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Inclusion Criteria for Subclasses of Functions and Gronwall's Inequality *

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Abstract

A normalized analytic function f is shown to be univalent in the open unit disk $\mathbb{D}$ if its second coefficient is sufficiently small and relates to its Schwarzian derivative through a certain inequality. New criteria for analytic functions to be in certain subclasses of functions are

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established in terms of the Schwarzian derivatives and the second coefficients. These include obtaining a sufficient condition for functions to be strongly α-Bazilevič of order β.

Keywords and Phrases: *Univalent functions, Bazilevič functions, Gronwall's inequality, Schwarzian derivative, Second coefficient.*

1. Introduction

Let $\mathcal{A}$ be the set of all normalized analytic functions f of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{A}$ is starlike if it maps $\mathbb{D}$ onto a starlike domain with respect to the origin, and f is convex if $f(\mathbb{D})$ is a convex domain. Analytically, these are respectively equivalent to the conditions $\operatorname{Re}(zf'(z)/f(z)) > 0$ and $1 + \operatorname{Re}(zf''(z)/f'(z)) > 0$ in $\mathbb{D}$. Denote by $\mathcal{ST}$ and $\mathcal{CV}$ the classes of starlike and convex functions respectively. More generally, for $0 \leq \alpha < 1$, a function $f \in \mathcal{A}$ is starlike of order α if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$, and is convex of order α if $1 + \operatorname{Re}(zf''(z)/f'(z)) > \alpha$. We denote these classes by $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$ respectively. For $0 < \alpha \leq 1$, let $\mathcal{SST}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions f satisfying the inequality

$$\left| \arg \frac{zf'(z)}{f(z)} \right| \leq \frac{\alpha\pi}{2}.$$

Functions in $\mathcal{SST}(\alpha)$ are called strongly starlike functions of order α.

The Schwarzian derivative $S(f, z)$ of a locally univalent analytic function f is defined by

$$S(f, z) := \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

The Schwarzian derivative is invariant under Möbius transformations. Also, the Schwarzian derivative of an analytic function f is identically zero if and only if it is a Möbius transformation.

Nehari showed that the univalence of an analytic function in $\mathbb{D}$ can be guaranteed if its Schwarzian derivative is dominated by a suitable positive function [10, Theorem I, p. 700]. In [9], by considering two particular positive functions, a bound on the Schwarzian derivative was obtained that would ensure univalence of an analytic function in $\mathcal{A}$. In fact, the following theorem was proved.

Theorem 1.1. *[9, Theorem II, p. 549] If $f \in \mathcal{A}$ satisfies*

$$|S(f,z)| \leq \frac{\pi^2}{2} \quad (z \in \mathbb{D}),$$

then $f \in \mathcal{S}$. The result is sharp for the function f given by $f(z) = (\exp(i\pi z) - 1)/i\pi$.

The problems of finding similar bounds on the Schwarzian derivatives that would imply univalence, starlikeness or convexity of functions were investigated by a number of authors including Gabriel [4], Friedland and Nehari [3], and Ozaki and Nunokawa [11]. Corresponding results related to meromorphic functions were dealt with in [4, 6, 9, 12]. For instance, Kim and Sugawa [8] found sufficient conditions in terms of the Schwarzian derivative for locally univalent meromorphic functions in the unit disk to possess specific geometric properties such as starlikeness and convexity. The method of proof in [8] was based on comparison theorems in the theory of ordinary differential equations with real coefficients.

Chiang [1] investigated strong-starlikeness of order α and convexity of functions f by requiring the Schwarzian derivative $S(f,z)$ and the second coefficient a_2 of f to satisfy certain inequalities. The following results were proved:

Theorem 1.2. *[1, Theorem 1, pp. 108-109] Let $f \in \mathcal{A}$, $0 < \alpha \leq 1$ and $|a_2| = \eta < \sin(\alpha\pi/2)$. Suppose*

$$\sup_{z \in \mathbb{D}} |S(f,z)| = 2\delta(\eta), \tag{1.1}$$

where $\delta(\eta)$ satisfies the inequality

$$\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + \sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \leq \frac{\alpha\pi}{2}.$$

Then $f \in \mathcal{SST}(\alpha)$. Further, $|\arg(f(z)/z)| \leq \alpha\pi/2$.

Theorem 1.3. *[1, Theorem 2, p. 109] Let $f \in \mathcal{A}$, and $|a_2| = \eta < 1/3$. Suppose (1.1) holds where $\delta(\eta)$ satisfies the inequality*

$$6\eta + 5(1+\eta)\delta e^{\delta/2} < 2.$$

Then

$$f \in \mathcal{CV}\left(\frac{2-6\eta-5(1+\eta)\delta e^{\delta/2}}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\right).$$

In particular, if $a_2 = 0$ and $2\delta \leq 0.6712$, then $f \in \mathcal{CV}$.

Chiang's proofs in [1] rely on Gronwall's inequality (see Lemma 2.1 below). In this paper, Gronwall's inequality is used to obtain sufficient conditions for analytic functions to be univalent. Also, certain inequalities related to the Schwarzian derivative and the second coefficient will be formulated that would ensure analytic functions to possess certain specific geometric properties. The sufficient conditions of convexity obtained in [1] will be seen to be a special case of our result, and similar conditions for starlikeness will also be obtained.

2. Consequences of Gronwall's Inequality

Gronwall's inequality and certain relationships between the Schwarzian derivative of f and the solution of the linear second-order differential equation $y'' + A(z)y = 0$ with $A(z) := S(f;z)/2$ will be revisited in this section. We first state *Gronwall's inequality*, which is needed in our investigation.

Lemma 2.1. *[7, p. 19] Suppose A and g are non-negative continuous real functions for $t \geq 0$. Let $k > 0$ be a constant. Then the inequality*

$$g(t) \leq k + \int_0^t g(s)A(s)ds$$

implies

$$g(t) \leq k \exp\left(\int_0^t A(s)ds\right) \quad (t > 0).$$

For the linear second-order differential equation $y'' + A(z)y = 0$ where $A(z) := \frac{1}{2}S(f;z)$ is an analytic function, suppose that u and v are two linearly independent solutions with initial conditions $u(0) = v'(0) = 0$ and $u'(0) = v(0) = 1$. Such solutions always exist and thus the function f can be represented by

$$f(z) = \frac{u(z)}{cu(z) + v(z)}, \quad (c := -a_2). \tag{2.1}$$

It is evident that

$$f'(z) = \frac{1}{(cu(z)+v(z))^2}. \tag{2.2}$$

Estimates on bounds for various expressions related to u and v were found in [1]. Indeed, using the integral representation of the fundamental solutions

$$\begin{aligned} u(z) &= z + \textstyle\int_0^z (\eta - z)A(\eta)u(\eta)d\eta, \\ v(z) &= 1 + \textstyle\int_0^z (\eta - z)A(\eta)v(\eta)d\eta, \end{aligned} \tag{2.3}$$

and applying Gronwall's inequality, Chiang obtained the following inequalities [1] which we list for easy reference:

$$|u(z)| < e^{\delta/2}, \tag{2.4}$$

$$\left|\frac{u(z)}{z} - 1\right| < \frac{1}{2}\delta e^{\delta/2}, \tag{2.5}$$

$$|cu(z) + v(z)| < (1+\eta)e^{\delta/2}, \tag{2.6}$$

$$|cu(z) + v(z) - 1| < \eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}. \tag{2.7}$$

For instance, by taking the path of integration $\eta(t) = te^{i\theta}$, $t \in [0, r]$, $z = re^{i\theta}$, Gronwall's inequality shows that, whenever $|A(z)| < \delta$ and $0 < r < 1$,

$$\begin{aligned} |u(z)| &\le 1 + \int_0^r (r-t)|A(te^{i\theta})|\,|u(te^{i\theta})|dt \\ &\le \exp(\int_o^r (r-t)|A(te^{i\theta})|dt) \le \exp(\delta/2). \end{aligned}$$

This proves inequality (2.4). Note that there was a typographical error in [1, Inequality (8), p. 112], and that inequality (2.5) is the right form.

3. Inclusion Criteria for Subclasses of Analytic Functions

The first result leads to sufficient conditions for univalence.

Theorem 3.1. *Let $0 < \alpha \le 1$, $0 \le \beta < 1$, $f \in \mathcal{A}$ and $|a_2| = \eta$, where α, β and η satisfy*

$$\sin^{-1}\left(\beta(1+\eta)^2\right) + 2\sin^{-1}\eta < \frac{\alpha\pi}{2}. \tag{3.1}$$

Suppose (1.1) *holds where* $\delta(\eta)$ *satisfies the inequality*

$$\sin^{-1}\left(\beta(1+\eta)^2 e^{\delta}\right) + 2\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \leq \frac{\alpha\pi}{2}. \tag{3.2}$$

Then $|\arg(f'(z)-\beta)| \leq \alpha\pi/2$.

Proof. Using a limiting argument as $\delta \to 0$, the condition (3.1) shows that there is a real number $\delta(\eta) \geq 0$ satisfying inequality (3.2). The representation of f' in terms of the linearly independent solutions of the differential equation $y'' + A(z)y = 0$ with $A(z) := S(f;z)/2$ as given by equation (2.2) yields

$$f'(z) - \beta = \frac{1 - \beta(c\,u(z) + v(z))^2}{(c\,u(z) + v(z))^2}. \tag{3.3}$$

In view of the fact that for $w \in \mathbb{C}$,

$$|w-1| \leq r \Leftrightarrow |\arg w| \leq \sin^{-1} r,$$

inequality (2.6) implies

$$|\arg[1 - \beta(c\,u(z) + v(z))^2]| \leq \sin^{-1}\left(\beta(1+\eta)^2 e^{\delta}\right). \tag{3.4}$$

Similarly, inequality (2.7) shows

$$|\arg[c\,u(z) + v(z)]| \leq \sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right). \tag{3.5}$$

Hence, it follows from (3.3), (3.4) and (3.5) that

$$\begin{aligned} |\arg(f'(z)-\beta)| &\leq |\arg[1 - \beta(c\,u(z) + v(z))^2]| + 2|\arg[c\,u(z) + v(z)]| \\ &\leq \sin^{-1}(\beta(1+\eta)^2 e^{\delta}) + 2\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \\ &\leq \frac{\alpha\pi}{2}, \end{aligned}$$

where the last inequality follows from (3.2). This completes the proof. $\square$

By taking $\beta = 0$ in Theorem 3.1, the following univalence criterion is obtained.

Corollary 3.1. *Let $f \in \mathcal{A}$, and $|a_2| = \eta < \sin(\alpha\pi/4)$, $0 < \alpha \leq 1$. Suppose* (1.1) *holds where $\delta(\eta)$ satisfies the inequality*

$$\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2} \leq \sin\left(\frac{\alpha\pi}{4}\right).$$

Then $|\arg f'(z)| \leq \alpha\pi/2$, and in particular $f \in \mathcal{S}$.

Example 3.1. Consider the univalent function g given by

$$g(z) = \frac{z}{1+cz}, \qquad |c| \leq 1, \quad z \in \mathbb{D}.$$

Since the Schwarzian derivative of an analytic function is zero if and only if it is a Möbius transformation, it is evident that $S(g,z) = 0$. Therefore the condition (1.1) is satisfied with $\delta = 0$. It is enough to take $\eta = |c|$ and to assume that η, α and β satisfy the inequality (3.1). Now

$$\begin{aligned} |\arg(g'(z) - \beta)| &= \left|\arg \frac{1}{(1+cz)^2} - \beta\right| \leq |\arg(1-\beta(1+cz)^2)| + 2|\arg(1+cz)| \\ &\leq \sin^{-1}(\beta(1+|c|)^2) + 2\sin^{-1}|c|. \end{aligned}$$

In view of the latter inequality, it is necessary to assume inequality (3.1) for g to satisfy $|\arg(g'(z) - \beta)| \leq \alpha\pi/2$.

Let $0 \leq \rho < 1$, $0 \leq \lambda < 1$, and α be a positive integer. A function $f \in \mathcal{A}$ is called an α-Bazilevič function of order ρ and type λ, written $f \in \mathcal{B}(\alpha, \rho, \lambda)$, if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)^{1-\alpha} g(z)^{\alpha}}\right) > \rho \quad (z \in \mathbb{D})$$

for some function $g \in \mathcal{ST}(\lambda)$. The following subclass of α-Bazilevič functions is of interest. A function $f \in \mathcal{A}$ is called strongly α-Bazilevič of order β if

$$\left|\arg\left(\left(\frac{z}{f(z)}\right)^{1-\alpha} f'(z)\right)\right| < \frac{\beta\pi}{2}, \quad (\alpha > 0;\ 0 < \beta \leq 1),$$

(see Gao [5]). For the class of strongly α-Bazilevič functions of order β, the following sufficient condition is obtained.

Theorem 3.2. *Let $\alpha > 0$, $0 < \beta \leq 1$, $f \in \mathcal{A}$ and $|a_2| = \eta$, where η, α and β satisfy*

$$\eta < \sin\left(\frac{\beta\pi}{2(1+\alpha)}\right).$$

Suppose (1.1) *holds where $\delta(\eta)$ satisfies the inequality*

$$|1-\alpha| \sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + (1+\alpha)\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \leq \frac{\beta\pi}{2}. \tag{3.6}$$

Then f is strongly α-Bazilevič of order β.

Proof. The condition $\eta < \sin(\beta\pi/2(1+\alpha))$ ensures that there is a real number $\delta(\eta)$ satisfying (3.6). Using (2.1) and (2.2) lead to

$$\begin{aligned} \left|\arg\left(\left(\frac{z}{f(z)}\right)^{1-\alpha} f'(z)\right)\right| &= \left|\arg\left(\left(\frac{u(z)}{z}\right)^{\alpha-1} (cu(z)+v(z))^{-(\alpha+1)}\right)\right| \\ &\leq |1-\alpha| \left|\arg\left(\frac{u(z)}{z}\right)\right| + |\alpha+1|\, |\arg(cu(z)+v(z))|. \end{aligned}$$

It now follows from (2.5), (3.5) and (3.6) that

$$\begin{aligned} \left|\arg\left(\left(\frac{z}{f(z)}\right)^{1-\alpha} f'(z)\right)\right| &\leq |1-\alpha|\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + (1+\alpha)\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \\ &\leq \frac{\beta\pi}{2}. \end{aligned}$$

□

For $\alpha \geq 0$, consider the class $R(\alpha)$ defined by

$$\mathcal{R}(\alpha) = \{f \in \mathcal{A} : \operatorname{Re}(f'(z) + \alpha z f''(z)) > 0,\ \alpha \geq 0\}.$$

For this class, the following sufficient condition is obtained.

Theorem 3.3. *Let $\alpha \geq 0$, $f \in \mathcal{A}$ and $|a_2| = \eta$, where η and α satisfy*

$$2\sin^{-1}\eta + \sin^{-1}\left(\frac{2\eta\alpha}{1-\eta}\right) < \frac{\pi}{2}. \tag{3.7}$$

Suppose (1.1) *holds where $\delta(\eta)$ satisfies the inequality*

$$2\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) + \sin^{-1}\left(\frac{4\alpha\big(\eta + (1+\eta)\delta e^{\delta/2}\big)}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\right) \leq \frac{\pi}{2}. \tag{3.8}$$

Then $f \in \mathcal{R}(\alpha)$.

Proof. Again it is easily seen from a limiting argument that the condition (3.7) guarantees the existence of a real number $\delta(\eta) \geq 0$ satisfying the inequality (3.8). It is sufficient to show that

$$\left|\arg\left(f'(z)\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right)\right| < \frac{\pi}{2}.$$

The equation (2.2) yields

$$\frac{zf''(z)}{f'(z)} = -2z\frac{cu'(z)+v'(z)}{cu(z)+v(z)}. \tag{3.9}$$

A simple calculation from (2.3) shows that

$$cu'(z)+v'(z) = c - \int_0^z A(\eta)[cu(\eta)+v(\eta)]d\eta,$$

and an application of (2.6) leads to

$$|cu'(z)+v'(z)| \leq \eta + (1+\eta)\delta e^{\delta/2}. \tag{3.10}$$

Use of (2.7) results in

$$|cu(z)+v(z)| \geq 1 - |cu(z)+v(z)-1| \geq 1-\eta-\frac{1}{2}(1+\eta)\delta e^{\delta/2}. \tag{3.11}$$

The lower bound in (3.11) is non-negative from the assumption made in (3.8). From (3.9), (3.10) and (3.11) , it is evident that

$$\begin{aligned}\left|\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)-1\right| &= \left|2z\alpha\frac{cu'(z)+v'(z)}{cu(z)+v(z)}\right| \\ &\leq \frac{2\alpha\big(\eta+(1+\eta)\delta e^{\delta/2}\big)}{1-\eta-\frac{1}{2}(1+\eta)\delta e^{\delta/2}}.\end{aligned}$$

Hence,

$$\left|\arg\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right| \leq \sin^{-1}\left(\frac{4\alpha\big(\eta+(1+\eta)\delta e^{\delta/2}\big)}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\right). \tag{3.12}$$

From (3.5) it follows that

$$|\arg f'(z)| = 2|\arg(cu(z)+v(z))| \leq 2\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right). \tag{3.13}$$

Using (2.2) and (3.5), the inequality (3.13) together with (3.12) and (3.8) imply that

$$\begin{aligned}&\left|\arg\left(f'(z)\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right)\right| \leq |\arg f'(z)| + \left|\arg\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right| \\ &\leq 2\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right)+\sin^{-1}\left(\frac{4\alpha\big(\eta+(1+\eta)\delta e^{\delta/2}\big)}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\right) \\ &\leq \frac{\pi}{2}. \end{aligned}$$

□

Theorem 3.4. *Let $f \in \mathcal{A}$, $|a_2| = \eta \leq 1/3$, and β, α be real numbers satisfying*

$$|\alpha|\sin^{-1}\eta + |\beta|\sin^{-1}\left(\frac{2\eta}{1-\eta}\right) < \frac{\pi}{2}. \tag{3.14}$$

Suppose (1.1) *holds where $\delta(\eta)$ satisfies the inequality*

$$\begin{aligned}|\alpha|\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right)+|\alpha|\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right)& \\ +|\beta|\sin^{-1}\left(\frac{4\big(\eta+(1+\eta)\delta e^{\delta/2}\big)}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\right) &\leq \frac{\pi}{2}.\end{aligned} \tag{3.15}$$

Then

$$\operatorname{Re}\left(\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta}\right) > 0. \tag{3.16}$$

Proof. The inequality (3.14) assures the existence of δ satisfying (3.15). From (2.1) and (2.2) it follows that

$$\frac{zf'(z)}{f(z)} = \frac{z}{u(z)}\frac{1}{cu(z)+v(z)}, \quad z \in \mathbb{D}. \tag{3.17}$$

By (2.5) and (3.5),

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| \leq \sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right)+\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right). \tag{3.18}$$

Using (3.12) with $\alpha = 1$, (3.18) and (3.15) lead to

$$\left|\arg\left(\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta}\right)\right| \leq |\alpha|\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| + |\beta|\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|$$
$$\leq |\alpha|\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + |\alpha|\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right)$$
$$+ |\beta|\sin^{-1}\frac{4\big(\eta+(1+\eta)\delta e^{\delta/2}\big)}{2-2\eta-(1+\eta)\delta e^{\delta/2}}$$
$$\leq \frac{\pi}{2}.$$

This shows that (3.16) holds. □

Remark 3.1. Theorem 3.4 yields the following interesting special cases.

(i) If $\alpha = 0$, $\beta = 1$, a sufficient condition for convexity is obtained. This case reduces to a result in [1, Theorem 2, p. 109].

(ii) For $\alpha = 1$, $\beta = 0$, a sufficient condition for starlikeness is obtained.

(iii) For $\alpha = -1$ and $\beta = 1$, then the class of functions satisfying (3.16) reduces to the class of functions

$$\mathcal{G} := \left\{ f \in \mathcal{A} \,\middle|\, \operatorname{Re}\left(\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right) > 0 \right\}.$$

This class $\mathcal{G}$ was considered by Silverman [14] and Tuneski [15].

Theorem 3.5. *Let $\beta \geq 0$, $f \in \mathcal{A}$ and $|a_2| = \eta$, where η satisfies*

$$\sin^{-1}(\eta) + \sin^{-1}\left(\frac{2\beta\eta}{1-\eta}\right) < \frac{\pi}{2}. \tag{3.19}$$

Suppose (1.1) *holds where $\delta(\eta)$ satisfies the inequality*

$$\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + \sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) + \sin^{-1}\left(\frac{4\beta\big(\eta+(1+\eta)\delta e^{\delta/2}\big)}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\right)$$
$$\leq \frac{\pi}{2}.$$

Then

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}+\beta\frac{z^2f''(z)}{f(z)}\right)>0. \tag{3.20}$$

The proof is similar to the proof of Theorem 3.4, and is therefore omitted. The inequality (3.19) is equivalent to the condition

$$\eta\left(1+\sqrt{(1-\eta)^2-4\beta^2\eta^2}+2\beta\sqrt{1-\eta^2}\right)<1.$$

For $\beta=1$, the above equation simplifies to

$$\eta^8-4\eta^7+12\eta^6-12\eta^5+6\eta^4+20\eta^3-4\eta^2-4\eta+1=0;$$

the value of the root η is approximately 0.321336. Functions satisfying inequality (3.20) were investigated by Ramesha *et al.* [13].

Consider the class $P(\gamma)$, $0\le\gamma\le1$, given by

$$P(\gamma):=\left\{f\in\mathcal{A}:\left|\arg\left((1-\gamma)\frac{f(z)}{z}+\gamma f'(z)\right)\right|<\frac{\pi}{2},\quad z\in\mathbb{D}\right\}.$$

The same approach applying Gronwall's inequality leads to the following result about the class $P(\gamma)$.

Theorem 3.6. *Let $0\le\gamma<1$, $f\in\mathcal{A}$ and $|a_2|=\eta$, where η and γ satisfy*

$$\sin^{-1}\left(\frac{\gamma}{1-\gamma}\frac{1}{\eta-1}\right)+\sin^{-1}\eta<\frac{\pi}{2}. \tag{3.21}$$

Suppose (1.1) *holds where $\delta(\eta)$ satisfies the inequality*

$$\begin{aligned}&\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right)+\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right)\\&+\sin^{-1}\left(\frac{2\gamma}{1-\gamma}\frac{1}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\frac{1}{1-2e^{\delta/2}}\right)\le\frac{\pi}{2}.\end{aligned} \tag{3.22}$$

Then $f\in P(\gamma)$.

Proof. Condition (3.21) assures the existence of a real number $\delta(\eta) \geq 0$ satisfying the inequality (3.22). A simple calculation from (2.3) and Lemma 2.1 shows that

$$\begin{aligned} |u(z)-1| &\leq |z-1| + \left| \int_0^z (\zeta - z)A(\zeta)u(\zeta)d\zeta \right| \\ &\leq 2e^{\delta/2}. \end{aligned}$$

The above inequality gives

$$\left|\frac{z}{u(z)}\right| \leq \frac{1}{|u(z)|} \leq \frac{1}{1-|u(z)-1|} \leq \frac{1}{1-2e^{\delta/2}}. \tag{3.23}$$

Therefore, for some $0 < \beta \leq \gamma/(1-\gamma)$, (3.17), (3.23) and (3.11) lead to

$$\begin{aligned} \left|1+\beta\frac{zf'(z)}{f(z)}-1\right| &= \beta\left|\frac{z}{u(z)}\right| \frac{1}{|cu(z)+v(z)|} \\ &\leq \frac{\beta}{1-2e^{\delta/2}} \frac{1}{1-\eta-\frac{1}{2}(1+\eta)\delta e^{\delta/2}} \\ &= \frac{2\beta}{1-2e^{\delta/2}} \frac{1}{2-2\eta-(1+\eta)\delta e^{\delta/2}}. \end{aligned}$$

Hence

$$\left|\arg\left(1+\beta\frac{zf'(z)}{f(z)}\right)\right| \leq \sin^{-1}\left(\frac{2\beta}{1-2e^{\delta/2}} \frac{1}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\right). \tag{3.24}$$

Also, (2.5) and (3.5) yield

$$\begin{aligned} \left|\arg\frac{f(z)}{z}\right| &= \left|\arg\frac{u(z)}{z(cu(z)+v(z))}\right| \\ &\leq \left|\arg\frac{u(z)}{z}\right| + |\arg(cu(z)+v(z))| \\ &\leq \sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + \sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right). \end{aligned} \tag{3.25}$$

Replacing β by $\gamma/(1-\gamma)$ in inequality (3.24), and using (3.25) and (3.22) yield

$$\begin{aligned}\left|\arg\left((1-\gamma)\frac{f(z)}{z}+\gamma f'(z)\right)\right| &\leq \left|\arg\frac{f(z)}{z}\right| + \left|\arg\left(1+\frac{\gamma}{1-\gamma}\frac{zf'(z)}{f(z)}\right)\right| \\ &\leq \sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + \sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \\ &\quad + \sin^{-1}\left(\frac{2\gamma}{1-\gamma}\frac{1}{1-2e^{\delta/2}}\frac{1}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\right) \\ &\leq \frac{\pi}{2},\end{aligned}$$

and hence $f \in P(\gamma)$. □

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