z)}{f(z)} + \alpha z^2f'''(z)/f(z) < (\alpha - 1)q(z) + \alpha q^2(z) - \alpha zq'(z),$$

then $-\frac{zf''(z)}{f(z)} \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := -\frac{zf'(z)}{f(z)}.$$  

Then a computation shows that

$$p(z) - \frac{zp'(z)}{p(z)} = -\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

Using (6.4) and (6.3), we have

$$\frac{zf(z)}{f(z)} + \alpha z^2f'(z)/f(z) = (\alpha - 1)p(z) + \alpha p^2(z) - \alpha zp'(z).$$

Define the function $\vartheta$ and $\varphi$ by

$$\vartheta(w) = (\alpha - 1)w + \alpha w^2 \quad \text{and} \quad \varphi(w) = -\alpha.$$

Then the functions $\vartheta$ and $\varphi$ are analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ in $\mathbb{C}$. Also the function $Q(z) := zq'(z)/\varphi(q(z)) = -\alpha zq'(z)$ is starlike in $\Delta$. Using (6.2), we see that the function $h(z) := \vartheta(q(z)) + Q(z) = (\alpha - 1)q(z) + \alpha q(z)^2 + Q(z)$ satisfies $\Re[zh'(z)/Q(z)] > 0$. The result now follows by an application of Lemma 6.1. ■
**Theorem 6.2.** Let \( q(z) \neq 0 \) be univalent in \( \Delta \) and \( q(0) = 1 \). Let \( zq'(z)/q(z)^2 \) be starlike in \( \Delta \). If \( f \in \Sigma \) and
\[
\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} < 1 - \frac{zq'(z)}{q(z)^2},
\]
then \( -zf'(z)/f(z) \prec q(z) \) and \( q(z) \) is the best dominant.

**Proof.** Let \( p(z) \) be defined by (6.3). From (6.4) and (6.3), we get
\[
1 - \frac{zp'(z)}{p(z)^2} = \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)}
\]
and the result follows by an application of Lemma 6.1.

The corresponding superordination results are obtained from Lemma 6.2 in a similar manner to Theorems 6.1 and 6.2. The proofs are omitted.

**Theorem 6.3.** Let \( \alpha \) be a nonzero complex number, \( q(z) \) be convex univalent in \( \Delta \). Assume that \( \Re \{ \frac{\alpha - 1}{\alpha} + 2q(z) \} < 0 \). If \( f \in \Sigma \), \(-zf'(z)/f(z) \in \mathcal{H}[1,1] \cap \mathcal{Q} \) and \( \frac{zf'(z)}{f(z)} + \alpha zq'(z) \) is univalent in \( \Delta \) and
\[
(\alpha - 1)q(z) + \alpha q^2(z) - \alpha zq'(z) \prec \frac{zf'(z)}{f(z)} + \alpha z^2 f''(z),
\]
then \( q(z) \prec -\frac{zf'(z)}{f(z)} \) and \( q(z) \) is the best subordinant.

**Theorem 6.4.** Let \( q(z) \neq 0 \) be univalent, \( q(0) = 1 \) and \( zq'(z)/q(z)^2 \) be starlike in \( \Delta \). If \( f \in \Sigma \), \(-zf'(z)/f(z) \in \mathcal{H}[1,1] \cap \mathcal{Q} \) and \( \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \) is univalent in \( \Delta \), then \( q(z) \prec -zf'(z)/f(z) \) and \( q(z) \) is the best subordinant.

**References**


Rosihan M. Ali
School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 USM Penang,
Malaysia
E-mail: rosihan@cs.usm.my

V. Ravichandran
Department of Mathematics,
University of Delhi,
Delhi 110 007,
India
E-mail: vravi@maths.du.ac.in
        vravi68@gmail.com