A Subclass of Harmonic Univalent Functions with Positive Coefficients defined by Dziok-Srivastava Operator

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Abstract

In this paper using the Dziok-Srivastava [4] operator, we introduce a subclass of the class $\mathcal{H}$ of complex valued Harmonic univalent functions $f = h + \bar{g}$, where $h$ is the analytic part and $g$ is the co-analytic part of $f$ in $|z| < 1$. Coefficient bounds, extreme points, inclusion results and closure under integral operator for this class are obtained.

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1 Introduction

Harmonic mappings have found applications in many diverse fields such as engineering, aerodynamics and other branches of applied Mathematics. Harmonic mappings in a domain $D \subseteq \mathbb{C}$ are univalent complex-valued harmonic functions $f = u + iv$ where both $u$ and $v$ are real harmonic.

The important work of Clunie and Sheil-Small [2] on the class consisting of complex-valued harmonic orientation preserving univalent functions $f$ defined on the open unit disk $U$ formed the basis for several investigations on different subclasses of harmonic univalent functions.

In any simply-connected domain $D$, it is known that [2] we can write $f = h + \bar{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and orientation preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$ [2].

Denote by $\mathcal{H}$ the family of harmonic functions

\begin{equation}
(1) \quad f = h + \bar{g}
\end{equation}

which are univalent and orientation preserving in the open unit disk $U = \{z : |z| < 1\}$ and $f$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \bar{g} \in \mathcal{H}$ the analytic functions $h$ and $g$ are given by

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=1}^{\infty} b_m z^m.$$  

Hence

\begin{equation}
(2) \quad f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=1}^{\infty} b_m z^m, \quad |b_1| < 1
\end{equation}

We note that the family $\mathcal{H}$ reduces to the well known class $S$ of normalized univalent functions if the co-analytic part of $f$ is identically zero, that is $g \equiv 0$. 

For complex numbers $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ ($\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, q$) the generalized hypergeometric function $[8] \ _pF_q(z)$ is defined by

$$pF_q(z) \equiv pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \cdots (\alpha_p)_m}{(\beta_1)_m \cdots (\beta_q)_m} \frac{z^m}{m!}$$

($p \leq q + 1; p, q \in N_0 = N \cup \{0\}; z \in U$),

where $N$ denotes the set of all positive integers and $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m = \begin{cases} 
1, & m = 0, \\
(a+1)(a+2)\ldots(a+m-1), & m \in N.
\end{cases}$$

Dziok and Srivastava [4] introduced an operator in their study of analytic functions associated with generalized hypergeometric functions. This Dziok-Srivastava operator is known to include many well-known operators as special cases.

Let $H(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q) : A \to A$ be a linear operator defined by

$$[(H(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q))(\phi)](z) = z \ _pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) \ast \phi(z)$$

$$= z + \sum_{m=2}^{\infty} \Gamma_m a_m z^m$$

where

$$\Gamma_m = \frac{(\alpha_1)_{m-1} \cdots (\alpha_p)_{m-1}}{(\beta_1)_{m-1} \cdots (\beta_q)_{m-1}} \frac{1}{m - 1!}$$

and $\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q$ are positive real numbers, such that $p \leq q + 1$; $p, q \in N \cup \{0\}$, and $(a)_m$ is the familiar Pochhammer symbol.

The linear operator $H(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q)$ or $H_p^q[\alpha_1, \beta_1]$ in short, is the Dziok-Srivastava operator ([4] & [12]), which includes several well known operators.
The Dziok-Srivastava operator when extended to the harmonic function $f = h + \bar{g}$ is defined by

$$H^p_q[\alpha_1, \beta_1]f(z) = H^p_q[\alpha_1, \beta_1]h(z) + \overline{H^p_q[\alpha_1, \beta_1]g(z)}$$

Denote by $V_H$ the subclass of $\mathcal{H}$ consisting of functions of the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{m=2}^{\infty} |a_m| z^m, \quad g(z) = \sum_{m=1}^{\infty} |b_m| z^m, \quad |b_1| < 1$$

Motivated by earlier works of [1, 3, 6, 7, 10, 11] on harmonic functions, we introduce here a new subclass $R_H([\alpha_1, \beta_1], \alpha, \beta)$ of $V_H$ using Dziok-Srivastava operator extended to harmonic functions.

We denote by $R_H([\alpha_1, \beta_1], \alpha, \beta)$, the subclass of $V_H$, consisting of functions of the form (8) satisfying the condition

$$\text{Re} \left\{ \alpha \left( \frac{H^p_q[\alpha_1, \beta_1]h(z) + H^p_q[\alpha_1, \beta_1]g(z)}{z} \right) + (H^p_q[\alpha_1, \beta_1]h(z))' + (H^p_q[\alpha_1, \beta_1]g(z))' - \alpha \right\} < \beta$$

where $\alpha \geq 0, 1 < \beta \leq 2$.

For $p = q + 1$, $\alpha_2 = \beta_1, \ldots, \alpha_p = \beta_q, \alpha_1 = 1, \alpha = 0$ the class $R_H([\alpha_1, \beta_1], \alpha, \beta)$ reduces to the class $R_H(\beta)$ studied in [3]. Further if the co-analytic part of $f = h + \bar{g}$ is zero that is $g \equiv 0$, the class $R_H([\alpha_1, \beta_1], \alpha, \beta)$ reduces to the class studied in [13].

In this paper extreme points, inclusion results and closure under integral operator for the class $R_H([\alpha_1, \beta_1], \alpha, \beta)$ are obtained.

## 2 Main Results

**Theorem 1.** A function $f$ of the form (8) is in $R_H([\alpha_1, \beta_1], \alpha, \beta)$ if and only if

$$\sum_{m=2}^{\infty} (\alpha + m) \Gamma_m |a_m| + \sum_{m=1}^{\infty} (\alpha + m) \Gamma_m |b_m| \leq \beta - 1$$
Proof. Let \( \sum_{m=2}^{\infty} (\alpha + m)\Gamma_m |a_m| + \sum_{m=1}^{\infty} (\alpha + m)\Gamma_m |b_m| \leq \beta - 1 \).

It suffices to prove that

\[
\begin{align*}
\alpha \left( \frac{H^p_q[\alpha_1, \beta_1]h(z) + H^p_q[\alpha_1, \beta_1]g(z)}{z} \right) \\
+ (H^p_q[\alpha_1, \beta_1]h(z))' + (H^p_q[\alpha_1, \beta_1]g(z))' - \alpha - 1 \\
\leq 1, \quad z \in U.
\end{align*}
\]

we have

\[
\begin{align*}
\frac{\alpha \left( \frac{H^p_q[\alpha_1, \beta_1]h(z) + H^p_q[\alpha_1, \beta_1]g(z)}{z} \right) }{\alpha \left( \frac{H^p_q[\alpha_1, \beta_1]h(z) + H^p_q[\alpha_1, \beta_1]g(z)}{z} \right) } \\
+ (H^p_q[\alpha_1, \beta_1]h(z))' + (H^p_q[\alpha_1, \beta_1]g(z))' - \alpha - 1 \\
\leq \frac{\sum_{m=2}^{\infty} (\alpha + m)\Gamma_m |a_m| z^{m-1} + \sum_{m=1}^{\infty} (\alpha + m)\Gamma_m |b_m| z^{m-1} }{\sum_{m=2}^{\infty} (\alpha + m)\Gamma_m |a_m| z^{m-1} + \sum_{m=1}^{\infty} (\alpha + m)\Gamma_m |b_m| z^{m-1} - (2\beta - 1) } \\
= \frac{\sum_{m=2}^{\infty} (\alpha + m)\Gamma_m |a_m| z^{m-1} + \sum_{m=1}^{\infty} (\alpha + m)\Gamma_m |b_m| z^{m-1} - (2\beta - 1) }{\sum_{m=2}^{\infty} (\alpha + m)\Gamma_m |a_m| z^{m-1} + \sum_{m=1}^{\infty} (\alpha + m)\Gamma_m |b_m| z^{m-1} - 2(\beta - 1) } \\
\leq 1.
\end{align*}
\]
which is bounded above by 1, by hypothesis and the sufficient part is proved.

Conversely, suppose that

\[
\Re \left\{ \sum_{m=2}^{\infty} (\alpha + m) \Gamma_m |a_m| z^{m-1} + \sum_{m=1}^{\infty} (\alpha + m) \Gamma_m |b_m| z^{m-1} \right\} \leq 2(\beta - 1) - \sum_{m=2}^{\infty} (\alpha + m) \Gamma_m |a_m| z^{m-1} - \sum_{m=1}^{\infty} (\alpha + m) \Gamma_m |b_m| z^{m-1}
\]

which is equivalent to

\[
\Re \left\{ \sum_{m=2}^{\infty} (\alpha + m) \Gamma_m |a_m| + \sum_{m=1}^{\infty} (\alpha + m) \Gamma_m |b_m| \right\} \leq 2(\beta - 1) - \sum_{m=2}^{\infty} (\alpha + m) \Gamma_m |a_m| - \sum_{m=1}^{\infty} (\alpha + m) \Gamma_m |b_m|
\]

which is bounded above by 1, by hypothesis and the sufficient part is proved.

Conversely, suppose that

\[
\Re \left\{ \sum_{m=2}^{\infty} (\alpha + m) \Gamma_m |a_m| z^{m-1} + \sum_{m=1}^{\infty} (\alpha + m) \Gamma_m |b_m| z^{m-1} \right\} < \beta,
\]

which is equivalent to

\[
\Re \left\{ \sum_{m=2}^{\infty} (\alpha + m) \Gamma_m |a_m| + \sum_{m=1}^{\infty} (\alpha + m) \Gamma_m |b_m| \right\} < \beta.
\]

The above condition must hold for all values of \(z, |z| = r < 1\). Upon choosing the values of \(z\) to be real and let \(z \to 1^-\), we obtain

\[
\sum_{m=2}^{\infty} (\alpha + m) \Gamma_m |a_m| + \sum_{m=1}^{\infty} (\alpha + m) \Gamma_m |b_m| \leq \beta - 1,
\]

which gives the necessary part. This completes the proof of the theorem.

\[\square\]

We now determine the extreme points of the closed convex hulls of \(R_H([\alpha_1, \beta_1], \alpha, \beta)\) denoted by \( \text{clco} \ R_H([\alpha_1, \beta_1], \alpha, \beta) \).
Theorem 2. A function $f(z) \in \text{clco } R_H([\alpha_1, \beta_1], \alpha, \beta)$ if and only if

$$f(z) = \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_m(z))$$

where $h_1(z) = z$, $h_m(z) = z + \frac{\beta - 1}{(\alpha + m) \Gamma_m} z^m$; $(m \geq 2)$, $g_m(z) = z + \frac{\beta - 1}{(\alpha + m) \Gamma_m} z^{-m}$; $(m \geq 1)$ and $\sum_{m=1}^{\infty} (X_m + Y_m) = 1$, $X_m \geq 0$ and $Y_m \geq 0$. In particular, the extreme points of $R_H([\alpha_1, \beta_1], \alpha, \beta)$ are $\{h_m\}$ and $\{g_m\}$.

Proof. For functions $f$ of the form (10) write

$$f(z) = \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_m(z))$$

$$= \sum_{m=1}^{\infty} (X_m + Y_m) z + \sum_{m=2}^{\infty} \frac{\beta - 1}{(\alpha + m) \Gamma_m} X_m z^m + \sum_{m=1}^{\infty} \frac{\beta - 1}{(\alpha + m) \Gamma_m} Y_m z^{-m}$$

$$= z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} B_m z^{-m},$$

where

$$A_m = \frac{\beta - 1}{(\alpha + m) \Gamma_m} X_m, \quad \text{and} \quad B_m = \frac{\beta - 1}{(\alpha + m) \Gamma_m} Y_m.$$

Therefore,

$$\sum_{m=2}^{\infty} \frac{(\alpha + m) \Gamma_m}{\beta - 1} A_m + \sum_{m=1}^{\infty} \frac{(\alpha + m) \Gamma_m}{\beta - 1} B_m$$

$$= \sum_{m=2}^{\infty} X_m + \sum_{m=1}^{\infty} Y_m$$

$$= 1 - X_1 \leq 1,$$

and hence $f(z) \in \text{clco } R_H([\alpha_1, \beta_1], \alpha, \beta)$. 
Conversely, suppose that $f(z) \in \text{clco } R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$. Setting

$$X_m = \frac{(\alpha + m)\Gamma_m}{\beta - 1} A_m; \quad (m \geq 2),$$

$$Y_m = \frac{(\alpha + m)\Gamma_m}{\beta - 1} B_m; \quad m \geq 1$$

where $\sum_{m=1}^{\infty} (X_m + Y_m) = 1$. We have

$$f(z) = z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} B_m z^{-m}, \quad A_m, B_m \geq 0$$

$$= z + \sum_{m=2}^{\infty} \frac{\beta - 1}{(\alpha + m)\Gamma_m} X_m z^m + \sum_{m=1}^{\infty} \frac{\beta - 1}{(\alpha + m)\Gamma_m} Y_m z^{-m}$$

$$= z + \sum_{m=2}^{\infty} (h_m(z) - z) X_m + \sum_{m=1}^{\infty} (g_m(z) - z) Y_m$$

$$= \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_m(z))$$

as required. \(\square\)

**Theorem 3.** Each function in the class $R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ maps a disk $U_r$ where $r < \inf \left\{ \frac{1}{m(\beta - 1 - (\alpha + 1)|b_1|)} \right\}^{\frac{1}{m-1}}$ onto convex domains for $\beta > 1 + (\alpha + 1)|b_1|$.

**Proof.** Let $f \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)$ and let $r$ be fixed, $0 < r < 1$. Then
\( r^{-1} f(rz) \in R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta) \) and we have

\[
\begin{align*}
\sum_{m=2}^{\infty} m^2(|a_m| + |b_m|) & = \sum_{m=2}^{\infty} m(|a_m| + |b_m|)(mr^{m-1}) \\
\leq & \sum_{m=2}^{\infty} m(|a_m| + |b_m|) \\
\leq & \sum_{m=2}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1}(|a_m| + |b_m|) \\
\leq & \beta - 1 - (\alpha + 1)|b_1| \\
\leq & 1
\end{align*}
\]

provided

\[
mr^{m-1} \leq \frac{1}{\beta - 1 - (\alpha + 1)|b_1|}
\]

or

\[
r < \inf_m \left\{ \frac{1}{m(\beta - 1 - (\alpha + 1)|b_1|)} \right\}^{1/m-1}.
\]

This completes the proof of theorem 3. \(\square\)

For our next theorem, we need to define the convolution of two harmonic functions.

For harmonic functions of the form

\[
f(z) = z + \sum_{m=2}^{\infty} |a_m|z^m + \sum_{m=1}^{\infty} |b_m|z^{-m}
\]

and

\[
F(z) = z + \sum_{m=2}^{\infty} |A_m|z^m + \sum_{m=1}^{\infty} |B_m|z^{-m},
\]

we define their convolution

\[
(11) \quad (f * F)(z) = f(z) * F(z) = z + \sum_{m=2}^{\infty} |a_mA_m|z^m + \sum_{m=1}^{\infty} |b_mB_m|z^{-m}
\]

Using this definition, we show that the class \(R_{\mathcal{H}}([\alpha_1, \beta_1], \alpha, \beta)\) is closed under convolution.
Theorem 4. For $1 < \beta \leq \delta \leq 2$, let $f \in R_H([\alpha, \beta_1], \alpha, \delta)$ and $F \in R_H([\alpha_1, \beta_1], \alpha, \beta)$. Then $f\ast F \in R_H([\alpha_1, \beta_1], \alpha, \beta) \subseteq R_H([\alpha_1, \beta_1], \alpha, \delta)$.

Proof. Let $f(z) = z + \sum_{m=2}^{\infty} |a_m|z^m + \sum_{m=1}^{\infty} |b_m|z^{-m} \in R_H([\alpha_1, \beta_1], \alpha, \delta)$ and $F(z) = z + \sum_{m=2}^{\infty} |A_m|z^m + \sum_{m=1}^{\infty} |B_m|z^{-m} \in R_H([\alpha_1, \beta_1], \alpha, \beta)$.

The convolution $(f \ast F)$ is given by (11).

We note that, for $F \in R_H([\alpha_1, \beta_1], \alpha, \delta)$, $|A_m| \leq 1$ and $|B_m| \leq 1$. Now we have

\[
\sum_{m=2}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} |a_m| |A_m| + \sum_{m=1}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} |b_m| |B_m| \\
\leq \sum_{m=2}^{\infty} \frac{(\alpha + m)}{\beta - 1} |a_m| + \sum_{m=1}^{\infty} \frac{(\alpha + m)}{\beta - 1} |b_m| \\
\leq 1, \quad (f \in R_H([\alpha_1, \beta_1], \alpha, \beta)
\]

Therefore $f \ast F \in R_H([\alpha_1, \beta_1], \alpha, \beta) \subseteq R_H([\alpha_1, \beta_1], \alpha, \delta)$. 

Next, we show that $R_H([\alpha_1, \beta_1], \alpha, \beta)$ is closed under convex combinations of its members.

Theorem 5. The class $R_H([\alpha_1, \beta_1], \alpha, \beta)$ is closed under convex combination.

Proof. For $i = 1, 2, 3, \ldots$, let $f_i \in R_H([\alpha_1, \beta_1], \alpha, \beta)$, where

\[
f_i(z) = z + \sum_{m=2}^{\infty} |a_{m,i}|z^m + \sum_{m=1}^{\infty} |b_{m,i}|z^{-m}.
\]

Then by theo 1, we have

\[
\sum_{m=2}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} |a_{m,i}| + \sum_{m=1}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} |b_{m,i}| \leq 1.
\]
For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of $f_i$ may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{m=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i a_{m,i} \right) z^m + \sum_{m=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i b_{m,i} \right) z^{-m}$$

Then by theo 1, we have

$$\sum_{m=2}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} \left( \sum_{i=1}^{\infty} t_i a_{m,i} \right) + \sum_{m=1}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} \left( \sum_{i=1}^{\infty} t_i b_{m,i} \right)$$

$$= \sum_{i=1}^{\infty} t_i \left( \sum_{m=2}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} |a_{m,i}| + \sum_{m=1}^{\infty} \frac{(\alpha + m)\Gamma_m}{\beta - 1} |b_{m,i}| \right)$$

$$\leq \sum_{i=1}^{\infty} t_i = 1.$$ 

Therefore, $\sum_{i=1}^{\infty} t_i f_i(z) \in R_H([\alpha_1, \beta_1], \alpha, \beta)$. \hfill \Box

Following Ruscheweyh [9], the $\delta$-neighborhood of $f$ is the set

$$N_\delta(f) = \left\{ F : F(z) = z + \sum_{m=2}^{\infty} |A_m|z^m + \sum_{m=1}^{\infty} |B_m|z^m \text{ and } \sum_{m=2}^{\infty} m(|a_m - A_m| + |b_m - B_m| + |b_1 - B_1| \leq \delta \right\}$$

**Theorem 6.** Let $f \in R_H([\alpha_1, \beta_1], \alpha, \beta)$ and $\delta = \beta - 1 - \alpha |b_1|$. Then $R_H([\alpha_1, \beta_1], \alpha, \beta) \subset N_\delta(I)$, where $I$ is the identity function $I(z) = z$.

**Proof.** Let $f \in R_H([\alpha_1, \beta_1], \alpha, \beta)$. 

We have
\[ |b_1| + \sum_{m=2}^{\infty} m(|a_m| + |b_m|) \]
\[ \leq |b_1| + \sum_{m=2}^{\infty} (\alpha + m) \Gamma_m (|a_m| + |b_m|) \]
\[ \leq |b_1| + \beta + \alpha - (1 + \alpha)(1 + |b_1|) \]
\[ = \beta - 1 - \alpha |b_1|. \]

Hence \( f(z) \in N_\delta(I) \).

\[ \square \]

3 Integral Operator

Now, we examine a closure property of the class \( R_H([\alpha_1, \beta_1], \alpha, \beta) \) under the generalized Bernardi-Libera-Livingston integral operator \( L_c(f) \) which is defined by

\[ L_c(f) = \frac{c + 1}{Z^c} \int_0^z t^{c-1} f(t) dt, \quad c > -1. \]

**Theorem 7.** Let \( f(z) \in R_H([\alpha_1, \beta_1], \alpha, \beta) \). Then \( L_c(f(z)) \in R_H([\alpha_1, \beta_1], \alpha, \beta) \).

**Proof.** From the representation of \( L_c(f(z)) \), it follows that

\[ L_c(f) = \frac{c + 1}{Z^c} \int_0^z t^{c-1} (h(t) + g(t)) dt \]
\[ = \frac{c + 1}{Z^c} \left( \int_0^z t^{c-1} \left( t + \sum_{m=2}^{\infty} a_m t^m \right) dt + \int_0^z t^{c-1} \left( \sum_{m=1}^{\infty} b_m t^m \right) dt \right) \]
\[ = z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} B_m z^m, \]

where, \( A_m = \frac{c + 1}{c + n} a_m, \) \( B_m = \frac{c + 1}{c + n} b_m. \)
Therefore,
\[ \sum_{m=1}^{\infty} \left( \frac{\alpha + m}{\beta - 1} \right) \left( \frac{c + 1}{c + n} \right) |a_m| + \left( \frac{\alpha + m}{\beta - 1} \right) \left( \frac{c + 1}{c + n} \right) |b_m| \right) \Gamma_m \]
\[ \leq \sum_{m=1}^{\infty} \left( \frac{\alpha + m}{\beta - 1} |a_m| + \frac{\alpha + m}{\beta - 1} |b_m| \right) \Gamma_m \]
\[ \leq 1, \]

since \( f(z) \in R_H([\alpha_1, \beta_1], \alpha, \beta) \), therefore by theo 1,
\[ L_c(f(z)) \in R_H([\alpha_1, \beta_1], \alpha, \beta). \]

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