Janowski starlikeness for a class of analytic functions

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1. Introduction and motivation

Let \( \mathcal{A} \) be the class of all analytic functions \( f \) defined on the open unit disk \( \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( f(0) = 0 = f'(0) - 1 \). If \( f \) and \( g \) are analytic in \( \Delta \), then \( f \) is subordinate to \( g \), written \( f(z) \prec g(z) \), if there is an analytic function \( w \), satisfying \( w(0) = 0 \) and \( |w(z)| < 1 \), such that \( f(z) = g(w(z)) \). In case \( g \) is univalent in \( \Delta \), then \( f \) is subordinate to \( g \) if and only if \( f(0) = g(0) \) and \( f(\Delta) \subseteq g(\Delta) \). Let \( A \) and \( B \) be complex numbers that satisfy the conditions \( |B| \leq 1 \) and \( A \neq B \), and let \( S^*(A,B) \) denote the class of Janowski starlike functions consisting of \( f \in \mathcal{A} \) satisfying the subordination

\[
\frac{zf'(z)}{f(z)} \preceq \frac{1 + Az}{1 + Bz}.
\]

Without loss of generality, it can be assumed that \( B \) is real. If \( A \) is also real with \( |A| \leq 1 \), the fact that \( S^*[A,B] = S^*[-A,-B] \) permits us to assume that \( B < A \). For \(-1 \leq B < A \leq 1 \), this class was introduced by Janowski and investigated in [1,2].

Several well-known subclasses of starlike functions are special cases of the class \( S^*[A,B] \) for suitable choices of the parameters \( A \) and \( B \); in particular, when \( 0 \leq \alpha < 1 \), \( S^*[1 - 2\alpha, -1] = S^*(\alpha) \) is the familiar class of starlike functions of order \( \alpha \). For \( A = 1 - 2\beta \), \( \beta > 1 \) and \( B = -1 \), denote the class \( S^*[1 - 2\beta, -1] \) by \( M(\beta) \). Equivalently, \( M(\beta) \) can be expressed in the form

\[
M(\beta) := \left\{ f \in \mathcal{A} : \Im \left( \frac{zf'(z)}{f(z)} \right) < \beta, (z \in \Delta) \right\}.
\]

The class \( M(\beta) \) was investigated by Uralegaddi et al. [3], while a subclass of \( M(\beta) \) was investigated by Owa and Srivastava [4]. It should be noted that functions in the class \( M(\beta) \) and in general \( S^*[A,B] \) need not be starlike. The class \( S^*[A,B] \) unifies the...
classes $\mathbb{S}^*(\alpha)$ and $\mathbb{M}(\beta)$; this will not happen if the assumption is only that $-1 \leq B < A \leq 1$. Ma and Minda [5] have earlier introduced and investigated the class $\mathbb{S}^*(\phi)$ of analytic functions $f \in \mathcal{A}$ for which
\[ \frac{zf'(z)}{f(z)} < \phi(z), \quad z \in \Delta, \]
where $\phi$ is an analytic function with positive real part on $\Delta$, $\phi(0) = 1, \phi'(0) > 0$, and $\phi$ maps $\Delta$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class $\mathbb{S}^*(\phi)$ contains many of the classes investigated in the literature such as functions that are starlike (of order $\alpha$), strongly starlike, parabolic starlike, and Janowski starlike (for real constants $A$ and $B$).

For $0 < \alpha \leq 1, \lambda > 0$, Tuneski and Irmak [6] introduced and studied the class
\[ \mathcal{G}_{\lambda,\alpha} := \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{zf'(z)} \right| < 1 - \alpha \right\}. \]
The class $\mathcal{G}_{\lambda,\alpha}$ also includes several other classes investigated earlier, for example,
\[ \mathcal{G}_{\lambda,1/2} = \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{zf'(z)} \right| < \frac{1}{2\lambda}, z \in \Delta \right\}, \]
\[ \mathcal{G}_{\lambda,1} = \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{zf'(z)} \right| < \frac{1}{\lambda}, z \in \Delta \right\}. \]
These or related classes were investigated in [7–14].

Using the theory of first-order differential subordination, Tuneski and Irmak [6] and Tuneski [15] obtained the following result of embedding the class $\mathcal{G}_{\lambda,\alpha}$ into the class $\mathbb{S}^*[A, B]$.

**Theorem 1** ([6, Theorem 2.4]). Let $f \in \mathcal{A}$, $-1 \leq B < A \leq 1$, and $(1 + |A|)/(3 + |A|) \leq \alpha \leq 1$. If
\[ \frac{f(z)}{zf'(z)} \left( 1 - \alpha + \alpha \frac{zf''(z)}{f'(z)} \right) < \alpha + (1 - 2\alpha) \frac{1 + Bz}{1 + Az} + \alpha \frac{z(A - B)}{(1 + Az)^2}, \]
then $f \in \mathbb{S}^*[A, B]$. This result is sharp.

As a consequence, the following result is obtained:

**Corollary 1** ([6, Corollary 2.4]). Let $-1 \leq B < A \leq 1$ and $(1 + |A|)/(3 + |A|) \leq \alpha \leq 1$. Then
\[ \lambda = (A - B) \frac{(2\alpha - 1)|A| - (1 - 3\alpha)}{(1 + |A|)^2} \tag{1.1} \]
is the greatest number such that $\mathcal{G}_{\lambda,\alpha} \subseteq \mathbb{S}^*[A, B]$.

Note that there was a typographic error in sign in the work of [6], and that expression (1.1) is the correct constant. We now introduce a class of analytic functions defined by means of subordination.

**Definition 1.** For complex constants $C$ and $D$ with $|D| \leq 1, C \neq D$, the class $\mathcal{G}_{\alpha}[C, D]$ consists of all functions $f \in \mathcal{A}$ satisfying the subordination
\[ \frac{f(z)}{zf'(z)} \left( 1 - \alpha + \alpha \frac{zf''(z)}{f'(z)} \right) < \left( 1 - \alpha \right) \frac{1 + Cz}{1 + Dz}. \]

For $0 < \alpha \leq 1, \lambda > 0$, the class $\mathcal{G}_{\alpha}[\lambda/(1 - \alpha), 0]$ reduced to the class $\mathcal{G}_{\lambda,\alpha}$ studied by Tuneski and Irmak [6]. In this paper, we investigate the more general inclusion $\mathcal{G}_{\alpha}[C, D] \subseteq \mathbb{S}^*[A, B]$. The following result will be required.

**Theorem 2** ([16, Theorem 3.4h, p.132]). Let $q$ be univalent in the unit disk $\Delta$ and $\vartheta$ and $\varphi$ be analytic in a domain $D$ containing $q(\Delta)$ with $\varphi(w) \neq 0$ whenever $w \in q(\Delta)$. Set $Q(z) := \varphi'(z)\varphi(q(z))$ and $H(z) := \vartheta(q(z)) + Q(z)$. Suppose that either $h$ is convex, or $Q$ is starlike univalent in $\Delta$. In addition, assume that $\Re[zH'(z)/Q(z)] > 0$ for $z \in \Delta$. If $p$ is analytic in $\Delta$ with $p(0) = q(0)$, $p(\Delta) \subseteq D$ and
\[ \vartheta(p(z)) + zp'(z)\varphi(p(z)) < \vartheta(q(z)) + zq'(z)\varphi(q(z)), \tag{1.2} \]
then $p(z) < q(z)$ and $q$ is the best dominant.
2. Main results

We begin with the following sufficient condition for a function $f \in A$ to satisfy the subordination $zf'(z)/f(z) < 1/q(z)$.

**Theorem 3.** Let $\alpha$ be a nonzero complex number. Let $q$ be univalent and $q(z) \neq 0$ in $\Delta$, $q(0) = 1$ and

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, \Re \left( \frac{1 - 2\alpha}{\alpha} \right) \right\}. \tag{2.1}$$

If $f \in A$ satisfies the subordination

$$\frac{zf'(z)}{f(z)} \left\{ 1 - \alpha + \alpha \frac{zf''(z)}{f'(z)} \right\} < \alpha + (1 - 2\alpha)q(z) - \alpha q'(z), \tag{2.2}$$

then

$$\frac{zf'(z)}{f(z)} < \frac{1}{q(z)}$$

and $1/q$ is the best dominant.

**Proof.** Let the function $p$ be defined by

$$p(z) = \frac{f(z)}{zf'(z)}. \tag{2.3}$$

A computation from (2.3) gives

$$zp'(z) = \frac{zf'(z)}{f(z)} - \left( 1 + \frac{zf''(z)}{f'(z)} \right).$$

and hence

$$1 + \frac{zf''(z)}{f'(z)} = -\frac{zp'(z)}{p(z)} + \frac{1}{p(z)}. \tag{2.4}$$

Now (2.3) and (2.4) yield

$$\frac{zf'(z)}{f(z)} \left\{ 1 - \alpha + \alpha \frac{zf''(z)}{f'(z)} \right\} = \alpha + (1 - 2\alpha)p(z) - \alpha q'(z). \tag{2.5}$$

Using (2.5), it follows that (2.2) becomes

$$\alpha + (1 - 2\alpha)p(z) - \alpha q'(z) \prec \alpha + (1 - 2\alpha)q(z) - \alpha q'(z),$$

or

$$(1 - 2\alpha)p(z) - \alpha q'(z) \prec (1 - 2\alpha)q(z) - \alpha q'(z). \tag{2.6}$$

Define the functions $\vartheta$ and $\psi$ by

$$\vartheta(w) = (1 - 2\alpha)w, \quad \psi(w) = -\alpha$$

so that (2.6) becomes (1.2). Since $\alpha \neq 0$, clearly $\psi(w) \neq 0$. Now let

$$Q(z) := q'(z)\psi(q(z)) = -\alpha q'(z)$$

$$h(z) := \vartheta(q(z)) + Q(z) = (1 - 2\alpha)q(z) - \alpha q'(z).$$

In view of (2.1), $Q$ is starlike and

$$\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ 1 + \frac{zq''}{q'} - \frac{1 - 2\alpha}{\alpha} \right\} > 0.$$

The result now follows by an application of Theorem 2. □

**Corollary 2.** Let $\alpha \in \mathbb{C}$, $-1 \leq B < A \leq 1$, and further assume that

$$\Re \left( \frac{1}{\alpha} \right) \leq \frac{3 + |A|}{1 + |A|}. \tag{2.7}$$
If \( f \in \mathcal{A} \) satisfies
\[
\left( 1 - \alpha + \alpha \frac{zf''(z)}{f'(z)} \right) < \alpha + (1 - 2\alpha) \frac{1 + Bz}{1 + Az} + \alpha \frac{z(A - B)}{(1 + Az)^2},
\]
then \( f \in S^*[A, B] \). The result is sharp.

**Proof.** Let the function \( q \) be defined by \( q(z) = (1 + Bz)/(1 + Az) \). This function \( q \) is convex univalent and (2.7) yields
\[
\Re \left( 1 + \frac{q''(z)}{q'(z)} \right) = \Re \left( \frac{1 - Az}{1 + Az} \right) > \frac{1 - |A|}{1 + |A|} \geq \max \left( 0, \Re \frac{1 - 2\alpha}{\alpha} \right).
\]
The result now follows from Theorem 3. □

**Remark 1.** When \( \alpha \) is real, Corollary 2 reduces to Theorem 1.

**Theorem 4.** Let \( A, B, C, D \) and \( \alpha \) be real numbers satisfying \( |D| \leq 1, C \neq D, |A| \leq 1, |B| \leq 1, A \neq B \) and \( \alpha \neq 0 \). Let \( I := (3\alpha - 1)^2 + (2\alpha - 1)^2A^2, J := 2(3\alpha - 1)(2\alpha - 1)A, K := (C - D)(1 - \alpha), L := A^3C(1 - \alpha) - AD\alpha(A - 2B) - ABD, \) and \( M := 2AC(1 - \alpha) - D(A + B) - Da(A - 3B) \). Further, when \( KL < 0 \) and \( |(A - B)^2J - 2(K + L)M| < -8KL \), assume that
\[
-16KL[(A - B)^2I - M^2(L - K)^2] - [(A - B)^2J - 2(K + L)M]^2 \geq 0
\]
while in all other cases, let
\[
|(A - B)^2J - 2(K + L)M| \leq (A - B)^2I - M^2 - (L + K)^2.
\]
Then \( g_\alpha[C, D] \subseteq S^*[A, B] \).

**Proof.** In view of Theorem 3, it is enough to show that
\[
g(z) := (1 - \alpha) \frac{1 + Cz}{1 + Dz} < \alpha + (1 - 2\alpha) \frac{1 + Bz}{1 + Az} + \alpha z(A - B) \frac{1 + Az}{(1 + Az)^2} := h(z).
\]
Since \( g \) is univalent, the subordination \( g(z) < h(z) \) is equivalent to the subordination
\[
z < g^{-1}(h(z)) := H(z).
\]
The proof will be completed by showing that \( |H(e^{i\theta})| \geq 1 \) for all \( \theta \in [0, 2\pi] \). First note that
\[
h(z) = \frac{1 - \alpha + [A + B + \alpha(A - 3B)]z + [A\alpha(A - 2B) + AB]z^2}{(1 + Az)^2},
\]
and
\[
g^{-1}(w) = \frac{w + \alpha - 1}{C(1 - \alpha) - Dw},
\]
so that
\[
H(z) = \frac{(A - B)[(3\alpha - 1) + (2\alpha - 1)Az]z}{K + Mz + Lz^2}.
\]
Writing \( t = \cos \theta \), it follows that
\[
|H(e^{i\theta})|^2 = \frac{(1 + Jt)(A - B)^2}{|Ke^{-i\theta} + M + Le^{i\theta}|^2} = \frac{(1 + Jt)(A - B)^2}{4KLT^2 + 2(K + L)M^2 + M^2 + (L - K)^2}.
\]
Now \( |H(e^{i\theta})|^2 \geq 1 \) provided \( at^2 + bt + c \geq 0 \), where \( a = -4KL, b = (A - B)^2J - 2(K + L)M, \) and \( c = (A - B)^2I - M^2 - (L - K)^2 \). Since
\[
\min_{|t| \leq 1} |at^2 + bt + c| = \begin{cases} 4ac - b^2, & a > 0, |b| < 2a, \\ 4a, & a + c - |b|, \end{cases}
\]
the inequality \( |H(e^{i\theta})| \geq 1 \) is satisfied provided the conditions stated in Theorem 4 hold. □
Remark 2. When $D = 0$, $C = \lambda/(1 - \alpha)$, we have $I = (3\alpha - 1)^2 + (2\alpha - 1)^2A^2$, $J = 2(3\alpha - 1)(2\alpha - 1)A$, $K = \lambda$, $L = \lambda A^2$, $M = 2\lambda$. Clearly $KL = \lambda^2A^2 \geq 0$. In this case, the condition in the hypothesis of Theorem 4 becomes

$$|(A - B)^2J - 4A\lambda^2(1 + A^2)| \leq (A - B)^2I - 4A^2\lambda^2 - \lambda^2[4A^2 + (A^2 - 1)^2].$$

A computation shows that

$$\lambda = (A - B)(2\alpha - 1)|A| - (1 - 3\alpha)

\frac{1}{(1 + |A|)^2}$$

provided $(1 + |A|)/(3 + |A|) \leq \alpha < 1$. Thus Theorem 4 reduces to [6, Corollary 2.4, p. 4].

Remark 3. In [17] and [18] a similar technique using Jack's lemma was used to investigate Janowski starlikeness of the Bernardi integral operator.

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References