General Mathematics Vol. 20, No. 2-3 (2012), 32–46

# A Subclass of Harmonic Univalent Functions with Positive Coefficients defined by Dziok-Srivastava Operator [1]

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**Abstract**

In this paper using the Dziok-Srivastava [4] operator, we introduce a subclass of the class $\mathcal{H}$ of complex valued Harmonic univalent functions $f = h + \bar{g}$, where $h$ is the analytic part and $g$ is the co-analytic part of $f$ in $|z| < 1$. Coefficient bounds, extreme points, inclusion results and closure under integral operator for this class are obtained.

**2010 Mathematics Subject Classification:** 30C45, 30C50.

**Key words and phrases:** Harmonic functions, hypergeometric functions, Dziok-Srivastava operator, extreme points, integral operator.

[1] *Received in final form 02 December, 2013*
*Accepted for publication (in final form) 02 December, 2013*

# 1 Introduction

Harmonic mappings have found applications in many diverse fields such as engineering, aerodynamics and other branches of applied Mathematics. Harmonic mappings in a domain $D \subseteq C$ are univalent complex-valued harmonic functions $f = u + iv$ where both $u$ and $v$ are real harmonic. The important work of Clunie and Sheil-Small [2] on the class consisting of complex-valued harmonic orientation preserving univalent functions $f$ defined on the open unit disk $U$ formed the basis for several investigations on different subclasses of harmonic univalent functions.

In any simply-connected domain $D$, it is known that [2] we can write $f = h + \bar{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and orientation preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$ [2].

Denote by $\mathcal{H}$ the family of harmonic functions

$$f = h + \bar{g} \tag{1}$$

which are univalent and orientation preserving in the open unit disk $U = \{z : |z| < 1\}$ and $f$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \bar{g} \in \mathcal{H}$ the analytic functions $h$ and $g$ are given by

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=1}^{\infty} b_m z^m.$$

Hence

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m}, \quad |b_1| < 1 \tag{2}$$

We note that the family $\mathcal{H}$ reduces to the well known class $S$ of normalized univalent functions if the co-analytic part of $f$ is identically zero, that is $g \equiv 0$.

For complex numbers $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ ($\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, q$) the generalized hypergeometric function [8] ${}_pF_q(z)$ is defined by

$$(3) \qquad {}_pF_q(z) \equiv {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \ldots (\alpha_p)_m}{(\beta_1)_m \ldots (\beta_q)_m} \frac{z^m}{m!}$$

$$(p \leq q+1; p, q \in N_0 = N \cup \{0\}; z \in U),$$

where $N$ denotes the set of all positive integers and $(a)_m$ is the Pochhammer symbol defined by

$$(4) \qquad (a)_m = \begin{cases} 1, & m = 0, \\ a(a+1)(a+2)\ldots(a+m-1), & m \in N. \end{cases}$$

Dziok and Srivastava [4] introduced an operator in their study of analytic functions associated with generalized hypergeometric functions. This Dziok-Srivastava operator is known to include many well-known operators as special cases.

Let $H(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q) : A \to A$ be a linear operator defined by

$$[(H(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q))(\phi)](z) = z \, {}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) * \phi(z)$$

$$(5) \qquad = z + \sum_{m=2}^{\infty} \Gamma_m a_m z^m$$

where

$$(6) \qquad \Gamma_m = \frac{(\alpha_1)_{m-1} \ldots (\alpha_p)_{m-1}}{(\beta_1)_{m-1} \ldots (\beta_q)_{m-1}} \frac{1}{m-1!}$$

and $\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q$ are positive real numbers, such that $p \leq q+1$; $p, q \in N \cup \{0\}$, and $(a)_m$ is the familiar Pochhammer symbol.

The linear operator $H(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q)$ or $H_q^p[\alpha_1, \beta_1]$ in short, is the Dziok-Srivastava operator ([4] & [12]), which includes several well known operators.

The Dziok-Srivastava operator when extended to the harmonic function $f = h + \bar{g}$ is defined by

$$H_q^p[\alpha_1, \beta_1]f(z) = H_q^p[\alpha_1, \beta_1]h(z) + \overline{H_q^p[\alpha_1, \beta_1]g(z)} \tag{7}$$

Denote by $V_{\mathcal{H}}$ the subclass of $\mathcal{H}$ consisting of functions of the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{m=2}^{\infty} |a_m| z^m, \quad g(z) = \sum_{m=1}^{\infty} |b_m| z^m, \quad |b_1| < 1 \tag{8}$$

Motivated by earlier works of [1, 3, 6, 7, 10, 11] on harmonic functions, we introduce here a new subclass $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ of $V_{\mathcal{H}}$ using Dziok-Srivastava operator extended to harmonic functions.

We denote by $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$, the subclass of $V_{\mathcal{H}}$, consisting of functions of the form (8) satisfying the condition

$$Re \left\{ \begin{array}{l} \alpha \left( \dfrac{H_q^p[\alpha_1, \beta_1]h(z) + H_q^p[\alpha_1, \beta_1]g(z)}{z} \right) \\ +(H_q^p[\alpha_1, \beta_1]h(z))' + (H_q^p[\alpha_1, \beta_1]g(z))' - \alpha \end{array} \right\} < \beta$$

where $\alpha \geq 0$, $1 < \beta \leq 2$.

For $p = q + 1$, $\alpha_2 = \beta_1, \ldots, \alpha_p = \beta_q$, $\alpha_1 = 1$, $\alpha = 0$ the class $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ reduces to the class $R_{\mathcal{H}}(\beta)$ studied in [3]. Further if the co-analytic part of $f = h + \bar{g}$ is zero that is $g \equiv 0$, the class $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ reduces to the class studied in [13].

In this paper extreme points, inclusion results and closure under integral operator for the class $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ are obtained.

## 2 Main Results

**Theorem 1.** *A function $f$ of the form (8) is in $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ if and only if*

$$\sum_{m=2}^{\infty} (\alpha + m)\Gamma_m |a_m| + \sum_{m=1}^{\infty} (\alpha + m)\Gamma_m |b_m| \leq \beta - 1 \tag{9}$$

*Proof.* Let $\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|+\sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|\leq\beta-1.$
It suffices to prove that

$$\left|\frac{\alpha\left(\dfrac{H_q^p[\alpha_1,\beta_1]h(z)+H_q^p[\alpha_1,\beta_1]g(z)}{z}\right)+(H_q^p[\alpha_1,\beta_1]h(z))'+(H_q^p[\alpha_1,\beta_1]g(z))'-\alpha-1}{\alpha\left(\dfrac{H_q^p[\alpha_1,\beta_1]h(z)+H_q^p[\alpha_1,\beta_1]g(z)}{z}\right)+(H_q^p[\alpha_1,\beta_1]h(z))'+(H_q^p[\alpha_1,\beta_1]g(z))'-\alpha-(2\beta-1)}\right|<1,\quad z\in U.$$

we have

$$\left|\frac{\alpha\left(\dfrac{H_q^p[\alpha_1,\beta_1]h(z)+H_q^p[\alpha_1,\beta_1]g(z)}{z}\right)+(H_q^p[\alpha_1,\beta_1]h(z))'+(H_q^p[\alpha_1,\beta_1]g(z))'-\alpha-1}{\alpha\left(\dfrac{H_q^p[\alpha_1,\beta_1]h(z)+H_q^p[\alpha_1,\beta_1]g(z)}{z}\right)+(H_q^p[\alpha_1,\beta_1]h(z))'+(H_q^p[\alpha_1,\beta_1]g(z))'-\alpha-(2\beta-1)}\right|$$
$$=\left|\frac{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1}+\sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1}}{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1}+\sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1}+1-(2\beta-1)}\right|$$
$$=\left|\frac{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1}+\sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1}}{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1}+\sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1}-2(\beta-1)}\right|$$

$$\leq \left| \frac{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1} + \sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1}}{2(\beta-1) - \sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1} - \sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1}} \right|$$

$$\leq \left| \frac{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m| + \sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|}{2(\beta-1) - \sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m| - \sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|} \right|$$

which is bounded above by 1, by hypothesis and the sufficient part is proved.
Conversely, suppose that

$$Re\left\{ \begin{array}{l} \alpha\left(\frac{H_q^p[\alpha_1,\beta_1]h(z) + H_q^p[\alpha_1,\beta_1]g(z)}{z}\right) \\ +(H_q^p[\alpha_1,\beta_1]h(z))' + (H_q^p[\alpha_1,\beta_1]g(z))' - \alpha \end{array} \right\} < \beta,$$

which is equivalent to

$$Re\left\{\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m|z^{m-1} + \sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m|z^{m-1} + 1\right\} < \beta.$$

The above condition must hold for all values of $z$, $|z| = r < 1$. Upon choosing the values of $z$ to be real and let $z \to 1^-$, we obtain

$$\sum_{m=2}^{\infty}(\alpha+m)\Gamma_m|a_m| + \sum_{m=1}^{\infty}(\alpha+m)\Gamma_m|b_m| \leq \beta - 1,$$

which gives the necessary part. This completes the proof of the theorem. □

We now determine the extreme points of the closed convex hulls of $R_{\mathcal{H}}([\alpha_1,\beta_1],\alpha,\beta)$ denoted by *clco* $R_{\mathcal{H}}([\alpha_1,\beta_1],\alpha,\beta)$.

**Theorem 2.** *A function* $f(z) \in clco\ R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ *if and only if*

$$f(z) = \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_m(z)) \tag{10}$$

*where* $h_1(z) = z$, $h_m(z) = z + \frac{\beta-1}{(\alpha+m)\Gamma_m} z^m$; $(m \geq 2)$, $g_m(z) = z + \frac{\beta-1}{(\alpha+m)\Gamma_m} z^{-m}$; $(m \geq 1)$ *and* $\sum_{m=1}^{\infty}(X_m + Y_m) = 1$, $X_m \geq 0$ *and* $Y_m \geq 0$. *In particular, the extreme points of* $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ *are* $\{h_m\}$ *and* $\{g_m\}$.

*Proof.* For functions $f$ of the form (10) write

$$\begin{aligned} f(z) &= \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_m(z)) \\ &= \sum_{m=1}^{\infty} (X_m + Y_m) z + \sum_{m=2}^{\infty} \frac{\beta-1}{(\alpha+m)\Gamma_m} X_m z^m + \sum_{m=1}^{\infty} \frac{\beta-1}{(\alpha+m)\Gamma_m} Y_m z^{-m} \\ &= z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} B_m z^{-m}, \end{aligned}$$

where

$$A_m = \frac{\beta-1}{(\alpha+m)\Gamma_m} X_m, \quad \text{and} \quad B_m = \frac{\beta-1}{(\alpha+m)\Gamma_m} Y_m$$

Therefore,

$$\begin{aligned} \sum_{m=2}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} A_m + \sum_{m=1}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} B_m \\ &= \sum_{m=2}^{\infty} X_m + \sum_{m=1}^{\infty} Y_m \\ &= 1 - X_1 \leq 1, \end{aligned}$$

and hence $f(z) \in clco\ R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$.

Conversely, suppose that $f(z) \in clco\ R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$.
Setting

$$X_m = \frac{(\alpha+m)\Gamma_m}{\beta-1} A_m; \quad (m \geq 2),$$
$$Y_m = \frac{(\alpha+m)\Gamma_m}{\beta-1} B_m; \quad m \geq 1$$

where $\sum_{m=1}^{\infty}(X_m + Y_m) = 1$. We have

$$\begin{aligned} f(z) &= z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} B_m z^{-m}, \quad A_m, B_m \geq 0 \\ &= z + \sum_{m=2}^{\infty} \frac{\beta-1}{(\alpha+m)\Gamma_m} X_m z^m + \sum_{m=1}^{\infty} \frac{\beta-1}{(\alpha+m)\Gamma_m} Y_m z^{-m} \\ &= z + \sum_{m=2}^{\infty}(h_m(z) - z) X_m + \sum_{m=1}^{\infty}(g_m(z) - z) Y_m \\ &= \sum_{m=1}^{\infty}(X_m h_m(z) + Y_m g_m(z)) \end{aligned}$$

as required. □

**Theorem 3.** *Each function in the class $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ maps a disk $U_r$ where $r < \inf_m \left\{ \frac{1}{m(\beta-1-(\alpha+1)|b_1|} \right\}^{\frac{1}{m-1}}$ onto convex domains for $\beta > 1 + (\alpha+1)|b_1|$.*

*Proof.* Let $f \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ and let $r$ be fixed, $0 < r < 1$. Then

$r^{-1}f(rz) \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ and we have

$$\begin{aligned}\sum_{m=2}^{\infty} m^2(|a_m| + |b_m|)r^{m-1} &= \sum_{m=2}^{\infty} m(|a_m| + |b_m|)(mr^{m-1}) \\ &\leq \sum_{m=2}^{\infty} m(|a_m| + |b_m|) \\ &\leq \sum_{m=2}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1}(|a_m| + |b_m|) \\ &\leq \beta - 1 - (\alpha + 1)|b_1| \\ &\leq 1\end{aligned}$$

provided

$$mr^{m-1} \leq \frac{1}{\beta - 1 - (\alpha + 1)|b_1|}$$

or

$$r < \inf_{m} \left\{ \frac{1}{m(\beta - 1 - (\alpha + 1)|b_1|} \right\}^{\frac{1}{m-1}}.$$

This completes the proof of theorem 3. □

For our next theorem, we need to define the convolution of two harmonic functions.

For harmonic functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} |a_m| z^m + \sum_{m=1}^{\infty} |b_m| z^{-m}$$

and

$$F(z) = z + \sum_{m=2}^{\infty} |A_m| z^m + \sum_{m=1}^{\infty} |B_m| z^{-m},$$

we define their convolution

$$(f * F)(z) = f(z) * F(z) = z + \sum_{m=2}^{\infty} |a_m A_m| z^m + \sum_{m=1}^{\infty} |b_m B_m| z^{-m} \tag{11}$$

Using this definition, we show that the class $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ is closed under convolution.

**Theorem 4.** *For* $1 < \beta \leq \delta \leq 2$, *let* $f \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \delta)$ *and* $F \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$. *Then* $f * F \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta) \subseteq R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \delta)$.

*Proof.* Let $f(z) = z + \sum_{m=2}^{\infty} |a_m| z^m + \sum_{m=1}^{\infty} |b_m| z^{-m} \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \delta)$ and $F(z) = z + \sum_{m=2}^{\infty} |A_m| z^m + \sum_{m=1}^{\infty} |B_m| z^{-m} \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$.
The convolution $(f * F)$ is given by (11).
We note that, for $F \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \delta)$, $|A_m| \leq 1$ and $|B_m| \leq 1$. Now we have

$$\begin{aligned}
&\sum_{m=2}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} |a_m||A_m| + \sum_{m=1}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} |b_m||B_m| \\
&\qquad \leq \sum_{m=2}^{\infty} \frac{(\alpha+m)}{\beta-1} |a_m| + \sum_{m=1}^{\infty} \frac{(\alpha+m)}{\beta-1} |b_m| \\
&\qquad \leq 1, \quad (f \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)
\end{aligned}$$

Therefore $f * F \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta) \subseteq R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \delta)$. □

Next, we show that $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ is closed under convex combinations of its members.

**Theorem 5.** *The class* $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ *is closed under convex combination.*

*Proof.* For $i = 1, 2, 3, \ldots$, let $f_i \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$, where

$$f_i(z) = z + \sum_{m=2}^{\infty} |a_{m,i}| z^m + \sum_{m=1}^{\infty} |b_{m,i}| z^{-m}.$$

Then by theo 1, we have

$$\sum_{m=2}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} |a_{m,i}| + \sum_{m=1}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} |b_{m,i}| \leq 1.$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of $f_i$ may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{m=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{m,i}| \right) z^m + \sum_{m=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{m,i}| \right) z^{-m}$$

Then by theo 1, we have

$$\begin{aligned}
&\sum_{m=2}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} \left( \sum_{i=1}^{\infty} t_i |a_{m,i}| \right) + \sum_{m=1}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} \left( \sum_{i=1}^{\infty} t_i |b_{m,i}| \right) \\
&\quad = \sum_{i=1}^{\infty} t_i \left( \sum_{m=2}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} |a_{m,i}| + \sum_{m=1}^{\infty} \frac{(\alpha+m)\Gamma_m}{\beta-1} |b_{m,i}| \right) \\
&\quad \leq \sum_{i=1}^{\infty} t_i = 1.
\end{aligned}$$

Therefore, $\sum_{i=1}^{\infty} t_i f_i(z) \in R_H([\alpha_1, \beta_1], \alpha, \beta)$. □

Following Ruscheweyh [9], the $\delta$-neighborhood of $f$ is the set

$$N_\delta(f) = \left\{ F : F(z) = z + \sum_{m=2}^{\infty} |A_m| z^m + \sum_{m=1}^{\infty} |B_m| \bar{z}^m \text{ and } \right.$$
$$\left. \sum_{m=2}^{\infty} m(|a_m - A_m| + |b_m - B_m| + |b_1 - B_1| \leq \delta \right\}$$

**Theorem 6.** *Let $f \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ and $\delta = \beta - 1 - \alpha |b_1|$. Then $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta) \subset N_\delta(I)$, where $I$ is the identity function $I(z) = z$.*

*Proof.* Let $f \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$.

We have

$$\begin{aligned}
&|b_1| + \sum_{m=2}^{\infty} m(|a_m| + |b_m|) \\
&\qquad \le |b_1| + \sum_{m=2}^{\infty} (\alpha + m)\Gamma_m(|a_m| + |b_m|) \\
&\qquad \le |b_1| + \beta + \alpha - (1+\alpha)(1+|b_1|) \\
&\qquad = \beta - 1 - \alpha|b_1|.
\end{aligned}$$

Hence $f(z) \in N_\delta(I)$. □

# 3 Integral Operator

Now, we examine a closure property of the class $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ under the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ which is defined by

$$L_c(f) = \frac{c+1}{Z^c} \int_0^z t^{c-1} f(t)dt, \quad c > -1.$$

**Theorem 7.** *Let $f(z) \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$. Then $L_c(f(z)) \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$.*

*Proof.* From the representation of $L_c(f(z))$, it follows that

$$\begin{aligned}
L_c(f) &= \frac{c+1}{Z^c} \int_0^z t^{c-1}(h(t) + \overline{g(t)})dt \\
&= \frac{c+1}{Z^c} \left( \int_0^z t^{c-1} \left( t + \sum_{m=2}^{\infty} a_m t^m \right) dt + \overline{\int_0^z t^{c-1} \left( \sum_{m=1}^{\infty} b_m t^m \right) dt} \right) \\
&= z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} B_m z^m,
\end{aligned}$$

where, $A_m = \dfrac{c+1}{c+n} a_m$, $B_m = \dfrac{c+1}{c+n} b_m$.

Therefore,

$$\begin{aligned}\sum_{m=1}^{\infty}&\left(\frac{(\alpha+m)}{\beta-1}\left(\frac{c+1}{c+n}\right)|a_m|+\frac{(\alpha+m)}{\beta-1}\left(\frac{c+1}{c+n}\right)|b_m|\right)\Gamma_m\\ &\leq\sum_{m=1}^{\infty}\left(\frac{(\alpha+m)}{\beta-1}|a_m|+\frac{(\alpha+m)}{\beta-1}|b_m|\right)\Gamma_m\\ &\leq 1,\end{aligned}$$

since $f(z)\in R_{\mathcal{H}}([\alpha_1,\beta_1],\alpha,\beta)$, therefore by theo 1,
$L_c(f(z))\in R_{\mathcal{H}}([\alpha_1,\beta_1],\alpha,\beta)$. □

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