Research Article

A Third-Order Differential Equation and Starlikeness of a Double Integral Operator

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1. Introduction

Let \( \mathcal{A} \) denote the class of all analytic functions \( f \) defined in the open unit disk \( U := \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( f(0) = 0 \), \( f'(0) = 1 \). Further, let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) consisting of univalent functions, and let \( \mathcal{S}^* \) be its subclass of starlike functions. A starlike function \( f \) is characterized analytically by the condition \( \Re(zf'(z)/f(z)) > 0 \) in \( U \), that is, the domain \( f(U) \) is starlike with respect to origin. For two functions \( f(z) = z + a_2 z^2 + \cdots \) and \( g(z) = z + b_2 z^2 + \cdots \) in \( \mathcal{A} \), the Hadamard product (or convolution) of \( f \) and \( g \) is the function \( f * g \) defined by

\[
(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. 
\]

For \( f \) and \( g \) in \( \mathcal{A} \), a function \( f \) is subordinate to \( g \), written as \( f(z) \prec g(z) \), if there is an analytic function \( w \) satisfying \( w(0) = 0 \) and \( |w(z)| < 1 \), such that \( f(z) = g(w(z)) \), \( z \in U \).
When \( g \) is univalent in \( U \), then \( f \) is subordinated to \( g \) which is equivalent to \( f(U) \subset g(U) \) and \( f(0) = g(0) \).

In a recent paper, Miller and Mocanu [1] investigated starlikeness properties of functions \( f \) defined by double integral operators of the form

\[
f(z) = \int_0^1 \int_0^1 W(s,t,z) ds \, dt.
\]  

(1.2)

In this paper, conditions on a different kernel \( W \) are investigated from the perspective of starlikeness. Specifically, we consider functions \( f \in \mathcal{A} \) given by the double integral operator of the form

\[
f(z) = \int_0^1 \int_0^1 G(zt^\mu s^\nu) t^{-\mu}s^{-\nu} ds \, dt.
\]  

(1.3)

In this case, it follows that

\[
f'(z) = \int_0^1 \int_0^1 g(zt^\mu s^\nu) ds \, dt,
\]  

(1.4)

where \( G'(z) = g(z) \). Further, the function \( f \) satisfies a third-order differential equation of the form

\[
f'(z) + azf''(z) + \gamma z^2 f'''(z) = g(z)
\]  

(1.5)

for appropriate parameters \( a \) and \( \gamma \). The investigation of such functions \( f \) can be seen as an extension to the study of the class

\[
R(\alpha, h) = \{ f \in \mathcal{A} : f'(z) + azf''(z) < h(z), \ z \in U \}.
\]  

(1.6)

The class \( R(\alpha, h) \) or its variations for an appropriate function \( h \) have been investigated in several works; see, for example, [2–10] and more recently [11, 12].

2. Results on Differential Subordination

We first recall the definition of best dominant solution of a differential subordination.

**Definition 2.1** (dominant and best dominant) [13]). Let \( \Psi : \mathbb{C}^3 \times U \to \mathbb{C} \), and let \( h \) be univalent in \( U \). If \( p \) is analytic in \( U \) and satisfies the differential subordination

\[
\Psi(p(z), zp'(z), z^2p''(z)) < h(z),
\]  

(2.1)

then \( p \) is called a solution of the differential subordination. A univalent function \( q \) is called a dominant if \( p < q \) for all \( p \) satisfying (2.1). A dominant \( \tilde{q} \) that satisfies \( \tilde{q} < q \) for all dominants \( q \) of (2.1) is said to be the best dominant of (2.1).
In the following sequel, we will assume that \( h \) is an analytic convex function in \( U \) with \( h(0) = 1 \). For \( \alpha \geq \gamma \geq 0 \), consider the third-order differential equation

\[
f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) = g(z), \quad g(z) < h(z). \quad (2.2)
\]

We will denote the class consisting of all solutions \( f \in \mathcal{A} \) as \( R(\alpha, \gamma, h) \), that is,

\[
R(\alpha, \gamma, h) = \{ f \in \mathcal{A} : f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) < h(z), \ z \in U \}. \quad (2.3)
\]

Let

\[
\mu = \frac{(\alpha - \gamma) - \sqrt{(\alpha - \gamma)^2 - 4\gamma}}{2}, \quad \nu + \mu = \alpha - \gamma, \ \mu \nu = \gamma. \quad (2.4)
\]

The discriminant is denoted by \( \Delta := (\alpha - \gamma)^2 - 4\gamma \). Note that \( \text{Re} \mu \geq 0 \) and \( \text{Re} \nu \geq 0 \).

We will rewrite the solution of

\[
f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) = g(z) \quad (2.5)
\]

in its equivalent integral form

\[
f'(z) = \int_0^1 \int_0^1 g(z t^\mu s^\nu) ds \ dt. \quad (2.6)
\]

It follows from relations (2.4) that

\[
g(z) = f'(z) + (\mu(1 + \nu) + \nu) z f''(z) + \mu \nu z^2 f'''(z) \]
\[
= \nu z^{1-1/\nu} \left( \mu z^{1+1/\nu} f''(z) + z^{1/\nu} f'(z) \right)' \quad (2.7)
\]
\[
= \nu z^{1-1/\nu} \left( \mu z^{1+1/\nu-1/\mu} \left( z^{1/\mu} f'(z) \right) \right)'.
\]

Thus,

\[
\mu z^{1+1/\nu-1/\mu} \left( z^{1/\mu} f'(z) \right)' = \frac{1}{\nu} \int_0^z w^{1/\nu-1} g(w) dw. \quad (2.8)
\]

Making the substitution \( w = z s^\nu \) in the above integral and integrating again, a change of variables yields

\[
f'(z) = \int_0^1 \int_0^1 g(z t^\mu s^\nu) ds \ dt. \quad (2.9)
\]
We will use the notation $\phi_\lambda$ for 

$$\phi_\lambda(z) = \int_0^1 \frac{dt}{1 - zt^\lambda} = \sum_{n=0}^\infty \frac{z^n}{1 + \lambda n}. \quad (2.10)$$

From [14] it is known that $\phi_\lambda$ is convex in $U$ provided $\text{Re} \lambda \geq 0$.

**Theorem 2.2.** Let $\mu$ and $\nu$ be given by (2.4), and

$$q(z) = \int_0^1 \int_0^1 h(zt^\mu s^\nu) dt \, ds. \quad (2.11)$$

Then the function $q(z) = (\phi_\nu \ast \phi_\mu) \ast h(z)$ is convex. If $f \in R(\alpha, \gamma, h)$, then

$$f'(z) < q(z) < h(z), \quad (2.12)$$

and $q$ is the best dominant.

**Proof.** It follows from (2.10) that

$$h(z) \ast \phi_\mu(z) = \int_0^1 \frac{1}{1 - zt^\mu} dt \ast h(z) = \int_0^1 h(zt^\mu) dt := k(z). \quad (2.13)$$

Thus,

$$h(z) \ast (\phi_\mu(z) \ast \phi_\nu(z)) = k(z) \ast \phi_\nu(z) = \int_0^1 k(zs^\nu) ds = \int_0^1 \int_0^1 h(zt^\mu s^\nu) dt \, ds = q(z). \quad (2.14)$$

Since the convolution of two convex functions is convex [15], the function $q$ is convex. Let

$$p(z) = f'(z) + \nu z f''(z). \quad (2.15)$$

Then,

$$p(z) + \mu z p'(z) = f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) < h(z). \quad (2.16)$$

It is known from [16] that

$$p(z) < \frac{1}{\mu z^{1/\mu}} \int_0^z \xi^{1/\mu - 1} h(\xi) d\xi = (\phi_\mu \ast h)(z) < h(z). \quad (2.17)$$

Similarly,

$$p(z) = f'(z) + \nu z f''(z) < (\phi_\mu \ast h)(z) \quad (2.18)$$
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implies

\[ f'(z) < (\phi_\nu * \phi_\mu * h)(z) \]
\[ = \sum_{n=0}^{\infty} \frac{z^n}{(1 + \nu n)(1 + \mu n)} * h(z) \]
\[ = \left( \int_0^1 \int_0^1 \frac{dt \, ds}{1 - z t^\mu s^\nu} \right) * h(z) \]
\[ = \int_0^1 \int_0^1 h(z t s^\nu) dt \, ds = q(z). \]  

The differential chain

\[ f' < q < \phi_\mu * h < h \]  

shows that \( q < h \). Since \( q(z) + \alpha z q'(z) + \gamma z^2 q''(z) = h(z) \), the function

\[ Q(z) = \int_0^z q(w) \, dw \]  

is a solution of the differential subordination \( f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) < h(z) \), and thus \( q < \tilde{q} \) for all dominants \( \tilde{q} \). Hence, \( q \) is the best dominant.

Remark 2.3. (1) When \( \gamma = 0 \), then \( \mu = 0 \) and \( \nu = \alpha \), and the above subordination reduces to the result of [16], that is,

\[ f'(z) + \alpha z f''(z) < h(z) \implies f'(z) < \int_0^1 h(z t^\alpha) dt. \]  

(2) The above proof also reveals that

\[ f \in R(\alpha, \gamma, h) \implies f \in R(0,0,h), \]  

that is, \( f'(z) < h(z) \).

Theorem 2.4. Let \( \mu, \nu, \) and \( q \) be as given in Theorem 2.2. If \( f \in R(\alpha, \gamma, h) \), then

\[ \frac{f(z)}{z} < \int_0^1 q(tz) dt \]
\[ = \int_0^1 \int_0^1 h(zr s^\nu) dr \, ds \, dt. \]
Proof. Let \( p(z) = f(z)/z \). Then
\[
p(z) + zp'(z) = f'(z) < q(z). \tag{2.25}
\]
With \( \phi_1 \) given by (2.10), this subordination implies
\[
p(z) = (\phi_1 \ast (p + zp'))(z) < (\phi_1 \ast q)(z) = \int_0^t q(tz)dt. \tag{2.26}
\]

In this paper, starlikeness properties will be investigated for functions \( f \) given by a double integral operator of the form (1.3).

3. Applications

First, we consider a class of convex univalent functions \( h \) so that \( h(U) \) is symmetric with respect to the real axis. Denote by \( R(\alpha, \gamma, A, B) \) the class
\[
R(\alpha, \gamma, A, B) = \left\{ f \in \mathcal{A} : f'(z) + \alpha zf''(z) + \gamma z^2 f'''(z) < \frac{1 + Az}{1 + Bz}, \ z \in U \right\}, \tag{3.1}
\]
where \(-1 \leq B < A \leq 1\), and let \( h(z; A, B) = (1 + Az)/(1 + Bz) \). When \( A = 1 - 2\beta \) and \( B = -1 \), let \( h_\beta(z) := h(z; 1 - 2\beta, -1) \). The class of \( R(\alpha, \gamma, h_\beta) \) is of particular significance, and we will simply denote it by
\[
R(\alpha, \gamma, h_\beta) := R(\alpha, \gamma, \beta)
\]
\[
= \left\{ f \in \mathcal{A} : f'(z) + \alpha zf''(z) + \gamma z^2 f'''(z) < \frac{1 + (1 - 2\beta)z}{1 - z}, \ z \in U \right\}. \tag{3.2}
\]

Equivalently,
\[
R(\alpha, \gamma, \beta) = \left\{ f \in \mathcal{A} : \text{Re}\left(f'(z) + \alpha zf''(z) + \gamma z^2 f'''(z)\right) > \beta \right\}. \tag{3.3}
\]

The following result is an immediate consequence of Theorems 2.2 and 2.4.

Theorem 3.1. Under the assumptions of Theorem 2.2, if
\[
f'(z) + \alpha zf''(z) + \gamma z^2 f'''(z) < \frac{1 + Az}{1 + Bz}, \tag{3.4}
\]
then
\[
f'(z) \begin{cases} q(z; A, B) < \frac{1 + Az}{1 + Bz}, & \text{if } B \neq 0, \\ q(z; A) < 1 + Az, & \text{if } B = 0, \end{cases} \tag{3.5}
\]
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where

\[ q(z; A, B) := 1 + (A - B) \sum_{n=1}^{\infty} \frac{(-B)^{n-1} z^n}{(1 + \mu_n)(1 + \nu_n)}, \] \hspace{1cm} (3.6)

\[ q(z; A) := 1 + \frac{Az}{1 + a} \]

is the best dominant. Further,

\[ \frac{f(z)}{z} < A - B \int_0^1 \int_0^1 \int_0^1 ds dt du \]
\[ \quad \times \left[ 1 + Bz\mu s \nu \right] \]
\[ \quad = 1 + (A - B) \sum_{n=1}^{\infty} \frac{(-B)^{n-1} z^n}{(1 + n)(1 + \mu_n)(1 + \nu_n)} \]

if \( B \neq 0 \), and

\[ \frac{f(z)}{z} < 1 + \frac{Az}{2(1 + a)} \] \hspace{1cm} (3.7)

if \( B = 0 \).

4. Starlikeness Property

Starlikeness properties of functions given by a double integral operator are investigated in this section. The following result will be required.

Lemma 4.1 (see [5]). If \( f \in A \) satisfies

\[ \Re(f'(z) + azf''(z)) > \frac{(-1/\alpha)}{1 - 1/\alpha} \int_0^1 t^{1/\alpha - 1} ((1 - t)/(1 + t)) dt, \quad z \in \mathcal{U}, \] \hspace{1cm} (4.1)

for \( \alpha \geq 1/3 \), then \( f \in S^* \). This result is sharp.

Theorem 4.2. Let \( \mu \) and \( \nu \) be given by (2.4) with \( \Delta \geq 0 \) and \( \nu \geq 1/3 \). If

\[ f(z) = \int_0^1 \int_0^1 G(zt^\mu s^\nu) t^\mu s^\nu ds dt, \] \hspace{1cm} (4.2)
where \( G'(z) < h_\beta(z) = h(z; 1 - 2\beta, -1) \), and \( \beta \) satisfies

\[
\beta = 1 - \frac{1}{2 \left( 1 - (1/\nu) \int_0^1 t^{1/\nu - 1} ((1 - t)/(1 + t)) dt \right) \left( 1 - \int_0^1 (dt/(1 + t^\nu)) \right)},
\]

(4.3)

then \( f \in \mathcal{S}^* \).

**Proof.** The function \( f \) satisfies

\[
f'(z) = \int_0^1 \int_0^1 g(z t^\mu s^\nu) ds \, dt,
\]

\[
G'(z) = g(z) < h_\beta(z),
\]

(4.4)

and thus

\[
f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) = g(z) < h_\beta(z).
\]

(4.5)

Now, \( \Re h_\beta(z) > \beta \) also implies that \( \Re g(z) > \beta \), and so

\[
\Re \left( f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \right) > \beta, \quad \beta < 1.
\]

(4.6)

It follows from the proof of Theorem 2.2 that

\[
f'(z) + \nu z f''(z) < (\phi_\mu * h_\beta)(z) := q_\mu(z),
\]

(4.7)

where

\[
q_\mu(z) = 2\beta - 1 + 2(1 - \beta) \int_0^1 \frac{dt}{1 - z^t^\mu}.
\]

(4.8)

Since

\[
\Re q_\mu(z) > 2\beta - 1 + 2(1 - \beta) \int_0^1 \frac{dt}{1 + t^\nu},
\]

(4.9)

an application of Lemma 4.1 yields the result.

**Corollary 4.3.** Let \( \alpha \geq 3 \) and

\[
\Re \left( f'(z) + \alpha z f''(z) + \frac{\alpha - 1}{2} z^2 f'''(z) \right) > \beta, \quad \beta < 1.
\]

(4.10)
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If $\beta$ satisfies
\[
\beta = 1 - \frac{1}{2(1 - \log 2) \left(1 - (2/ (\alpha - 1)) \int_0^1 t^{2/ (\alpha - 1)} ((1 - t)/(1 + t)) dt\right)},
\] (4.11)
then $f \in S^*$. 

**Proof.** In this case, $\mu = 1$, $\nu = (\alpha - 1)/2$, and the result now follows from Theorem 4.2.  

**Example 4.4.** If
\[
\text{Re} \left( f'(z) + 3zf''(z) + z^2f'''(z) \right) > \beta
\] (4.12)
and $\beta$ satisfies
\[
\beta = \frac{4(1 - \log 2)^2 - 1}{4(1 - \log 2)^2} \approx -1.65509,
\] (4.13)
then $f \in S^*$. 

**Theorem 4.5.** Let $f, g \in R(\alpha, \gamma, \beta)$ and let $\mu$ and $\nu$ be given by (2.4) with $\Delta \geq 0$. If $\beta$ satisfies
\[
\beta = 1 - \frac{1}{4 \left(1 - \int_0^1 \int_0^1 \int_0^1 ds dt du / (1 + ut^{\mu}s^{\nu})\right)},
\] (4.14)
then $f \ast g \in R(\alpha, \gamma, \beta)$. 

**Proof.** Clearly,
\[
(f \ast g)'(z) + az(f \ast g)''(z) + \gamma z^2(f \ast g)'''(z) = \left(\left(f' + azf'' + \gamma z^2f'''\right) \ast \frac{g}{z}\right)(z).
\] (4.15)

Since $f \in R(\alpha, \gamma, \beta)$, substituting $A = 1 - 2\beta$ and $B = -1$ in (3.7) gives
\[
\text{Re} \frac{g(z)}{z} > 2\beta - 1 + 2(1 - \beta) \int_0^1 \int_0^1 ds dt du / 1 + ut^{\mu}s^{\nu} = \frac{1}{2}.
\] (4.16)
Hence, it follows that
\[
\text{Re} \left( (f \ast g)'(z) + az(f \ast g)''(z) + \gamma z^2(f \ast g)'''(z) \right) > \beta.
\] (4.17)
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