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ON CERTAIN SUBCLASS OF ANALYTIC AND UNIVALENT FUNCTIONS BASED ON RUSCHEWEYH DERIVATIVES AND HADAMARD PRODUCT

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Abstract

Let S denote the class of functions $f(z)$ analytic and univalent in the unit disc $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. In this paper we introduce a new subclass of S based on Ruscheweyh derivative and Hadamard product. Coefficient estimates, extreme points, distortion theorem, closure theorem, radius of starlikeness and convexity, radii of close-to-convexity, inclusion property and integral operators are determined for functions in this subclass.

1. Introduction

Let S denote the class of function $f(z)$ analytic and univalent in the unit disc $\Delta = \{z/|z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. The Hadamard product of two functions $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ and $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$ in S is given by

$$(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m. \tag{1}$$

Key Words : *Ruscheweyh, Derivatives, Hadamard product.*

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Let $D^{\alpha} f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z)$, $\alpha \geq -1$. Ruscheweyh [5] observed that $D^n f(z) = \frac{z(z^{n-1} f(z))^n}{n!}$ when $n \in N_0 = \{0, 1, 2, \ldots\}$. This symbol $D^n f(z)$, $n \in N_0$, was called the n^{th} Ruscheweyh derivative of $f(z)$.

Several subclasses of S have been introduced and studied by using either the Hadamard product or Ruscheweyh derivatives by many authors [1, 2, 3, 4, 6, 9].

Definition 1.1 : A function f of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{2}$$

is said to be in $S_n(\phi, \psi, \alpha, \lambda)$, $0 \leq \alpha < 1$, $0 \leq \lambda < 1$ if

$$\Re \left\{ \frac{\frac{D^{n+1}(f*\phi)(z)}{D^n(f*\psi)(z)}}{\lambda \frac{D^{n+1}(f*\phi)(z)}{D^n(f*\psi)(z)} + (1-\lambda)} \right\} > \alpha \tag{3}$$

where $\phi(z) = z + \sum_{m=2}^{\infty} \lambda_m z^m$; $\psi(z) = z + \sum_{m=2}^{\infty} \mu_m z^m$; $\lambda_m \geq 0, \mu_m \geq 0$, $\lambda_m \geq \mu_m$; $m = 2, 3, \ldots$ and $(f * \psi)(z) \neq 0$.

Let T denote the subclass of S consisting of functions of the form $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geq 0$ and let $TS_n(\phi, \psi, \alpha, \lambda) = S_n(\phi, \psi, \alpha, \lambda) \cap T$. The family $TS_n(\phi, \psi, \alpha, \lambda)$ is of special interest for it contain many well-known as well as new classes of T for suitable choices of $\phi(z), \psi(z), \alpha$ and λ. We provide necessary and sufficient coefficient condition, extreme points, distortion theorem, closure theorem, radius of starlikeness and convexity, radii of close-to-convexity, inclusion property and integral operators for functions in $TS_n(\phi, \psi, \alpha, \lambda)$.

2. Coefficient Inequalities

In this section, we find a necessary and sufficient condition for a functions to be in $TS_n(\phi, \psi, \alpha, \lambda)$ and consequently calculate coefficient estimates for functions in $TS_n(\phi, \psi, \alpha, \lambda)$.

Theorem 2.1 : A function $f(z) = z - \sum_{m=2}^{\infty} a_m z^m \in TS_n(\phi, \psi, \alpha, \lambda)$ if and only if

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!} K_m |a_m| \leq 1 - \alpha, \ n \in N_0 \tag{4}$$

where $K_m = (1-\alpha\lambda)(m+n)\lambda_m - \alpha(1-\lambda)(n+1)\mu_m$, ϕ and ψ are as given in Definition 1.1, and $0 \leq \alpha < 1$, $0 \leq \lambda < 1$.

Proof : Assume that $f(z) \in TS_n(\phi, \psi, \alpha, \lambda)$. Then

$$\begin{aligned}
&\Re\left\{\frac{\frac{D^{n+1}(f*\phi)(z)}{D^n(f*\psi)(z)}}{\lambda\frac{D^{n+1}(f*\phi)(z)}{D^n(f*\psi)(z)} + (1-\lambda)}\right\} \\
&= \Re\left\{\frac{D^{n+1}(f*\phi)(z)}{\lambda D^{n+1}(f*\phi)(z) + (1-\lambda)D^n(f*\psi)(z)}\right\} \\
&= \Re\left\{\frac{1 - \sum\limits_{m=2}^{\infty}\frac{(m+n)!}{(m-1)!(n+1)!}\lambda_m a_m z^{m-1}}{1 - \sum\limits_{m=2}^{\infty}\frac{(m+n-1)!}{(m-1)!(n+1)!}\left[\lambda(m+n)\lambda_m + (1-\lambda)(n+1)\mu_m\right]a_m z^{m-1}}\right\} \\
&> \alpha,\ z \in \Delta. \qquad (5)
\end{aligned}$$

Let $z \to 1-$ through real values, from (5), we obtain

$$\begin{aligned}
&1 - \sum_{m=2}^{\infty}\frac{(m+n)!}{(m-1)!(n+1)!}\lambda_m a_m \\
&\quad > \alpha - \alpha\sum_{m=2}^{\infty}\frac{(m+n-1)!}{(m-1)!(n+1)!}\left[\lambda(m+n)\lambda_m + (1-\lambda)(n+1)\mu_m\right]a_m \\
\text{or} \quad &\sum_{m=2}^{\infty}\frac{(m+n-1)!}{(m-1)!(n+1)!}K_m a_m \leq 1-\alpha,
\end{aligned}$$

where $K_m = (1-\alpha\lambda)(m+n)\lambda_m - \alpha(1-\lambda)(n+1)\mu_m$.

Conversely, assume that (4) holds.

Then we have,

$$\begin{aligned}
&\left|\frac{\frac{D^{n+1}(f*\phi)(z)}{D^n(f*\psi)(z)}}{\lambda\frac{D^{n+1}(f*\phi)(z)}{D^n(f*\psi)(z)} + (1-\lambda)} - 1\right| \\
&= \left|\frac{D^{n+1}(f*\phi)(z)}{\lambda D^{n+1}(f*\phi)(z) + (1-\lambda)D^n(f*\psi)(z)} - 1\right| \\
&= \left|\frac{(1-\lambda)D^{n+1}(f*\phi)(z) - (1-\lambda)D^n(f*\psi)(z)}{\lambda D^{n+1}(f*\phi)(z) + (1-\lambda)D^n(f*\psi)(z)}\right| \\
&\leq \frac{(1-\lambda)\sum_{m=2}^{\infty}\frac{(m+n-1)!}{(m-1)!(n+1)!}\left[(m+n)\lambda_m - (n+1)\mu_m\right]|a_m|}{1 - \sum_{m=2}^{\infty}\frac{(m+n-1)!}{(m-1)!(n+1)!}\left[\lambda(m+n)\lambda_m + (1-\lambda)(n+1)\mu_m\right]|a_m|}.
\end{aligned}$$

This shows that the value of $\frac{\frac{D^{n+1}(f*\phi)(z)}{D^n(f*\psi)(z)}}{\lambda\frac{D^{n+1}(f*\phi)(z)}{D^n(f*\psi)(z)}+(1-\lambda)}$ lies in a circle centered at $w = 1$ whose radius is $1-\alpha$. This implies that $f(z) \in TS_n(\phi, \psi, \alpha, \lambda)$.

Corollary 2.1 : If $f \in TS_n(\phi, \psi, \alpha, \lambda)$, then $a_m \leq \frac{(m-1)!(n+1)!(1-\alpha)}{(m+n-1)!K_m}$, $m = 2, 3, \ldots$ and $n \in N_0$. The equality holds, for each m, for functions of the form $f_m(z) = z - \frac{(m-1)!(n+1)!(1-\alpha)}{(m+n-1)!K_m} z^m$, $z \in \Delta$.

Remark 2.1 :

1. For $0 \leq \alpha < 1$, $\lambda = 0$,
   $TS_n(\phi, \psi, \alpha, \lambda) = S_p^n(\phi, \psi, \alpha, \beta)$ with $\beta = 0$ [9].

2. For $0 \leq \alpha < 1$, $\lambda = 0$, $\phi(z) = \psi(z) = \frac{z}{1-z}$ and $n = 0$,
   $TS_n(\phi, \psi, \frac{\alpha+1}{2}, \lambda) = TS_p(\alpha)$ [8].

3. For $0 \leq \alpha < 1$, $\lambda = 0$, $\phi(z) = \psi(z) = \frac{z}{1-z}$ and $n = 0$,
   $TS_n(\phi, \psi, \frac{\alpha+1}{2}, \lambda) = TS_p^g(\alpha)$ [10].

4. For $0 \leq \alpha < 1$, $\lambda = 0$, $n = 0$, $\phi(z) = \psi(z) = \frac{z}{1-z}$;
   $TS_n(\phi, \psi, \alpha, \lambda) = T^*(\alpha)$ [7].

3. Distortion Theorem

Theorem 3.1 : Let $f \in TS_n(\phi, \psi, \alpha, \lambda)$ and $K_m = (\lambda+1)(m+n)\lambda_m - (n+1)(\alpha+\lambda)\mu_m$, $m = 2, 3, \ldots$, then

$$r - \frac{1-\alpha}{\min\left\{\frac{(m+n-1)!}{(m-1)!(n+1)!}K_m\right\}} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{\min\left\{\frac{(m+n-1)!}{(m-1)!(n+1)!}K_m\right\}} r^2,$$

$|z| = r < 1$. The result is sharp for $f(z) = z + \frac{1-\alpha}{\min\left\{\frac{(m+n-1)!}{(m-1)!(n+1)!}K_m\right\}} r^2$.

Proof : Let $|z| = r$. For $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{m=2}^{\infty} |a_m||z|^m \\ &\leq r + \sum_{m=2}^{\infty} a_m r^m \end{aligned}$$

$$
\begin{aligned}
&\leq \quad r + \sum_{m=2}^{\infty} a_m r^2 \\
&\leq \quad r + r^2 \sum_{m=2}^{\infty} a_m .
\end{aligned}
$$

Since

$$\min\left\{\frac{(m+n-1)!}{(m-1)!(n+1)!}K_m\right\} \leq \frac{(m+n-1)!}{(m-1)!(n+1)!}K_m$$

we have

$$\min\left\{\frac{(m+n-1)!}{(m-1)!(n+1)!}K_m\right\}\sum_{m=2}^{\infty} a_m \leq \sum_{m=2}^{\infty}\frac{(m+n-1)!}{(m-1)!(n+1)!} \leq 1-\alpha$$

or

$$\sum_{m=2}^{\infty} a_m \leq \frac{1-\alpha}{\min\left\{\frac{(m+n-1)!}{(m-1)!(n+1)!}K_m\right\}}$$

This gives $|f(z)| \leq r + \frac{1-\alpha}{\min\left\{\frac{(m+n-1)!}{(m-1)!(n+1)!}K_m\right\}} r^2$.

Similarly, we have $|f(z)| \geq r - \frac{1-\alpha}{\min\left\{\frac{(m+n-1)!}{(m-1)!(n+1)!}K_m\right\}} r^2$.

4. Closure Theorem

Theorem 4.1 : The class $TS_n(\phi,\psi,\alpha,\lambda)$ is closed under convex linear combination.

Proof : Let $f,\ g \in\ TS_n(\phi,\psi,\alpha,\lambda)$ and let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $a_m \geq 0$, $b_m \geq 0$. For μ such that $0 \leq \mu \leq 1$, it is sufficient to show that the function h, defined by $h(z) = (1-\mu)f(z) + \mu g(z)$, $z \in \Delta$ belongs to $TS_n(\phi,\psi,\alpha,\lambda)$.

Since $h(z) = z - \sum_{m=2}^{\infty}[(1-\mu)a_m + \mu b_m]z^m$, applying Theorem 2.1, we obtain,

$$
\begin{aligned}
&\sum_{m=2}^{\infty}\frac{(m+n-1)!K_m}{(m-1)!(n+1)!}[(1-\mu)a_m + \mu b_m] \\
&\leq \quad (1-\mu)\sum_{m=2}^{\infty}\frac{(m+n-1)!K_m}{(m-1)!(n+1)!}a_m + \mu\sum_{m=2}^{\infty}\frac{(m+n-1)!K_m}{(m-1)!(n+1)!}b_m \\
&\leq \quad (1-\mu)(1-\alpha) + \mu(1-\alpha) \\
&= \quad 1-\alpha .
\end{aligned}
$$

This implies that $h \in TS_n(\phi, \psi, \alpha, \lambda)$.

We now determine the extreme points of $TS_n(\phi, \psi, \alpha, \lambda)$.

Theorem 4.2 : Let $f_1(z) = z$, $f_m(z) = z - \frac{(m-1)!(n+1)!(1-\alpha)}{(m+n-1)!K_m} z^m$, $m = 2, 3, \ldots$, $z \in \Delta$ and $n \in N_0$. Then $f \in TS_n(\phi, \psi, \alpha, \lambda)$ if and only if it can be expressed as $f(z) = \sum_{m=1}^{\infty} \rho_m f_m(z)$, $\rho_m \geq 0$ and $\sum_{m=1}^{\infty} \rho_m = 1$.

Proof : Suppose that

$$\begin{aligned} f(z) &= \sum_{m=1}^{\infty} \rho_m f_m(z) \\ &= z - \sum \frac{(m-1)!(n+1)!(1-\alpha)}{(m+n-1)!K_m} \rho_m z^m \\ &= z - \sum_{m=2}^{\infty} t_m z^m. \end{aligned}$$

Therefore $f \in TS_n(\phi, \psi, \alpha, \lambda)$, since

$\sum_{m=2}^{\infty} \frac{(m+n-1)!K_m t_m}{(m-1)!(n+1)!(1-\alpha)} = \sum_{m=2}^{\infty} \rho_m = 1 - \rho_1 < 1.$

Conversely, If $f \in TS_n(\phi, \psi, \alpha, \lambda)$, by Corollary 2.1, we have $a_m \leq \frac{(m-1)!(n+1)!(1-\alpha)}{(m+n-1)!K_m}$, $m = 2, 3, \ldots$. We may set $\rho_m = \frac{(m+n-1)!K_m}{(m-1)!(n+1)!(1-\alpha)} a_m$, $m = 2, 3, \ldots$, $n \in N_0$ and $\rho_1 = 1 - \sum_{m=2}^{\infty} \rho_m$.

Then

$$\begin{aligned} f(z) &= z - \sum_{m=2}^{\infty} a_m z^m \\ &= z - \sum_{m=2}^{\infty} \frac{(m-1)!(n+1)!(1-\alpha)}{(m+n-1)!K_m} \rho_m z^m \\ &= z - \sum_{m=2}^{\infty} \rho_m [z - f_m(z)] \\ &= \left(1 - \sum_{m=2}^{\infty} \rho_m\right)(z) + \sum_{m=2}^{\infty} \rho_m f_m(z) \\ &= \sum_{m=1}^{\infty} \rho_m f_m(z). \end{aligned}$$

Corollary 4.1 : The extreme points of $TS_n(\phi, \psi, \alpha, \lambda)$ are the functions $f_m(z)$, $m = 1, 2, \ldots$.

5. Radius of Starlikeness and Convexity

Now, we determine the largest disc in which functions in $TS_n(\phi, \psi, \alpha, \lambda)$ are starlike and convex of order δ $(0 \leq \delta < 1)$ in Δ for all admissible choice of $\phi(z), \psi(z), \alpha, \lambda$ and n.

Theorem 5.1 : If $f \in TS_n(\phi, \psi, \alpha, \lambda)$, then f is starlike of order δ, $0 \leq \delta < 1$ for $|z| < r_1$, where $r_1 = \inf_m \left\{ \frac{(m+n-1)!(1-\delta)K_m}{(m-1)!(n+1)!(m-\delta)(1-\alpha)} \right\}^{\frac{1}{m-1}}$, $m = 2, 3, \ldots$ and $n \in N_0$.

Proof : For $0 \leq \delta < 1$, it is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta.$$

We have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{m=2}^{\infty} (m-1) a_m |z|^{m-1}}{1 - \sum_{m=2}^{\infty} a_m |z|^{m-1}} < 1 - \delta$$

$$\text{or} \qquad \sum_{m=2}^{\infty} \frac{m-\delta}{1-\delta} a_m |z|^{m-1} < 1. \tag{6}$$

It is easy to see that (6) holds if

$$|z|^{m-1} \leq \frac{(m+n-1)!(1-\delta)K_m}{(m-1)!(n+1)!(m-\delta)(1-\alpha)}.$$

This completes the proof.

Upon noting the fact that f is convex if and only if zf' is starlike, we obtain

Theorem 5.2 : If $f \in TS_n(\phi, \psi, \alpha, \lambda)$, then f is convex of order δ, $0 \leq \delta < 1$ for $|z| < r_2$,

$r_2 = \inf_m \left\{ \frac{(m+n-1)!(1-\delta)K_m}{m!(n+1)!(m-\delta)(1-\alpha)} \right\}^{\frac{1}{m-1}}$, $m = 2, 3, \ldots$ and $n \in N_0$.

6. Radii of Close-to-convexity

Theorem 6.1 : Let the function $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $(a_m \geq 0)$ be in the class $TS_n(\phi, \psi, \alpha, \lambda)$. Then $f(z)$ is close-to-convex of order δ $(0 \leq \delta < 1)$ in $|z| < r_1(n, \lambda, \alpha, \delta)$, where

$$r_1(n, \lambda, \alpha, \delta) = \operatorname*{Inf}_m \left\{ \frac{(m+n-1)!(1-\delta)K_m}{(m-1)!(n+1)!(m-\delta)(1-\alpha)} \right\}^{\frac{1}{m-1}},$$

$m = 2, 3, \ldots$ and $n \in N_0$.

The result is sharp with the extremal function $f(z)$ given by

$$f(z) = z - \frac{(m-1)!(n+1)!(1-\alpha)}{(m+n-1)!K_m} z^m, \; z \in \Delta.$$

Proof : It is sufficient to show that $|f'(z) - 1| \leq 1 - \delta$ $(0 \leq \delta < 1)$, $|z| < r_1$.

We have $|f'(z) - 1| - \left| \sum_{m=2}^{\infty} m a_m z^{m-1} \right| \leq \sum_{m=2}^{\infty} m a_m |z|^{m-1}$.

Thus

$$|f'(z) - 1| \leq 1 - \delta \text{ if } \sum_{m=2}^{\infty} \left(\frac{m}{1-\delta} \right) a_m |z|^{m-1} \leq 1. \tag{7}$$

But Theorem 2.1 confirms that

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!K_m}{(m-1)!(n+1)!(1-\alpha)} a_m \leq 1. \tag{8}$$

Hence (7) will be true if $\frac{m|z|^{m-1}}{1-\delta} \leq \frac{(m+n-1)!K_m}{(m-1)!(n+1)!(1-\alpha)}$

or if

$$|z| \leq \left[\frac{(m+n-1)!K_m}{m!(n+1)!(1-\alpha)} \right]^{1/m-1} \quad (m \geq 2). \tag{9}$$

The theorem now follows easily from (9).

7. Inclusion Property of the Class $TS_n(\phi, \psi, \alpha, \lambda)$

Theorem 7.1 : Let $0 \leq \alpha < 1$, $0 \leq \lambda_1 \leq \lambda_2$ and $n \in N_0$.

Then $TS_n(\phi, \psi, \alpha, \lambda_1) \subseteq TS_n(\phi, \psi, \alpha, \lambda_2)$.

Proof : Let $f(z) \in TS_n(\phi, \psi, \alpha, \lambda_1)$.

Then by Theorem 2.1, we have,

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!} K_m |a_m| \leq 1 - \alpha$$

$$\begin{aligned} \text{where } K_m &= (1-\alpha\lambda_1)(m+n)\lambda_m - \alpha(1-\lambda_1)(n+1)\mu_m \\ &\geq (1-\alpha\lambda_2)(m+n)\lambda_m - \alpha(1-\lambda_2)(n+1)\mu_m \\ &= K'_m. \end{aligned}$$

Therefore

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!} K'_m |a_m| \leq \sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!} K_m |a_m| \leq 1 - \alpha.$$

This shows that $f(z) \in TS_n(\phi, \psi, \alpha, \lambda_2)$ and hence $TS_n(\phi, \psi, \alpha, \lambda_1) \subseteq S_n(\phi, \psi, \alpha, \lambda_2)$.

8. Integral Operators

Theorem 8.1 : Let the function $f(z) = z - \sum_{m=2}^{\infty} a_m z^m \in TS_n(\phi, \psi, \alpha, \lambda)$ and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \tag{10}$$

also belongs to the class $TS_n(\phi, \psi, \alpha, \lambda)$.

Proof : From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{m=2}^{\infty} b_m z^m, \tag{11}$$

$$\text{where } b_m = \left(\frac{c+1}{c+m}\right) a_m. \tag{12}$$

Therefore,

$$\begin{aligned} \sum_{m=2}^{\infty} \frac{(m+n-1)! K_m}{(m-1)!(n+1)!} b_m &= \sum_{m=2}^{\infty} \frac{(m+n-1)! K_m}{(m-1)!(n+1)!} \left(\frac{c+1}{c+m}\right) a_m \\ &\leq \sum_{m=2}^{\infty} \frac{(m+n-1)! K_m}{(m-1)!(n+1)!} a_m \\ &\leq 1 - \alpha, \text{ since } f(z) \in TS_n(\phi, \psi, \alpha, \lambda). \end{aligned}$$

Hence by theorem 2.1, $F(z) \in TS_n(\phi, \psi, \alpha, \lambda)$.

Theorem 8.2 : Let c be a real number such that $c > -1$. If $F(z) \in TS_n(\phi, \psi, \alpha, \lambda)$, then the function $f(z)$ defined by (10) is univalent in $|z| < R^*$, where

$$R^* = \inf_m \left\{ \frac{(c+1)(m+n-1)! K_m}{(c+k)(1-\alpha)(m-1)!(n+1)!} \right\}^{\frac{1}{m-1}}, \ (m \geq 2). \tag{13}$$

The result is sharp.

Proof : Let $F(z) = z - \sum_{m=2}^{\infty} a_m z^m \ (a_m \geq 0)$.

It follows from (10) that

$$\begin{aligned} f(z) &= \frac{z^{1-c} \left[z^c F(z)\right]'}{c+1} \\ &= z - \sum_{m=2}^{\infty} \left(\frac{c+k}{c+1}\right) a_m z^m \ (c > -1). \end{aligned} \tag{14}$$

In order to obtain the required result, it suffices to show that $|f'(z)-1| < 1$ in $|z| < R^*$. Now

$$|f'(z) - 1| \leq \sum_{m=2}^{\infty} \frac{m(c+m)}{c+1} a_m |z|^{m-1}.$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{m=2}^{\infty} \frac{m(c+m)}{c+1} a_m |z|^{m-1} < 1. \tag{15}$$

Hence by using (8), (15) will be satisfied if

$$\frac{m(c+m)}{c+1}|z|^{m-1} \leq \frac{(m+n-1)!K_m}{(m-1)!(n+1)!(1-\alpha)}, \quad m \geq 2$$

or if

$$|z| \leq \left[\frac{(c+1)(m+n-1)!K_m}{m!(n+1)!(1-\alpha)(c+m)}\right]^{1/m-1} \quad (m \geq 2).$$

Therefore $f(z)$ is univalent in $|z| < R^*$. Sharpness follows if we take $f(z) = z - \frac{(1-\alpha)(c+m)m!(n+1)!}{(c+1)(m+n-1)!K_m} z^m, \quad m \geq 2.$

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