Integral operators on Ma–Minda type starlike and convex functions

Rosihan M. Ali a,∗, V. Ravichandran b

a School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia
b Department of Mathematics, University of Delhi, Delhi-110007, India

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A B S T R A C T
Two integral operators on the classes consisting of normalized p-valent Ma–Minda type starlike and convex functions are considered. Functions in these classes have the form

\[ \frac{zf'(z)}{f(z)} \prec p\varphi(z) \] and

\[ 1 + \frac{zf''(z)}{f'(z)} \prec p\varphi(z) \]

respectively, where \( \varphi \) is a convex function with \( \varphi(0) = 1 \). It is shown that the first of these operators maps starlike functions into convex functions, while the convex mappings are shown to be closed under the second integral operator.

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1. Introduction and motivation

Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) be the open unit disk in the complex plane and let \( \mathcal{A} \) denote the class of all functions \( f \) analytic in \( D \) and normalized by the conditions \( f(0) = 0, \) and \( f'(0) = 1 \). An analytic function \( f \) is subordinate to an analytic function \( g \), written \( f \prec g \) (\( z \in \mathbb{D} \)), if there exists a function \( w \), analytic in \( \mathbb{D} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), such that \( f = g(w(z)) \). When the function \( g \) is univalent in \( D \), the subordination \( f \prec g \) is equivalent to \( f(0) = g(0) \) and \( f(D) \subset g(D) \). A function \( f \in \mathcal{A} \) is starlike if \( f(D) \) is a starlike domain with respect to 0, and a function \( f \in \mathcal{A} \) is convex if \( f(D) \) is a convex domain. Analytically, these requirements are respectively equivalent to the conditions

\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0, \quad \text{and} \quad \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0. \]

In terms of subordination, these conditions are expressed respectively in the forms

\[ \frac{zf'(z)}{f(z)} < \frac{1 + z}{1 - z}, \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} < \frac{1 + z}{1 - z}. \]

Ma and Minda [1] gave a unified presentation of various subclasses of starlike and convex functions by replacing the superordinate function \( (1 + z)/(1 - z) \) with a more general function \( \varphi \). This analytic function \( \varphi \) has positive real part with \( \varphi(0) = 1 \), and maps the unit disk \( D \) onto a region starlike with respect to 1. Ma and Minda introduced the following classes that includes several well-known starlike and convex mappings as special cases:

\[ \mathcal{ST}(\varphi) := \{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \} \]

* Corresponding author.
E-mail addresses: rosihan@cs.usm.my (R.M. Ali), vravi@maths.du.ac.in (V. Ravichandran).

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and
\[ C^\mathcal{V}(\phi) := \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} < \phi(z) \right\}. \]

Let \( A_p \) be the class of all \( p \)-valent analytic functions \( f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots \) in the open unit disk \( \mathbb{D} \). The class \( A_1 \) will be denoted by \( A \). Following Ma and Minda [1], the following classes of \( p \)-valent starlike and convex functions were introduced and investigated in [2].

**Definition 1** ([2]). Let \( \psi \) be an analytic univalent function in \( \mathbb{D} \) with \( \psi(0) = 1 \). The class \( C^\mathcal{V}_p(\psi) \) consists of functions \( f \in A_p \) satisfying
\[
\frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \psi(z) \quad (z \in \mathbb{D}),
\]
and the class \( \mathcal{S}^\mathcal{T}_p(\psi) \) consists of functions \( f \in A_p \) satisfying
\[
\frac{1}{p} \frac{zf'(z)}{f(z)} < \psi(z) \quad (z \in \mathbb{D}).
\]

Let \( \varphi_p : \mathbb{D} \to \mathbb{C} \) be the function defined by
\[
\varphi_p(z) = \frac{1 + (1 - 2\beta)z}{1 - z}, \quad \beta \neq 1.
\]

When \( \beta < 1 \), \( \varphi_p(\mathbb{D}) \) is the half-plane defined by \( \text{Re} w > \beta \), while in the case \( \beta > 1 \), \( \varphi_p(\mathbb{D}) \) is the half-plane defined by \( \text{Re} w < \beta \). Thus for \( \beta < 1 \), the classes \( \mathcal{S}^\mathcal{T}_p(\varphi_p) \) and \( C^\mathcal{V}_p(\varphi_p) \) reduce to the familiar classes of \( p \)-valent starlike and convex functions of order \( \beta \):
\[
\mathcal{S}^\mathcal{T}_p(\beta) := \left\{ f \in A_p : \frac{1}{p} \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \beta \right\},
\]
\[
C^\mathcal{V}_p(\beta) := \left\{ f \in A_p : \frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta \right\}.
\]

Similarly, for \( \beta > 1 \), the classes \( \mathcal{S}^\mathcal{T}_p(\varphi_p) \) and \( C^\mathcal{V}_p(\varphi_p) \) reduce respectively to the equivalent classes
\[
\mathcal{M}_p(\beta) := \left\{ f \in A_p : \frac{1}{p} \text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \beta \right\},
\]
\[
\mathcal{N}_p(\beta) := \left\{ f \in A_p : \frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \beta \right\}.
\]

For \( p = 1 \), these classes were considered by Breaz [3], Nishiwaki and Owa [4], Owa and Nishiwaki [5], Owa and Srivastava [6], and Uralegaddi et al. [7].

Next let \( \varphi_{\lambda,\mu} : \mathbb{D} \to \mathbb{C} \) be the conformal mapping of \( \mathbb{D} \) onto the domain
\[
\Omega_{\lambda,\mu} := \{ w \in \mathbb{C} : \text{Re} w - \mu \geq \lambda |w - 1| \},
\]
and normalized by \( \varphi_{\lambda,\mu}(0) = 1 \). Then the classes \( \mathcal{S}^\mathcal{T}_p(\varphi_{\lambda,\mu}) \) and \( C^\mathcal{V}_p(\varphi_{\lambda,\mu}) \) reduce to the classes \( \mathcal{S}^\mathcal{T}_p(\lambda, \mu) \) and \( C^\mathcal{V}_p(\lambda, \mu) \) of \( p \)-valent starlike and convex functions associated with parabolic starlike and uniformly convex functions. The class \( C^\mathcal{V}_p(\lambda, \mu) \) was investigated by Yang and Owa [8], and Frasin [9]. In fact the classes \( C_p(\lambda, \mu) \) and \( UC_p(\beta, k) \) investigated by Frasin [9] are essentially the same: \( C_p(\lambda, \mu) = UC_p(p\mu, \lambda) \). We shall consider only the former class in this paper, which in our notation is the class \( C^\mathcal{V}_p(\lambda, \mu) \).

For \( a_i \geq 0 \) and \( f_i \in A_p \), define the following respective integral operators:
\[
F_p(z) = \int_0^z pt^{p-1} \prod_{i=1}^n \left( \frac{f_i(t)}{t^p} \right)^{a_i} \text{dt}, \quad (1.1)
\]
\[
G_p(z) = \int_0^z pt^{p-1} \prod_{i=1}^n \left( \frac{f_i'(t)}{t^{p-1}} \right)^{a_i} \text{dt}. \quad (1.2)
\]

In this paper, the above defined integral operators are investigated for the classes of \( p \)-valent Ma–Minda type starlike and convex functions. It is shown that \( F_p \) defined by (1.1) transforms a Ma–Minda type starlike function into a Ma–Minda type convex function. It is also shown that the Ma–Minda type convex functions are closed under the operator \( G_p \) given by (1.2). In the special case \( p = 1 \), the results obtained here include several earlier works found in the literature.
2. Convexity of the integral operators

**Theorem 2.1.** Let $\alpha_i \geq 0$, and $f_i \in A_p$, $i = 1, 2, \ldots, n$. Let $F_p$ be given by (1).

1. If $f_i \in \mathcal{S}(\beta)$, $\beta_i < 1$, then $F_p \in CV_p(\gamma)$ where $\gamma := 1 - (1 - \beta) \sum_{i=1}^{n} \alpha_i$. In particular, if $\sum_{i=1}^{n} \alpha_i \leq 1$, then $F_p \in CV_p(0)$.
2. If $f_i \in M_p(\beta_i)$, $\beta_i > 1$, then $F_p \in N_p(\gamma)$ where $\gamma := 1 + (\beta - 1) \sum_{i=1}^{n} \alpha_i$. 

**Proof.** Since

$$F_p'(z) = p^{-1} \prod_{i=1}^{n} \left( \frac{f_i(z)}{z^p} \right)^{\alpha_i},$$

it follows that

$$\frac{1}{p} \left( 1 + \frac{zF_p''(z)}{F_p'(z)} \right) = \left( 1 - \sum_{i=1}^{n} \alpha_i \right) + \sum_{i=1}^{n} \alpha_i \frac{1}{p} \left( \frac{zf_i''(z)}{f_i(z)} \right).$$

The desired results are now evident from the definitions of the above classes. □

**Corollary 2.1.** Let $\alpha_i \geq 0$, and $f_i \in A_p$, $i = 1, 2, \ldots, n$. Let $F_p$ be given by (1).

1. If $f_i \in \mathcal{S}(\beta)$, $\beta < 1$, then $F_p \in CV_p(\gamma)$ where $\gamma := 1 - (1 - \beta) \sum_{i=1}^{n} \alpha_i$. In particular, if $\sum_{i=1}^{n} \alpha_i \leq 1$, then $F_p \in CV_p(\beta)$.
2. If $f_i \in M_p(\beta_i)$, $\beta > 1$, then $F_p \in N_p(\gamma)$ where $\gamma := 1 + (\beta - 1) \sum_{i=1}^{n} \alpha_i$.

Given a complex number $b \neq 0$, the classes of $p$-valent starlike and convex functions of complex order $b$ and type $\beta$ ($\beta < 1$), are defined as below:

$$\mathcal{S}(b, \beta) := \left\{ f \in A_p : \Re \left( 1 + \frac{1}{b} \left( \frac{1}{b} \frac{zf''(z)}{f'(z)} - 1 \right) \right) > \beta \right\},$$

$$CV_p(b, \beta) := \left\{ f \in A_p : \Re \left( 1 + \frac{1}{b} \left( \frac{1}{b} \frac{zf'(z)}{f(z)} - 1 \right) \right) > \beta \right\}.$$ 

For $p = 1$, these classes were considered by [10–13]. It is clear that $\mathcal{S}(b, \beta) = \mathcal{S}(b(1 - \beta), 0)$ and $CV_p(b, \beta) = CV_p(b(1 - \beta), 0)$. Similarly, for $\beta > 1$, we define the following classes:

$$\mathcal{M}_p(b, \beta) := \left\{ f \in A_p : \Re \left( 1 + \frac{1}{b} \left( \frac{1}{b} \frac{zf''(z)}{f'(z)} - 1 \right) \right) < \beta \right\},$$

$$N_p(b, \gamma) := \left\{ f \in A_p : \Re \left( 1 + \frac{1}{b} \left( \frac{1}{b} \frac{zf'(z)}{f(z)} - 1 \right) \right) < \beta \right\}.$$ 

**Theorem 2.1** extends to the above defined classes as shown in the following result:

**Theorem 2.2.** Let $\alpha_i \geq 0$, and $f_i \in A_p$, $i = 1, 2, \ldots, n$. Let $F_p$ be given by (1).

1. If $f_i \in \mathcal{S}(\beta)$, $\beta_i < 1$, then $F_p \in CV_p(\gamma)$ where $\gamma := 1 - \sum_{i=1}^{n} \alpha_i (1 - \beta_i)$.
2. If $f_i \in M_p(\beta_i)$, $\beta_i > 1$, then $F_p \in N_p(\gamma)$ where $\gamma := 1 + \sum_{i=1}^{n} \alpha_i (\beta_i - 1)$.

**Proof.** The result follows by noting that

$$1 + \frac{1}{b} \left( \frac{1}{b} \frac{zf''(z)}{f'(z)} - 1 \right) = \left( 1 - \sum_{i=1}^{n} \alpha_i \right) + \sum_{i=1}^{n} \alpha_i \left( 1 + \frac{1}{b} \left( \frac{1}{b} \frac{zf''(z)}{f'(z)} - 1 \right) \right).$$

**Remark 2.1.** **Theorem 2.2**(1) extends the work of Bulut [12]. In particular, when $p = 1$, **Theorem 2.2**(1) reduces to Theorem 1 in [12].

**Theorem 2.3.** Let $\alpha_i \geq 0$, and $f_i \in A_p$, $i = 1, 2, \ldots, n$. Let $G_p$ be given by (1.2).

1. If $f_i \in CV_p(\beta)$, $\beta_i < 1$, then $G_p \in CV_p(\gamma)$ where $\gamma := 1 - \sum_{i=1}^{n} \alpha_i (1 - \beta_i)$. In particular, if $\sum_{i=1}^{n} \alpha_i (1 - \beta_i) \leq 1$, then $G_p \in CV_p(0)$.
2. If $f_i \in N_p(\beta_i)$, $\beta_i > 1$, then $G_p \in N_p(\gamma)$ where $\gamma := 1 + \sum_{i=1}^{n} \alpha_i (\beta_i - 1)$.
Proof. Since
\[ G_p(z) = p^{z-p-1} \prod_{i=1}^{n} \left( \frac{f'_i(z)}{p^{z-p-1}} \right)^{\alpha_i}, \]
it follows that
\[ \frac{1}{p} \left( 1 + \frac{2G'_p(z)}{G_p(z)} \right) = \left( 1 - \sum_{i=1}^{n} \alpha_i \right) + \sum_{i=1}^{n} \alpha_i \frac{1}{p} \left( 1 + \frac{2f''_i(z)}{f'_i(z)} \right). \]
The desired results follow directly from the definitions of the classes. □

Corollary 2.2. Let \( \alpha_i \geq 0 \), and \( f_i \in \mathcal{A}_p \), \( i = 1, 2, \ldots, n \). Let \( G_p \) be given by (1.2).

1. If \( f_i \in \mathcal{C} \mathcal{V}_p(\beta), \beta < 1 \), then \( G_p \in \mathcal{C} \mathcal{V}_p(\gamma) \) where \( \gamma := 1 - (1 - \beta) \sum_{i=1}^{n} \alpha_i \). In particular, if \( \sum_{i=1}^{n} \alpha_i \leq 1 \), then \( G_p \in \mathcal{C} \mathcal{V}_p(\beta) \).

2. If \( f_i \in \mathcal{N}_p(\beta), \beta > 1 \), then \( G_p \in \mathcal{N}_p(\gamma) \) where \( \gamma := 1 + (\beta - 1) \sum_{i=1}^{n} \alpha_i \).

In general, the following result is obtained:

Theorem 2.4. Let \( \alpha_i \geq 0 \), and \( f_i \in \mathcal{A}_p \), \( i = 1, 2, \ldots, n \). Let \( G_p \) be given by (1.2).

1. If \( f_i \in \mathcal{C} \mathcal{V}_p(\beta), \beta < 1 \), then \( G_p \in \mathcal{C} \mathcal{V}_p(\gamma) \) where \( \gamma := 1 - \sum_{i=1}^{n} \alpha_i(1 - \beta_i) \).

2. If \( f_i \in \mathcal{N}_p(\beta), \beta > 1 \), then \( G_p \in \mathcal{N}_p(\gamma) \) where \( \gamma := 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1) \).

Proof. The results follow from the equation
\[ 1 + \frac{1}{\beta} \left( 1 + \frac{2G'_p(z)}{G_p(z)} \right) = \left( 1 - \sum_{i=1}^{n} \alpha_i \right) + \sum_{i=1}^{n} \alpha_i \left( 1 + \frac{1}{\beta} \left( 1 + \frac{2f''_i(z)}{f'_i(z)} \right) - 1 \right). \]

Remark 2.2. For \( p = 1 \), Theorem 2.2(1) reduces to Theorem 3 in [12].

As applications of our results, the following results are obtained for the class \( \mathcal{C} \mathcal{V}_p(\lambda, \mu) \).

Theorem 2.5. For \( i = 1, 2, \ldots, n \), let \( \alpha_i \geq 0 \), \( \mu_i \geq \lambda_i \geq 0 \) and
\[ \gamma := 1 - \sum_{i=1}^{n} \alpha_i \frac{1 - \mu_i}{1 + \lambda_i}. \]

1. If \( f_i \in \mathcal{T}_p(\lambda_i, \mu_i) \), then \( F_p \in \mathcal{C} \mathcal{V}_p(\gamma) \).

2. If \( f_i \in \mathcal{N}_p(\lambda_i, \mu_i) \), then \( G_p \in \mathcal{C} \mathcal{V}_p(\gamma) \).

Proof. We first prove that \( \mathcal{T}_p(\lambda, \mu) \subset \mathcal{T}_p((\mu + \lambda)/(1 + \lambda)). \) Let \( f \in \mathcal{T}_p(\lambda, \mu) \). Then the quantity \( w := zf'(z)/(pf(z)) \) satisfies
\[ \text{Re } w - \mu \geq \lambda |w - 1|. \]
The inequality
\[ \text{Re } w - \mu \geq -\lambda \text{Re}(w - 1) \]
yields
\[ \text{Re } w \geq \frac{\mu + \lambda}{1 + \lambda}. \]

Thus \( f \in \mathcal{T}_p(\sum_{i=1}^{n} \frac{\alpha_i(1 - \mu_i)}{1 + \lambda_i}) \). Now since \( f_i \in \mathcal{T}_p(\lambda_i, \mu_i) \), then \( f_i \in \mathcal{T}_p((\mu_i + \lambda_i)/(1 + \lambda_i)) \), and the results of the theorem now follows from an application of Theorem 2.1(1).

The proof of the second part of the theorem follows similarly from Theorem 2.3(1). □

Remark 2.3. Since
\[ 1 - \sum_{i=1}^{n} \frac{\alpha_i(1 - \mu_i)}{1 + \lambda_i} \geq 1 - \sum_{i=1}^{n} \alpha_i(1 - \mu_i), \]
Theorem 2.5(2) improves the corresponding result of Frasin [9, Theorem 3.6]. It should be pointed out that the result obtained by Frasin is independent of the parameters \( \lambda_i \), where as these parameters play an important role in our Theorem 2.5(2).
Next let $-1 \leq B \leq A \leq 1$, and $\varphi_{A,B}$ be given by
\[ \varphi_{A,B}(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{D}). \]

Let $\delta \mathcal{T}_p(A,B) := \delta \mathcal{T}_p(\varphi_{A,B})$ and $\mathcal{CV}_p(A,B) := \mathcal{CV}_p(\varphi_{A,B})$. It can be shown that $\delta \mathcal{T}_p(A,B) \subset \delta \mathcal{T}_p((1-A)/(1-B))$.

Using this fact, the following theorem is evident:

**Theorem 2.6.** Let $\alpha_i \geq 0$, $-1 < B_i < A_i \leq 1$, $i = 1, 2, \ldots, n$, and
\[ \gamma := 1 - \sum_{i=1}^{n} \alpha_i - B_i. \]

1. If $f_i \in \delta \mathcal{T}_p(A_i,B_i)$, then $F_p \in \mathcal{CV}_p(\gamma)$.
2. If $f_i \in \mathcal{CV}_p(A_i,B_i)$, then $G_p \in \mathcal{CV}_p(\gamma)$.

### 3. Closure property of integral operators

For $i = 1, 2, \ldots, n$, let $\alpha_i \geq 0$, $-1 < B_i < A_i \leq 1$, and $\sum_{i=1}^{n} \alpha_i \leq 1$. For $f_i \in A_p$, let $F_p$ be given by (1.1). By Corollary 2.1, if $f_i \in \delta \mathcal{T}_p(\beta)$, then $F_p \in \mathcal{CV}_p(\beta)$. We prove this in a more general setting in the following theorem:

**Theorem 3.1.** For $i = 1, 2, \ldots, n$, let $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i \leq 1$. Let $\varphi$ be convex in $\mathbb{D}$ with $\varphi(0) = 1$. If $f_i \in \delta \mathcal{T}_p(\varphi)$, then $F_p \in \mathcal{CV}_p(\varphi)$.

**Proof.** As shown in the proof of Theorem 2.1, it follows that
\[ \frac{1}{p} \left( 1 + \frac{z f_i^p(z)}{F_p(z)} \right) = \left( 1 - \sum_{i=1}^{n} \alpha_i \right) + \sum_{i=1}^{n} \alpha_i \frac{1}{p} \left( \frac{z f_i(z)}{f_i(z)} \right). \]

The assumption that $f_i \in \delta \mathcal{T}_p(\varphi)$, yields
\[ \frac{1}{p} \frac{z f_i'(z)}{f_i(z)} < \varphi(z), \]

and thus
\[ \frac{1}{p} \frac{z f_i'(z)}{f_i(z)} \in \varphi(\mathbb{D}), \]

for every $z \in \mathbb{D}$. Since $\varphi$ is convex, the convex combination of $1$ and $\frac{1}{p} \frac{z f_i'(z)}{f_i(z)}$ ($i = 1, 2, \ldots, n$), is again in $\varphi(\mathbb{D})$. This shows that
\[ \frac{1}{p} \left( 1 + \frac{z f_i^p(z)}{F_p(z)} \right) = \left( 1 - \sum_{i=1}^{n} \alpha_i \right) + \sum_{i=1}^{n} \alpha_i \frac{1}{p} \left( \frac{z f_i'(z)}{f_i(z)} \right) \in \varphi(\mathbb{D}), \]

or
\[ \frac{1}{p} \left( 1 + \frac{z f_i^p(z)}{F_p(z)} \right) < \varphi(z). \]

Shanmugam and Ravichandran [14] have shown that if the $f_i$’s are uniformly convex functions and $\alpha_i$’s are real numbers such that $\alpha_i \geq 0$, and $\sum_{i=1}^{n} \alpha_i \leq 1$, then the function
\[ \int_{0}^{x} \prod_{i=1}^{n} |f_i'(| \zeta |)|^p d\zeta \]

is also uniformly convex. This result was extended to parabolic starlike functions of order $\rho$ by Aghalary and Kulkarni [15]. This result is indeed valid even for a more general class of functions:

**Theorem 3.2.** For $i = 1, 2, \ldots, n$, let $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i \leq 1$. Let $\varphi$ be convex in $\mathbb{D}$ with $\varphi(0) = 1$. If $f_i \in \mathcal{CV}_p(\varphi)$, then $G_p \in \mathcal{CV}_p(\varphi)$.

The proof is similar to Theorem 3.1, and is therefore omitted.
Remark 3.1. For $i = 1, 2, \ldots, n$, let $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i \leq 1$. Let $\varphi$ be convex in $\mathbb{D}$ with $\varphi(0) = 1$. If $f_i \in \mathcal{CV}_p(\varphi)$, then it follows from Theorem 3.2 that

$$z^p \prod_{i=1}^{n} \left( \frac{f_i'(z)}{pz^{p-1}} \right)^{\alpha_i} \in \mathcal{ST}_p(\varphi).$$

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