Advances in Applied Mathematical Analysis.
ISSN 0973-5313 Volume 9 Number 1 (2014), pp. 1-9

http://www.ripublication.com

A New Subclass of Harmonic Univalent Functions of Complex Order based on Convolution

Mugur Acu[1], R. Ezhilarasi[2], T.V. Sudharsan[2], K.G. Subramanian[3]

[1]*Department of Mathematics and Informatics, University "Lucian Blaga" of Sibiu, Str. Dr. I. Ratiu, No. 5–7, 550012 – Sibiu, Romania*
Email : acu_mugur@yahoo.com
[2]*Department of Mathematics, SIVET College, Chennai – 600 073, India*
Email : ezhilarasi2008@ymail.com, tvsudharsan@rediffmail.com
[3]*School of Computer Sciences, Universiti Sains Malaysia 11800 Penang, Malaysia*
Email : kgsmani1948@yahoo.com

Abstract

In this paper, we introduce a new subclass of harmonic univalent functions of complex order defined by convolution which includes several well known subclasses of harmonic univalent functions as well as various new ones. We also derive the coefficient inequality, extreme points, distortion theorem, convolution conditions and convex combination for this class.

Keywords: Harmonic, complex order, univalent, convolution.

2000 Mathematics Subject Classification: 30C45, 30C50.

1. Introduction

Clunie and Sheil-Small [5] investigated the class S_H, consisting of complex-valued harmonic sense-preserving univalent functions f in a simply connected domain $D \subseteq C$ defined on the open unit disc $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = f_z(0) - 1 = 0$.

Each function $f \in S_H$ can be expressed as $f = h + \overline{g}$ where h and g are analytic in D and

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1 \tag{1.1}$$

The work initiated by Clunie and Sheil-Small on the class S_H formed the basis for several related papers on S_H and its subclasses (see for example Ahuja [1] and Duren [6]).

In this note, we introduce a new subclass $S_H(\phi, \psi, b, \lambda, \beta)$ of S_H consisting of functions $f = h + \overline{g} \in S_H$ that satisfy the condition

$$\operatorname{Re}\left\{\frac{h(z) * \phi(z) - \overline{g(z) * \psi(z)}}{z'[(1-\lambda)z + \lambda(h(z) + \overline{g(z)})]}\right\} > 1 - \beta|b|. \tag{1.2}$$

or equivalently,

$$\left|\frac{1}{b}\left[\frac{h(z) * \phi(z) - \overline{g(z) * \psi(z)}}{z'[(1-\lambda)z + \lambda(h(z) + \overline{g(z)})]} - 1\right]\right| < \beta. \tag{1.3}$$

where $0 < \beta \le 1$, $0 \le \lambda \le 1$, b, a non-zero complex number with $|b| \le 1$, $\phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$ and $\psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ are analytic in Δ with the conditions $\lambda_n \ge 0$, $\mu_n \ge 0$ and $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, $0 \le r < 1$, $0 \le \theta \le 2\pi$. The operator '*' stands for the Hadamard product or convolution of two power series.

We further let $S_{\overline{H}}(\phi, \psi, b, \lambda, \beta)$ denote the subclass of $S_H(\phi, \psi, b, \lambda, \beta)$ consisting of functions $f = h + \overline{g} \in S_H$ such that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \qquad |b_1| < 1 \tag{1.4}$$

Remark 1.1.

$S_{\overline{H}}(\phi, \psi, 1, 1, \beta) = TS_H(\phi, \psi, 1-\beta)$ [7].

$S_{\overline{H}}\left(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}, 1, 1, \beta\right) = T_H(1-\beta)$ [8].

Different subclasses of harmonic univalent functions based on convolution have been studied by several authors (see [2, 3, 4, 9]).

In this paper, we obtain coefficient bounds, extreme points and distortion bounds for functions in $S_{\overline{H}}(\phi, \psi, b, \lambda, \beta)$.

2 Coefficient Bounds

We now obtain a sufficient coefficient condition for harmonic functions in $S_H(\phi, \psi, b, \lambda, \beta)$.

Theorem 2.1.

Let $f = h + \overline{g}$, where h and g are given by (1.1). If

$$\sum_{n=2}^{\infty} \frac{[\lambda_n - \lambda(1-\beta|b|)]}{\beta|b|}|a_n| + \sum_{n=1}^{\infty} \frac{[\mu_n + \lambda(1-\beta|b|)]}{\beta|b|}|b_n| \le 1 \tag{2.1}$$

where $0 < \beta \le 1$, $0 \le \lambda \le 1$, b, a non-zero complex number with $|b| \le 1$,

$\phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$, $\psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ are analytic in Δ with the conditions $\lambda_n \geq 0$, $\mu_n \geq 0$, $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, $0 \leq r < 1$, $0 \leq \theta \leq 2\pi$ and $n\beta|b| \leq [\lambda_n - \lambda(1-\beta\,|b|)] \leq [\mu_n + \lambda(1-\beta\,|b|)]$ then f is harmonic univalent in Δ and $f \in S_H(\phi, \psi, b, \lambda, \beta)$.

Proof.

We first note that f is locally univalent and sense preserving in Δ. This is because for $|b| \leq 1$ and from the hypothesis of Theorem 2.1,

$$\begin{aligned} h'(z) &\geq 1 - \sum_{n=2}^{\infty} n|a_n|r^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n| \\ &\geq 1 - \sum_{n=2}^{\infty} \frac{\lambda_n - \lambda(1-\beta|b|)}{\beta|b|}|a_n| \\ &\geq \sum_{n=1}^{\infty} \frac{\mu_n + \lambda(1-\beta|b|)}{\beta|b|}|b_n| \\ &\geq \sum_{n=1}^{\infty} nb_n > \sum_{n=1}^{\infty} n|b_n|r^{n-1} \geq |g'(z)| \end{aligned}$$

To show that f is univalent in Δ, we show that $f(z_1) \neq f(z_2)$ whenever $z_1 \neq z_2$. Suppose $z_1, z_2 \in \Delta$ so that $z_1 \neq z_2$. Since the unit disc Δ is simply connected and convex, we have $z(t) = (1-t)z_1 + t\, z_2$ in Δ where $0 \leq t \leq 1$. Then we write

$$f(z_2) - f(z_1) = \int_0^1 [(z_2 - z_1)h'(z(t)) + \overline{(z_2 - z_1)g'(z(t))}]\,dt.$$

On dividing throughout by $z_2 - z_1 \neq 0$ and taking only the real parts, we obtain

$$\begin{aligned} \mathrm{Re}\frac{f(z_2)-f(z_1)}{z_2 - z_1} &= \int_0^1 \mathrm{Re}\left[h'(z(t)) + \frac{\overline{z_2 - z_1}}{z_2 - z_1}\overline{g'(z(t))}\right]dt \\ &> \int_0^1 [\mathrm{Re}\,h'(z(t)) - |g'(z(t))|]\,dt \end{aligned} \tag{2.2}$$

On the other hand

$$\begin{aligned} \mathrm{Re}\,h'(z) - |g'(z)| \geq \mathrm{Re}\,h'(z) - \sum_{n=1}^{\infty} n|b_n| &\geq 1 - \sum_{n=2}^{\infty} n|a_n| - \sum_{n=1}^{\infty} n|b_n| \\ &\geq 1 - \sum_{n=2}^{\infty} \frac{[\lambda_n - \lambda(1-\beta|b|)]}{\beta|b|}|a_n| - \sum_{n=1}^{\infty} \frac{[\mu_n + \lambda(1-\beta|b|)]}{\beta|b|}|b_n| \\ &\geq 0, \qquad \text{by (2.1).} \end{aligned}$$

Therefore this together with inequality (2.2) implies the univalence of f. Next, we show that $f \in S_H(\phi, \psi, b, \lambda, \beta)$. To do so, we need to show that when (2.1) holds then (1.2) also holds true.

Using the fact that $\mathrm{Re}\,\omega > \delta$ if and only if $|1 - \delta + \omega| > |1 + \delta - \omega|$, it suffices to show that

$$\left|(1+\beta|b|)[z'((1-\lambda)z+\lambda(h(z)+\overline{g(z)}))]+[[h(z)*\phi(z)-\overline{g(z)*\psi(z)}]-z'[(1-\lambda)z+\lambda(h(z)+\overline{g(z)})]]\right|$$

$$-\left|(1-\beta|b|)[z'((1-\lambda)z+\lambda(h(z)+\overline{g(z)}))]-[[h(z)*\phi(z)-\overline{g(z)*\psi(z)}]-z'[(1-\lambda)z+\lambda(h(z)+\overline{g(z)})]]\right|$$

$$>0$$

Now,

$$\left|(1+\beta|b|)[z'((1-\lambda)z+\lambda(h(z)+\overline{g(z)}))]+[[h(z)*\phi(z)-\overline{g(z)*\psi(z)}]-z'[(1-\lambda)z+\lambda(h(z)+\overline{g(z)})]]\right|$$

$$-\left|(1-\beta|b|)[z'((1-\lambda)z+\lambda(h(z)+\overline{g(z)}))]-[[h(z)*\phi(z)-\overline{g(z)*\psi(z)}]-z'[(1-\lambda)z+\lambda(h(z)+\overline{g(z)})]]\right|$$

$$=\left|(1+\beta|b|)z+\sum_{n=2}^{\infty}[\lambda_n+\lambda\beta|b|]a_nz^n-\sum_{n=1}^{\infty}[\mu_n-\lambda\beta|b|]b_n\overline{z}^n\right|$$

$$-\left|(1-\beta|b|)z-\sum_{n=2}^{\infty}[\lambda_n+\lambda\beta|b|-2\lambda]a_nz^n+\sum_{n=1}^{\infty}[\mu_n-\lambda\beta|b|+2\lambda]b_n\overline{z}^n\right|$$

$$\geq(1+\beta|b|)|z|-\sum_{n=2}^{\infty}[\lambda_n+\lambda\beta|b|]|a_n||z|^n-\sum_{n=1}^{\infty}[\mu_n-\lambda\beta|b|]|b_n||z|^n$$

$$-(1-\beta|b|)|z|-\sum_{n=2}^{\infty}[\lambda_n+\lambda\beta|b|-2\lambda]|a_n||z|^n-\sum_{n=1}^{\infty}[\mu_n-\lambda\beta|b|+2\lambda]|b_n||z|^n$$

$$=2\beta|b|-\sum_{n=2}^{\infty}2[\lambda_n-\lambda(1-\beta|b|)]|a_n||z|^{n-1}-\sum_{n=1}^{\infty}2[\mu_n+\lambda(1-\beta|b|)]|b_n||z|^{n-1}$$

$$>2\beta|b|\left\{1-\sum_{n=2}^{\infty}\frac{[\lambda_n-\lambda(1-\beta|b|)]}{\beta|b|}|a_n|-\sum_{n=1}^{\infty}\frac{[\mu_n+\lambda(1-\beta|b|)]}{\beta|b|}|b_n|\right\} \tag{2.3}$$

$$>0$$

This last expression is non-negative by (2.1), which completes the proof of the theorem.

The harmonic univalent functions

$$f(z)=z+\sum_{n=2}^{\infty}\frac{\beta|b|}{[\lambda_n-\lambda(1-\beta|b|)]}X_nz^n+\sum_{n=1}^{\infty}\frac{\beta|b|}{[\mu_n+\lambda(1-\beta|b|)]}\overline{Y_nz^n}, \tag{2.4}$$

where $\sum_{n=2}^{\infty}|X_n|+\sum_{n=1}^{\infty}|Y_n|=1$, shows that the coefficient bound given by (2.1) is sharp. This is because

$$\sum_{n=2}^{\infty}\frac{[\lambda_n-\lambda(1-\beta|b|)]}{\beta|b|}|a_n|+\sum_{n=1}^{\infty}\frac{[\mu_n+\lambda(1-\beta|b|)]}{\beta|b|}|b_n|$$

$$=\sum_{n=2}^{\infty}\frac{[\lambda_n-\lambda(1-\beta|b|)]}{\beta|b|}\frac{\beta|b|}{[\lambda_n-\lambda(1-\beta|b|)]}|X_n|+\sum_{n=1}^{\infty}\frac{[\mu_n+\lambda(1-\beta|b|)]}{\beta|b|}\frac{\beta|b|}{[\mu_n+\lambda(1-\beta|b|)]}|Y_n|$$

$$=\sum_{n=2}^{\infty}|X_n|+\sum_{n=1}^{\infty}|Y_n|=1. \qquad \square$$

We next show that the condition (2.1) is also necessary for functions of the form (1.4) to be in the class $S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$.

Theorem 2.2.

Let $f=h+\overline{g}$, where h and g are given by (1.4), then $f(z)\in S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[\lambda_n-\lambda(1-\beta|b|)]}{\beta|b|}|a_n|+\sum_{n=1}^{\infty}\frac{[\mu_n+\lambda(1-\beta|b|)]}{\beta|b|}|b_n|\le 1 \tag{2.5}$$

where $0 < \beta \le 1$, $0 \le \lambda \le 1$, b, a non-zero complex number with $|b| \le 1$, $\phi(z)=z+\sum_{n=2}^{\infty}\lambda_n z^n$, $\psi(z)=z+\sum_{n=2}^{\infty}\mu_n z^n$ are analytic in Δ with the conditions $\lambda_n \ge 0$, $\mu_n \ge 0$, $z'=\frac{\partial}{\partial\theta}(z=re^{i\theta})$, $0 \le r < 1$, $0 \le \theta \le 2\pi$ and $n\beta|b| \le [\lambda_n - \lambda(1-\beta|b|)] \le [\mu_n + \lambda(1-\beta|b|)]$ and $|b_n| > |a_n|$, for every $n \ge 2$.

Proof.

Since $S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)\subset S_H(\phi,\psi,b,\lambda,\beta)$, we only need to prove the only if part. We show that if (2.5) does not hold then f is not in $S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$. Let $f \in S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$ then from (1.2), we have

$$\operatorname{Re}\left\{\frac{h(z)*\phi(z)-\overline{g(z)*\psi(z)}}{z'[(1-\lambda)z+\lambda(h(z)+\overline{g(z)})]}\right\}>1-\beta|b|,$$

equivalently,

$$\operatorname{Re}\left\{\frac{h(z)*\phi(z)-\overline{g(z)*\psi(z)}-(1-\beta|b|)[z'[(1-\lambda)z+\lambda(h(z)+\overline{g(z)})]]}{z'[(1-\lambda)z+\lambda(h(z)+\overline{g(z)})]}\right\}$$

$$\ge \operatorname{Re}\frac{\beta|b|z-\sum_{n=2}^{\infty}[\lambda_n-\lambda(1-\beta|b|)]|a_n|z^n-\sum_{n=1}^{\infty}[\mu_n+\lambda(1-\beta|b|)]|b_n|\overline{z}^n}{z-\sum_{n=2}^{\infty}\lambda|a_n|z^n+\sum_{n=1}^{\infty}\lambda|b_n|\overline{z}^n}$$

$$\ge 0$$

The above condition must hold for all values of z, $|z| = r < 1$ and any b such that $0 < |b| < 1$. Choose z to be in the positive real axis where $z = r < 1$. Thus the above condition becomes for $|b_n| > |a_n|$, for every $n \ge 2$

$$\frac{\beta|b|-\sum_{n=2}^{\infty}[\lambda_n-\lambda(1-\beta|b|)]|a_n|r^{n-1}-\sum_{n=1}^{\infty}[\mu_n+\lambda(1-\beta|b|)]|b_n|r^{n-1}}{1-\sum_{n=2}^{\infty}\lambda|a_n|r^{n-1}+\sum_{n=1}^{\infty}\lambda|b_n|r^{n-1}}\ge 0 \tag{2.6}$$

If the condition (2.5) does not hold, then the numerator in (2.6) is negative for $r \to 1$. This contradicts (2.6). Hence the proof is complete. □

3 Distortion Bounds

In this section, distortion bounds for the class $S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$ are obtained.

Theorem 3.1.
Let the function f(z) of the form (1.4) be in the class $S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$. Then for $|z| = r < 1$,

$$|f(z)| \le (1+|b_1|)r + \left[\frac{\beta|b|}{\lambda_2-\lambda(1-\beta|b|)} - \frac{\mu_1+\lambda(1-\beta|b|)}{\lambda_2-\lambda(1-\beta|b|)}|b_1|\right]r^2 \tag{3.1}$$

and

$$|f(z)| \ge (1-|b_1|)r - \left[\frac{\beta|b|}{\lambda_2-\lambda(1-\beta|b|)} - \frac{\mu_1+\lambda(1-\beta|b|)}{\lambda_2-\lambda(1-\beta|b|)}|b_1|\right]r^2 \tag{3.2}$$

The equalities in (3.1) and (3.2) are attained for the function f(z) is given by

$$f(z) = (1+|b_1|)\bar{z} + \left[\frac{\beta|b|}{\lambda_2-\lambda(1-\beta|b|)} - \frac{\mu_1+\lambda(1-\beta|b|)}{\lambda_2-\lambda(1-\beta|b|)}|b_1|\right]\bar{z}^2$$

and

$$f(z) = (1-|b_1|)\bar{z} - \left[\frac{\beta|b|}{\lambda_2-\lambda(1-\beta|b|)} - \frac{\mu_1+\lambda(1-\beta|b|)}{\lambda_2-\lambda(1-\beta|b|)}|b_1|\right]\bar{z}^2.$$

Proof.
Let $f(z) \in S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$. Then, we have

$$\begin{aligned}
|f(z)| &\le (1+|b_1|)\,r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)\,r^n \\
&\le (1+|b_1|)\,r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)\,r^2 \\
&\le (1+|b_1|)\,r + \frac{\beta|b|}{\lambda_2-\lambda(1-\beta|b|)}\sum_{n=2}^{\infty}\left(\frac{[\lambda_2-\lambda(1-\beta|b|)]}{\beta|b|}|a_n| + \frac{[\lambda_2-\lambda(1-\beta|b|)]}{\beta|b|}|b_n|\right)r^2 \\
&\le (1+|b_1|)\,r + \frac{\beta|b|}{\lambda_2-\lambda(1-\beta|b|)}\sum_{n=2}^{\infty}\left(\frac{[\lambda_2-\lambda(1-\beta|b|)]}{\beta|b|}|a_n| + \frac{[\mu_n+\lambda(1-\beta|b|)]}{\beta|b|}|b_n|\right)r^2 \\
&\le (1+|b_1|)\,r + \frac{\beta|b|}{\lambda_2-\lambda(1-\beta|b|)}\left(1 - \frac{[\mu_1+\lambda(1-\beta|b|)]}{\beta|b|}|b_1|\right)r^2, \quad \text{by}\,(2.5) \\
&\le (1+|b_1|)\,r + \left(\frac{\beta|b|}{\lambda_2-\lambda(1-\beta|b|)} - \frac{[\mu_1+\lambda(1-\beta|b|)]}{\lambda_2-\lambda(1-\beta|b|)}|b_1|\right)r^2
\end{aligned}$$

The lower bound can be similarly proved. □

4 Extreme Points

In this section, we determine the extreme points of the closed convex hull clco $S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$ of $S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$.

Theorem 4.1.
Let f(z) be given by (1.4). Then $f \in S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$ if and only if

$f(z)=\sum_{n=1}^{\infty}(X_n h_n(z)+Y_n g_n(z))$, where $h_1(z) = z$, $h_n(z)=z-\frac{\beta|b|}{\lambda_n-\lambda(1-\beta|b|)}z^n$, $n \ge 2$ and $g_n(z)=z+\frac{\beta|b|}{\mu_n+\lambda(1-\beta|b|)}\bar{z}^n$, $n \ge 1$, $X_n \ge 0$, $Y_n \ge 0$, $\sum_{n=1}^{\infty}(X_n+Y_n)=1$.

Proof.
Let

$$f(z)=\sum_{n=1}^{\infty}(X_n h_n(z)+Y_n g_n(z))$$

$$=\sum_{n=1}^{\infty}(X_n+Y_n)z-\sum_{n=2}^{\infty}\frac{\beta|b|}{\lambda_n-\lambda(1-\beta|b|)}X_n z^n+\sum_{n=1}^{\infty}\frac{\beta|b|}{\mu_n+\lambda(1-\beta|b|)}Y_n\bar{z}^n$$

$$f(z)=z-\sum_{n=2}^{\infty}\frac{\beta|b|}{\lambda_n-\lambda(1-\beta|b|)}X_n z^n+\sum_{n=1}^{\infty}\frac{\beta|b|}{\mu_n+\lambda(1-\beta|b|)}Y_n\bar{z}^n$$

From (2.5), we have

$$\sum_{n=2}^{\infty}\frac{[\lambda_n-\lambda(1-\beta|b|)]}{\beta|b|}|a_n|+\sum_{n=1}^{\infty}\frac{[\mu_n+\lambda(1-\beta|b|)]}{\beta|b|}|b_n|$$

$$=\sum_{n=2}^{\infty}\frac{[\lambda_n-\lambda(1-\beta|b|)]}{\beta|b|}\frac{\beta|b|}{[\lambda_n-\lambda(1-\beta|b|)]}X_n+\sum_{n=1}^{\infty}\frac{[\mu_n+\lambda(1-\beta|b|)]}{\beta|b|}\frac{\beta|b|}{[\mu_n+\lambda(1-\beta|b|)]}Y_n$$

$$=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n=\sum_{n=1}^{\infty}(X_n+Y_n)-X_1\le 1$$

then $f\in S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$.

Conversely, if $f\in S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$, then

$$X_n=\frac{[\lambda_n-\lambda(1-\beta|b|)]}{\beta|b|}|a_n|,\quad n\ge 2 \qquad \text{and}$$

$$Y_n=\frac{[\mu_n+\lambda(1-\beta|b|)]}{\beta|b|}|b_n|,\quad n\ge 1$$

where $\sum_{n=1}^{\infty}(X_n+Y_n)=1$. Then

$$f(z)=h(z)+\overline{g(z)}=z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\bar{z}^n$$

$$=z-\sum_{n=2}^{\infty}\frac{\beta|b|}{[\lambda_n-\lambda(1-\beta|b|)]}X_n z^n+\sum_{n=1}^{\infty}\frac{\beta|b|}{[\mu_n+\lambda(1-\beta|b|)]}Y_n\bar{z}^n$$

$$=z+\sum_{n=2}^{\infty}(h_n(z)-z)X_n+\sum_{n=1}^{\infty}(g_n(z)-z)Y_n$$

$$=\sum_{n=1}^{\infty}(X_n h_n(z)+Y_n g_n(z)).$$

This completes the proof of Theorem 4.1. □

Now, we prove that the class $S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$ is closed under convex combinations.

Theorem 4.2.

Let $0 \le t_i \le 1$ for i = 1, 2, ... and $\sum_{i=1}^{\infty} t_i = 1$. If the functions $f_i(z)$ defined by

$$f_i(z) = z - \sum_{n=2}^{\infty} \left|a_{n,i}\right| z^n + \sum_{n=1}^{\infty} \left|b_{n,i}\right| \overline{z}^n, \qquad (z \in \Delta, i = 1,2,...) \tag{4.1}$$

are in the class $S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$, then $\sum_{i=1}^{\infty} t_i f_i(z)$ of the form

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} c_i \left|a_{n,i}\right|\right) z^n + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i \left|b_{n,i}\right|\right)\overline{z}^n \tag{4.2}$$

is in the class $S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$.

Proof.

Since $f_i(z) \in S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$, it follows from Theorem 2.2 that

$$\sum_{n=2}^{\infty} \frac{\beta|b|}{[\lambda_n - \lambda(1-\beta|b|)]}\left|a_{n,i}\right| + \sum_{n=1}^{\infty} \frac{\beta|b|}{[\mu_n + \lambda(1-\beta|b|)]}\left|b_{n,i}\right| \le 1 \tag{4.3}$$

for every i = 1, 2, ... Hence

$$\sum_{n=2}^{\infty}\left(\frac{[\lambda_n - \lambda(1-\beta|b|)]}{\beta|b|}\sum_{i=1}^{\infty} t_i\left|a_{n,i}\right|\right) + \sum_{n=1}^{\infty}\left(\frac{[\mu_n + \lambda(1-\beta|b|)]}{\beta|b|}\sum_{i=1}^{\infty} t_i\left|b_{n,i}\right|\right)$$

$$= \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty}\frac{[\lambda_n - \lambda(1-\beta|b|)]}{\beta|b|}\left|a_{n,i}\right| + \sum_{n=1}^{\infty}\frac{[\mu_n + \lambda(1-\beta|b|)]}{\beta|b|}\left|b_{n,i}\right|\right)$$

$$\le \sum_{i=1}^{\infty} t_i \le 1.$$

By Theorem 2.2, it follows that $\sum_{i=1}^{\infty} t_i f_i(z) \in S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$. This proves that the class $S_{\overline{H}}(\phi,\psi,b,\lambda,\beta)$ is closed under convex combinations. □

References

[1] Ahuja, O.P, 2005, "Planar harmonic univalent and related mappings", J. Inequal. Pure Appl. Math., 6(4), Art. 122, 18pp.

[2] Ali, R.M., Adolf Stephen, B., and Subramanian, K.G., 2010, “Subclasses of harmonic mappings defined by convolution”, Applied Mathematics Letters, 23, pp. 1243−1247.

[3] Aouf, M.K, Mostafa, A.O., Shamandy, A.A., and Wagdy, A.K., 2012, “A study on certain class of harmonic functions of complex order associated with convolution”, LE MATEMATICHE, LXVII, Fasc. II, pp. 169−183.

[4] Breaz, D., Murugusundaramoorthy, G., and Vijaya, K., 2010, “Certain class of λ starlike harmonic functions associated with a convolution structure”, Studia Univ “Babes-Bolyai”, Mathematica, LV(3), pp. 69–82.

[5] Clunie, J., and Sheil-Small, T., 1984, “Harmonic univalent functions”, Ann. Acad. Sci. Fenn. Ser. A I Math. 9, pp. 3–25.

[6] Duren, P.L., 1992, “A survey of harmonic mappings in the plane”, Texas, Tech. Univ. Math. Ser., 18, pp. 1−15.

[7] Frasin, B.A., 2006, “Comprehensive family of harmonic univalent functions”, SUT Journal of Mathematics, 42(1), pp. 145−155.

[8] Jahangiri, J.M., 1999, “Harmonic functions starlike in the unit disc”, J. Math. Anal. Appl., 235, pp. 470–477.

[9] Murugusundaramoorthy, G., Uma, K., and Acu, M., 2010, “Harmonic functions prestarlike in the unit disc”, General Mathematics, 18(4), pp. 149−161.