Tamsui Oxford Journal of Information and Mathematical Sciences **29(2)** (2013) 267-284
Aletheia University

Applications of Differential Subordination for Functions with Fixed Second Coefficient to Geometric Function Theory*

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Received September 6, 2012, Accepted June 17, 2013.

Abstract

The theory of second order differential subordination of S. S. Miller and P. T. Mocanu [Differential Subordinations, Dekker, New York, 2000] was recently extended to functions with fixed initial coefficient

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*2010 *Mathematics Subject Classification*. Primary 30C45, 30C80.

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by R. M. Ali, S. Nagpal and V. Ravichandran [Second-order differential subordination for analytic functions with fixed initial coefficient, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), 611–629] and applied to obtain several generalization of classical results in geometric function theory. In this paper, further applications of this subordination theory is given. In particular, several sufficient conditions related to starlikeness, convexity, close-to-convexity of normalized analytic functions are derived.

Keywords and Phrases: *Analytic functions, Starlike functions, Convex functions, Subordination, Fixed second coefficient.*

1. Introduction

For univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the famous Bieberbach theorem shows that $|a_2| \leq 2$ and this bound for the second coefficient yields the growth and distortion bounds as well as covering theorem. In view of the influence of the second coefficient in the properties of univalent functions, several authors have investigated functions with fixed second coefficient. For a brief survey of the various developments, mainly on radius problems, from 1920 to this date, see the recent work by Ali *et al.* [2]. The theory of first-order differential subordination was developed by Miller and Mocanu and a very comprehensive account of the theory and numerous applications can be found in their monograph [9]. Ali *et al.* [4] have extended this well-known theory of differential subordination to the functions with preassigned second coefficient. Nagpal and Ravichandran [10] have applied the results in [4] to obtain several extensions of well-known results to the functions with fixed second coefficient. In this paper, we continue their investigation by deriving several sufficient conditions for starlikeness of functions with fixed second coefficient.

For convenience, let $\mathcal{A}_{n,b}$ denote the class of all functions $f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots$ where $n \in \mathbb{N} = \{1, 2, \dots\}$ and b is a fixed non-negative real number. For fixed $\mu \geq 0$, and $n \in \mathbb{N}$, let $\mathcal{H}_{\mu,n}$ consists of analytic functions p on $\mathbb{D}$ of the form

$$p(z) = 1 + \mu z^n + p_{n+1}z^{n+1} + \cdots \quad (z \in \mathbb{D}). \tag{1.1}$$

Let Ω be a subset of $\mathbb{C}$ and the class $\Psi_{\mu,n}[\Omega]$ consists of those functions $\psi : \mathbb{C}^2 \to \mathbb{C}$ that are continuous in a domain $D \subset \mathbb{C}^2$ with $(1, 0) \in D$, $\psi(1, 0) \in \Omega$, and satisfy the admissibility condition: $\psi(i\rho,\sigma)\notin\Omega$ whenever $(i\rho,\sigma)\in D$, $\rho\in\mathbb{R}$, and

$$\sigma\leq-\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2). \tag{1.2}$$

When $\Omega=\{w:\operatorname{Re}w>0\}$, let $\Psi_{\mu,n}:=\Psi_{\mu,n}[\Omega]$. The following theorem is needed to prove our main results.

Theorem 1.1. [4, Theorem 3.4] *Let $p\in\mathcal{H}_{\mu,n}$ with $0<\mu\leq 2$. Let $\psi\in\Psi_{n,\mu}$ with associated domain D. If $(p(z),zp'(z))\in D$ and $\operatorname{Re}\psi(p(z),zp'(z))>0$, then $\operatorname{Re}p(z)>0$ for $z\in\mathbb{D}$.*

For $\alpha\neq 1$, let

$$\mathcal{S}^*(\alpha):=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\frac{1+(1-2\alpha)z}{1-z}\right\}.$$

The function $p_\alpha(z):=(1+(1-2\alpha)z)/(1-z)$ maps $\mathbb{D}$ onto $\{w\in\mathbb{C}:\operatorname{Re}w>\alpha\}$ for $\alpha<1$ and onto $\{w\in\mathbb{C}:\operatorname{Re}w<\alpha\}$ for $\alpha>1$. Therefore, for $\alpha<1$, $\mathcal{S}^*(\alpha)$ is the class of starlike functions of order α consisting of functions $f\in\mathcal{A}$ for which $\operatorname{Re}(zf'(z)/f(z))>\alpha$. For $\alpha>1$, $\mathcal{S}^*(\alpha)$ reduces to the class $\mathcal{M}(\alpha)$ consisting of $f\in\mathcal{A}$ satisfying $\operatorname{Re}(zf'(z)/f(z))<\alpha$. The latter class $\mathcal{M}(\alpha)$ and its subclasses were investigated in [3, 15, 22, 25, 26]. For $0\leq\alpha<1$, $\mathcal{S}^*(\alpha)$ consists of only univalent functions while for other values of α, the class contains non-univalent functions. Other classes can be unified in a similar manner by subordination.

Motivated by the works of Lewandowski, Miller and Złotkiewicz [5], several authors [7, 8, 11, 15, 13, 14, 17, 18, 19, 23, 27] have investigated the functions f for which $zf'(z)/f(z)\cdot(\alpha zf''(z)/f'(z)+1)$ lies in certain region in the right half-plane. For $\alpha\geq 0$ and $\beta<1$, Ravichandran *et al.* [21] have shown that a function f of the form $f(z)=z+a_{n+1}z^{n+1}+\cdots$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right)>\alpha\beta\left(\beta+\frac{n}{2}-1\right)+\beta-\frac{\alpha n}{2} \tag{1.3}$$

is starlike of order β. In the first result of Theorem 2.1, we obtain the corresponding result for $f\in\mathcal{A}_{n,b}$.

For function p of the form $p(z)=1+p_1z+p_2z^2+\cdots$, Nunokawa *et al.* [12] showed that for certain analytic function w, with $w(0)=\alpha$, $\alpha p^2(z)+\beta zp'(z)\prec$

$w(z)$ implies $\operatorname{Re} p(z) > 0$, where $\beta > 0$, $\alpha \geq -\beta/2$. See also [20]. Lemma 2.6 investigates the conditions for similar class of functions.

For complex numbers β and γ, the differential subordination

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z),$$

where q is analytic and h is univalent with $q(0) = h(0)$, is popularly known as Briot-Bouquet differential subordination. This particular differential subordination has a significant number of important applications in the theory of analytic functions (for details see [9]). The importance of Briot-Bouquet differential subordination inspired many researchers to work in this area and many generalizations and extensions of the Briot-Bouquet differential subordination have recently been obtained. Ali *et al.* [1] obtained several results related to the Briot-Bouquet differential subordination. In Lemmas 2.2 and 2.5, the Briot-Bouquet differential subordination is investigated for functions with fixed second coefficient.

2. Subordinations for starlikeness and univalence

For $\beta \neq 1$, Theorem 2.1 provides several sufficient conditions for $f \in \mathcal{S}^*(\beta)$; in particular, for $0 \leq \beta < 1$, these are sufficient conditions for starlikeness of order β. Theorem 2.2 is the meromorphic analogue of Theorem 2.1. Theorem 2.3 gives sufficient conditions for the subordination $f'(z) \prec (1+(1-2\beta)z)/(1-z)$ to hold. For $\beta = 0$, this latter condition is sufficient for the close-to-convexity and hence univalence of the function f.

Theorem 2.1. *Let $\alpha \geq 0$, $\beta \geq 0$, $\beta \neq 1$, and $0 < \mu = nb \leq 2$. Let δ_1, δ_2, δ_3 and δ_4 be given by*

$$\delta_1 = -\frac{\alpha}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + (1-\alpha)\beta + \alpha\beta^2,$$

$$\delta_2 = -\frac{1}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + \beta,$$

$$\delta_3 = \begin{cases} \frac{-\alpha\beta}{2(1-\beta)}\left(n + \frac{2-\mu}{2+\mu}\right) + \beta, & if \quad 0 \leq \beta \leq \frac{1}{2}, \\ \frac{-\alpha}{2\beta}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + \beta, & if \quad \frac{1}{2} \leq \beta, \end{cases}$$

$$\delta_4 = \begin{cases} \frac{-\beta}{2(1-\beta)}\left(n + \frac{2-\mu}{2+\mu}\right), & if \quad 0 \leq \beta < \frac{1}{2}, \\ \frac{-1}{2\beta}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right), & if \quad \frac{1}{2} \leq \beta. \end{cases}$$

If $f \in \mathcal{A}_{n,b}$ satisfies one of the following subordinations

$$\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right) \prec \frac{1+(1-2\delta_1)z}{1-z}, \tag{2.1}$$

$$\frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{1+(1-2\delta_2)z}{1-z}, \tag{2.2}$$

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+(1-2\delta_3)z}{1-z}, \tag{2.3}$$

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec -\frac{2\delta_4 z}{1-z} \tag{2.4}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Our next theorem gives sufficient conditions for meromorphic functions to be starlike in the punctured unit disk $\mathbb{D}^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$. Precisely, we consider the class $\Sigma_{n,b}$ of all analytic functions defined on $\mathbb{D}^*$ of the form

$$f(z) = \frac{1}{z} + bz^n + a_{n+1}z^{n+1} + \cdots \quad (b \leq 0).$$

Theorem 2.2. *Let $\alpha \geq 0$, $\beta \geq 0$, $\beta \neq 1$, and $0 < \mu = -(n+1)b \leq 2$. Let δ_1, δ_2, δ_3 and δ_4 be given by*

$$\delta_1 = -\frac{\alpha}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + (1-\alpha)\beta + \alpha\beta^2,$$

$$\delta_2 = -\frac{1}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + \beta,$$

$$\delta_3 = \begin{cases} \frac{-\alpha\beta}{2(1-\beta)}\left(n + \frac{2-\mu}{2+\mu}\right) + \beta, & if \quad 0 \leq \beta \leq \frac{1}{2}, \\ \frac{-\alpha}{2\beta}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + \beta, & if \quad \frac{1}{2} \leq \beta, \end{cases}$$

$$\delta_4 = \begin{cases} \frac{-\beta}{2(1-\beta)}\left(n + \frac{2-\mu}{2+\mu}\right), & if \quad 0 \leq \beta < \frac{1}{2}, \\ \frac{-1}{2\beta}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right), & if \quad \frac{1}{2} \leq \beta. \end{cases}$$

If $f \in \Sigma_{n,b}$ satisfies one of the following subordinations

$$\frac{zf'(z)}{f(z)}\left(2\alpha\frac{zf'(z)}{f(z)} - \alpha\frac{zf''(z)}{f'(z)} - 1\right) \prec \frac{1+(1-2\delta_1)z}{1-z}, \tag{2.5}$$

$$-\frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{1+(1-2\delta_2)z}{1-z}, \tag{2.6}$$

$$\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - (1+\alpha)\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\delta_3)z}{1-z}, \tag{2.7}$$

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec -\frac{2\delta_4 z}{1-z} \tag{2.8}$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Theorem 2.3. *Let $\alpha \geq 0$, $\beta \geq 0$, $\beta \neq 1$, and $0 < \mu = (n+1)b \leq 2$. Let δ_1, δ_2, δ_3 and δ_4 be given as in Theorem 2.1. If $f \in \mathcal{A}_{n,b}$ satisfies one of the following subordinations*

$$f'(z)\left[\alpha\left(\frac{zf''(z)}{f'(z)} + f'(z) - 1\right) + 1\right] \prec \frac{1+(1-2\delta_1)z}{1-z}, \tag{2.9}$$

$$f'(z) + zf''(z) \prec \frac{1+(1-2\delta_2)z}{1-z}, \tag{2.10}$$

$$\alpha\frac{zf''(z)}{f'(z)} + f'(z) \prec \frac{1+(1-2\delta_3)z}{1-z}, \tag{2.11}$$

$$\frac{zf''(z)}{f'(z)} \prec -\frac{2\delta_4 z}{1-z} \tag{2.12}$$

then

$$f'(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

The proof of these theorems follows from the following series of lemmas.

Lemma 2.1. *For $\alpha \geq 0$, $\beta \geq 0$, $\beta \neq 1$, $\gamma > 0$, and $0 < \mu \leq 2$, let*

$$\delta := -\frac{\gamma}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + (1-\alpha)\beta + \alpha\beta^2.$$

If $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$(1-\alpha)p(z) + \alpha p^2(z) + \gamma zp'(z) \prec \frac{1+(1-2\delta)z}{1-z}, \tag{2.13}$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let $0 \leq \beta < 1$. Note that δ given in the hypothesis clearly satisfies $\delta < 1$. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = (p(z)-\beta)/(1-\beta)$. Then q is analytic and $(1-\beta)q(z) + \beta = p(z)$. By using this, the inequality (2.13) can be written as

$$\begin{aligned}\operatorname{Re}\Big[&(1-\beta)(1-\alpha+2\alpha\beta)q(z) + \alpha(1-\beta)^2q^2(z)\\ &+\gamma(1-\beta)zq'(z) + (1-\alpha)\beta + \alpha\beta^2 - \delta\Big] > 0.\end{aligned}$$

Define the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\psi(r,s) = (1-\beta)(1-\alpha+2\alpha\beta)r + \alpha(1-\beta)^2r^2 + \gamma(1-\beta)s + (1-\alpha)\beta + \alpha\beta^2 - \delta.$$

For $\rho \in \mathbb{R}$, $n \geq 1$ and σ satisfying (1.2), it follows that

$$\begin{aligned}
&\operatorname{Re}\psi(i\rho,\sigma)\\
&= \operatorname{Re}\left[(1-\beta)(1-\alpha+2\alpha\beta)i\rho - \alpha(1-\beta)^2\rho^2 + \gamma(1-\beta)\sigma + (1-\alpha)\beta + \alpha\beta^2 - \delta\right]\\
&= \gamma(1-\beta)\sigma - \alpha(1-\beta)^2\rho^2 + (1-\alpha)\beta + \alpha\beta^2 - \delta\\
&\leq \gamma(1-\beta)\left[-\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right] - \alpha(1-\beta)^2\rho^2 + (1-\alpha)\beta + \alpha\beta^2 - \delta\\
&= -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2) - \alpha(1-\beta)^2(\rho^2+1) + \alpha(1-\beta)^2\\
&\quad + (1-\alpha)\beta + \alpha\beta^2 - \delta\\
&= -(1+\rho^2)\left[\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right) + \alpha(1-\beta)^2\right]\\
&\quad + \alpha(1-\beta)^2 + (1-\alpha)\beta + \alpha\beta^2 - \delta\\
&\leq -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right) + (1-\alpha)\beta + \alpha\beta^2 - \delta.
\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$. By Theorem 1.1, $\operatorname{Re} q(z) > 0$ or equivalently $\operatorname{Re} p(z) > \beta$. For $\beta > 1$, the proof is similar. □

Lemma 2.2. *For $\beta \geq 0$, $\beta \neq 1$, $\gamma > 0$, and $0 < \mu \leq 2$, let*

$$\delta := -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right) + \beta.$$

If $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$p(z) + \gamma z p'(z) \prec \frac{1+(1-2\delta)z}{1-z},$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Replace $\alpha = 0$ in Lemma 2.1 to yield the result. □

Lemma 2.3. *For $\alpha > 0$, $\beta \geq 0$, $\beta \neq 1$, and $0 < \mu \leq 2$, let*

$$\delta = \begin{cases} \frac{-\alpha\beta}{2(1-\beta)}\left(n + \frac{2-\mu}{2+\mu}\right) + \beta := \delta_2, & if \quad 0 \leq \beta \leq \frac{1}{2}, \\ \frac{-\alpha}{2\beta}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right) + \beta := \delta_1, & if \quad \frac{1}{2} \leq \beta. \end{cases}$$

If the function $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$p(z) + \alpha\frac{zp'(z)}{p(z)} \prec \frac{1+(1-2\delta)z}{1-z} \tag{2.14}$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let $0 \leq \beta < 1$. As in the proof of Lemma 2.1, let $q : \mathbb{D} \to \mathbb{C}$ be given by $q(z) = (p(z)-\beta)/(1-\beta)$. Then inequality (2.14) can be written as

$$\operatorname{Re}\left[(1-\beta)q(z) + \beta + \frac{\alpha(1-\beta)}{(1-\beta)q(z)+\beta}zq'(z) - \delta\right] > 0. \tag{2.15}$$

Define the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\psi(r,s) = (1-\beta)r + \frac{\alpha(1-\beta)}{(1-\beta)r+\beta}s + \beta - \delta.$$

Then $\operatorname{Re}\psi(q(z), zq'(z)) > 0$ and $\operatorname{Re}\psi(1,0) > 0$. We now show that $\psi \in \Psi_{\mu,n}$. For $\rho \in \mathbb{R}$, $n \geq 1$ and σ satisfying (1.2), it follows that

$$\begin{aligned} \operatorname{Re}\psi(i\rho,\sigma) &= \operatorname{Re}\left[(1-\beta)i\rho + \frac{\alpha(1-\beta)}{(1-\beta)i\rho+\beta}\sigma + \beta - \delta\right] \\ &= \operatorname{Re}\left[(1-\beta)i\rho + \frac{\alpha\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma - \frac{\alpha(1-\beta)^2 i\rho}{\beta^2+(1-\beta)^2\rho^2}\sigma + \beta - \delta\right] \\ &= \frac{\alpha\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma + \beta - \delta \\ &\leq \frac{\alpha\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\left[-\frac{1}{2}\left(n + \frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right] + \beta - \delta \\ &= -\frac{\alpha\beta}{2}(1-\beta)\left(n + \frac{2-\mu}{2+\mu}\right)\left(\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}\right) + \beta - \delta. \end{aligned}$$

For $1/2 \leq \beta$, the expression

$$\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}$$

attains minimum at $\rho = 0$ and therefore

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\alpha\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{\beta^2}+\beta-\delta_1\\ &= \frac{-\alpha}{2\beta}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\beta-\delta_1.\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$.

For $0 \leq \beta \leq 1/2$,

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\alpha\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{(1-\beta)^2}+\beta-\delta_2\\ &= \frac{-\alpha\beta}{2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta-\delta_2.\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$ or $\psi \in \Psi_{\mu,n}$. Thus Theorem 1.1 implies $\operatorname{Re} q(z) > 0$ or equivalently $\operatorname{Re} p(z) > \beta$. The proof of the case $\beta > 1$ is similar. □

Lemma 2.4. *For $\beta \geq 0$, $\beta \neq 1$ and $0 < \mu \leq 2$, let*

$$\delta = \begin{cases} \frac{-\beta}{2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right), & if \quad 0 \leq \beta < \frac{1}{2},\\ \frac{-1}{2\beta}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right), & if \quad \frac{1}{2} \leq \beta.\end{cases}$$

If the function $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$\frac{zp'(z)}{p(z)} \prec -\frac{2\delta z}{1-z},$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. We consider the case $0 \leq \beta < 1$. The case $\beta > 1$ is similar. Let $q(z) = (p(z)-\beta)/(1-\beta)$ or $(1-\beta)q(z)+\beta = p(z)$. Then

$$\frac{zp'(z)}{p(z)} = \frac{(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta}. \tag{2.16}$$

Define $\psi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\psi(r,s) = \frac{(1-\beta)s}{(1-\beta)r+\beta} - \delta.$$

Then $\psi(r,s)$ is continuous on $(\mathbb{C} - \{-\beta/(1-\beta)\}) \times \mathbb{C}$. For $\rho \in \mathbb{R}$, $n \geq 1$ and σ satisfying (1.2), it follows that

$$\begin{aligned}
\operatorname{Re}\psi(i\rho,\sigma) &= \operatorname{Re}\left(\frac{(1-\beta)}{(1-\beta)i\rho+\beta}\sigma - \delta\right)\\
&= \operatorname{Re}\left(\frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma - \frac{(1-\beta)^2 i\rho}{\beta^2+(1-\beta)^2\rho^2}\sigma - \delta\right)\\
&= \frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma - \delta\\
&\leq \frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\left[-\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right] - \delta\\
&= -\frac{\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\left(\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}\right) - \delta.
\end{aligned}$$

For $1/2 \leq \beta$, the expression

$$\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}$$

attains its minimum at $\rho = 0$ and therefore

$$\begin{aligned}
\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{\beta^2} - \delta\\
&= \frac{-1}{2\beta}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right) - \delta.
\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$.

For $0 \leq \beta \leq 1/2$,

$$\begin{aligned}
\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\beta}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{(1-\beta)^2} - \delta\\
&= \frac{-\beta}{2(1-\beta)}\left(n+\frac{2-\mu}{2+\mu}\right) - \delta.
\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$. Thus Theorem 1.1 implies $\operatorname{Re} q(z) > 0$ or equivalently $\operatorname{Re} p(z) > \beta$. $\square$

Lemma 2.5. *For $\alpha > 0$, $\beta \geq 0$, $\beta \neq 1$, $\gamma > -\alpha\beta$ and $0 < \mu \leq 2$, let*

$$\delta = \begin{cases} \frac{-1}{2}\frac{(1-\beta)}{(\alpha\beta+\gamma)}\left(n + \frac{2-\mu}{2+\mu}\right) + \beta, & if \quad \gamma \geq \alpha(1-2\beta), \\ \frac{-1}{2}\frac{(\alpha\beta+\gamma)}{\alpha^2(1-\beta)}\left(n + \frac{2-\mu}{2+\mu}\right) + \beta, & if \quad \gamma \leq \alpha(1-2\beta). \end{cases}$$

If the function $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$p(z) + \frac{zp'(z)}{\alpha p(z) + \gamma} \prec \frac{1 + (1-2\delta)z}{1-z},$$

then

$$p(z) \prec \frac{1 + (1-2\beta)z}{1-z}.$$

Proof. We consider the case $0 \leq \beta < 1$. The case $\beta > 1$ is similar. Define $q(z) = (p-\beta)/(1-\beta)$ or $(1-\beta)q + \beta = p(z)$. Then

$$p(z) + \frac{zp'(z)}{\alpha p(z) + \gamma} = (1-\beta)q(z) + \beta + \frac{(1-\beta)}{\alpha[(1-\beta)q(z) + \beta] + \gamma} zq'(z). \quad (2.17)$$

Define $\psi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\psi(r, s) = (1-\beta)r + \frac{(1-\beta)}{\alpha(1-\beta)r + \alpha\beta + \gamma} s + \beta - \delta.$$

Thus $\psi(r, s)$ is continuous and for $\rho \in \mathbb{R}$, $n \geq 1$ and σ satisfying (1.2), it follows that

$$\begin{aligned} \operatorname{Re}\psi(i\rho, \sigma) &= \operatorname{Re}\left[(1-\beta)i\rho + \frac{(1-\beta)}{\alpha(1-\beta)i\rho + \alpha\beta + \gamma}\sigma + \beta - \delta\right] \\ &= \frac{(1-\beta)(\alpha\beta+\gamma)}{(\alpha\beta+\gamma)^2 + \alpha^2(1-\beta)^2\rho^2}\sigma + \beta - \delta \\ &\leq \frac{(1-\beta)(\alpha\beta+\gamma)}{(\alpha\beta+\gamma)^2 + \alpha^2(1-\beta)^2\rho^2}\left[-\frac{1}{2}\left(n + \frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right] + \beta - \delta \\ &= \frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n + \frac{2-\mu}{2+\mu}\right)\left(\frac{1+\rho^2}{(\alpha\beta+\gamma)^2 + \alpha^2(1-\beta)^2\rho^2}\right) + \beta - \delta \end{aligned}$$

For $\gamma \leq \alpha(1-2\beta)$,

$$\begin{aligned} \operatorname{Re}\psi(i\rho, \sigma) &\leq \frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n + \frac{2-\mu}{2+\mu}\right)\frac{1}{\alpha^2(1-\beta)^2} + \beta - \delta \\ &= \frac{-1}{2}\frac{(\alpha\beta+\gamma)}{\alpha^2(1-\beta)}\left(n + \frac{2-\mu}{2+\mu}\right) + \beta - \delta. \end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$.

For $\gamma \geq \alpha(1-2\beta)$, the expression

$$\frac{1+\rho^2}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}$$

attains minimum at $\rho = 0$ and therefore

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma) &\leq \frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n+\frac{2-\mu}{2+\mu}\right)\frac{1}{(\alpha\beta+\gamma)^2}+\beta-\delta \\ &= \frac{-1}{2}\frac{(1-\beta)}{(\alpha\beta+\gamma)}\left(n+\frac{2-\mu}{2+\mu}\right)+\beta-\delta.\end{aligned}$$

Thus $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$, and result follows from Theorem 1.1. □

Lemma 2.6. *For $\beta \geq 0$, $\beta \neq 1$, $\gamma > 0$, and $0 < \mu \leq 2$. If the function $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination*

$$p^2(z)+\gamma z p'(z) \prec \frac{1+(1-2\delta)z}{1-z}, \tag{2.18}$$

where

$$\delta := -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\beta^2,$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. We consider the case $0 \leq \beta < 1$. The case $\beta > 1$ is similar. Define $q(z) = (p(z)-\beta)/(1-\beta)$ or $(1-\beta)q(z)+\beta = p(z)$. Using this it can be shown that inequality (2.18) can be written as

$$\operatorname{Re}\left[((1-\beta)q(z)+\beta)^2+\gamma(1-\beta)zq'(z)-\delta\right] > 0.$$

Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ be defined by

$$\psi(r,s) = [(1-\beta)r+\beta]^2+\gamma(1-\beta)s-\delta.$$

For $\rho \in \mathbb{R}$, $n \geq 1$ and σ satisfying (1.2), it follows that

$$\begin{aligned}
&\operatorname{Re}\psi(i\rho,\sigma)\\
&= \operatorname{Re}\left[((1-\beta)i\rho+\beta)^2+\gamma(1-\beta)\sigma-\delta\right]\\
&= -(1-\beta)^2\rho^2+\beta^2+\gamma(1-\beta)\sigma-\delta\\
&\leq \gamma(1-\beta)\left[-\frac{1}{2}\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)\right]+\beta^2-(1-\beta)^2\rho^2-\delta\\
&= -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)(1+\rho^2)-(1-\beta)^2(\rho^2+1)+(1-\beta)^2+\beta^2-\delta\\
&= -(1+\rho^2)\left[\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+(1-\beta)^2\right]+(1-\beta)^2+\beta^2-\delta\\
&\leq -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2-\mu}{2+\mu}\right)+\beta^2-\delta.
\end{aligned}$$

Hence $\operatorname{Re}\psi(i\rho,\sigma) \leq 0$, or $\psi \in \Psi_{\mu,n}$, and Theorem 1.1 implies $\operatorname{Re} q(z) > 0$ or equivalently $\operatorname{Re} p(z) > \beta$. $\square$

Proof of Theorem 2.1. For a given function $f \in \mathcal{A}_{n,b}$, let the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = zf'(z)/f(z)$. Then computation shows that $p(z) = 1 + nbz^n + \cdots \in \mathcal{H}_{\mu,n}$ where $\mu = nb$. Further calculations yield

$$\begin{aligned}
&\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right) = (1-\alpha)p(z)+\alpha p^2(z)+\alpha zp'(z),\\
&\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) = p(z)+zp'(z),\\
&(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) = p(z)+\alpha\frac{zp'(z)}{p(z)},\\
&1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)}.
\end{aligned}$$

Hence the result follows from Lemmas 2.1–2.4. $\square$

Proof of Theorem 2.2. Let $f \in \Sigma_{n,b}$, and define the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(0) = 1$ and $p(z) = -zf'(z)/f(z)$ for $z \in \mathbb{D}^*$. Then $p(z) = 1-(n+1)bz^{n+1}+\cdots \in \mathcal{H}_{\mu,n}$ with $\mu = -(n+1)b$. Simple computations shows that

$$
\begin{aligned}
&\frac{zf'(z)}{f(z)}\left(2\alpha\frac{zf'(z)}{f(z)}-\alpha\frac{zf''(z)}{f'(z)}-1\right)=(1-\alpha)p(z)+\alpha p^2(z)+\alpha zp'(z),\\
&-\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)=p(z)+zp'(z),\\
&\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-(1+\alpha)\frac{zf'(z)}{f(z)}=p(z)+\alpha\frac{zp'(z)}{p(z)},\\
&1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=\frac{zp'(z)}{p(z)}.
\end{aligned}
$$

Hence the result follows from Lemmas 2.1–2.4. □

Proof of Theorem 2.3. For $f\in\mathcal{A}_{n,b}$, let the function $p:\mathbb{D}\to\mathbb{C}$ be defined by $p(z)=f'(z)$. Then $p(z)=1+(n+1)bz^n+(n+2)a_{n+2}z^{n+1}+\cdots\in\mathcal{H}_{\mu,n}$ with $\mu=(n+1)b$. Also, we have the following:

$$
\begin{aligned}
&f'(z)\left(\alpha\left(\frac{zf''(z)}{f'(z)}+f'(z)-1\right)+1\right)=(1-\alpha)p(z)+\alpha p^2(z)+\alpha zp'(z),\\
&f'(z)+\alpha zf''(z)=p(z)+\alpha zp'(z),\\
&\alpha\frac{zf''(z)}{f'(z)}+f'(z)=p(z)+\alpha\frac{zp'(z)}{p(z)},\\
&\frac{zf''(z)}{f'(z)}=\frac{zp'(z)}{p(z)}.
\end{aligned}
$$

Hence the result follows from Lemmas 2.1–2.4. □

Remark 2.1.

(i) For $\beta=0$, the condition (2.9)–(2.12) gives a sufficient condition for close-to-convexity and hence for univalence.

(ii) If $\mu=2$, result (2.1) reduces to [21, Theorem 2.1]. If $\mu=2$, and $f'(z)$ is considered as $f(z)/z$, result (2.10) reduces to [21, Theorem 2.4]. Inequality (2.11) reduces to [24, Theorem 2, p. 182] in the case when $\mu=2$, $n=1$ and $\beta=1/2$. Furthermore, if $\mu=2$, $n=1$ and $\beta=(\alpha+1)/2$, result (2.12) reduces to [16, Theorem 1].

Acknowledgements. The research of the first and last authors are supported respectively by FRGS grant and MyBrain MyPhD programme of the Ministry of Higher Education, Malaysia. The authors are thankful to the referee for his useful comments.

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