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A Class of Harmonic Multivalent Functions Defined by an Integral Operator

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Abstract

A new class of harmonic multivalent functions defined by an integral operator is introduced. Coefficient inequalities, extreme points, distortion bounds, inclusion results and closure under an integral operator for this class are obtained.

Keywords: *Harmonic functions, multivalent functions, integral operator.*

1 Introduction

Harmonic mappings are important in different applied fields of study [1]. Harmonic mappings in a simply connected domain $D \subseteq C$ are univalent complex valued harmonic functions $f = u + iv$ where both u and v are real harmonic in D.

Let S_H denote the family of harmonic functions $f = h + \overline{g}$ [6], which are univalent and sense-preserving in the open unit disc $\Delta = \{z : |z| < 1\}$ where h and g are analytic in D and f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Subclasses of harmonic functions have been studied by many authors (See for example, Aouf et al. [2], Atshan and Kulkarni [3], Chandrashekar et al. [5], Cotîrlă [7], Jahangiri [9, 10], Jahangiri and Ahuja [11], Jahangiri et al. [12]).

The class $H_p(n)$ $(p, n \in N = \{1, 2, \dots\})$, consisting of all p-valent harmonic functions $f = h + \overline{g}$ that are orientation preserving in Δ was defined by Ahuja and Jahangiri [11] where h and g are of the form

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \quad |b_p| < 1. \qquad (1)$$

An integral operator I^n was introduced by Salagean [14] which is given below in a slightly modified form as stated by [7].

(i) $I^0 f(z) = f(z)$;

(ii) $I^1 f(z) = If(z) = p \int_0^z f(t) t^{-1} dt$;

(iii) $I^n f(z) = I(I^{n-1} f(z))$, $n \in N$, $f \in A$
where $A = \{f \in H : f(z) = z + a_2 z^2 + \dots\}$ and $H = H(\Delta)$, the class of holomorphic functions in Δ.

The modified Salagean integral operator of $f = h + \overline{g}$ given by (1) is defined [7] as

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}, \qquad (2)$$

where

$$I^n h(z) = z^p + \sum_{k=2}^{\infty} \left(\frac{p}{k+p-1} \right)^n a_{k+p-1} z^{k+p-1} \quad \text{and}$$
$$I^n g(z) = \sum_{k=1}^{\infty} \left(\frac{p}{k+p-1} \right)^n b_{k+p-1} z^{k+p-1}$$

For $0 \leq \beta < 1$, $0 \leq t \leq 1$, $n \in N$, $z \in \Delta$, let $H_p(n, \beta, t)$ denote the family of harmonic functions of the form (1) such that

$$Re \left(\frac{I^n f(z)}{(1-t) z^p + t I^{n+1} f(z)} \right) > \beta \qquad (3)$$

where I^n is defined by (2).

Let $\overline{H_p}(n, \beta, t)$ denote the subclass consists of harmonic functions $f_n = h + \overline{g_n}$ in $H_p(n, \beta, t)$ so that h and g_n are of the form

$$h(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} \text{ and } g_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1} \qquad (4)$$

where $a_{k+p-1}, b_{k+p-1} \geq 0$ and $|b_p| < 1$.

Remark 1.1 *The class* $\overline{H_p}(n, \beta, t)$ *reduces to the class* $\overline{H_p}(n, \beta)$ [7] *and to the class* $\overline{H_p}(n+1, n, \beta, 0)$ [8], *when* $t = 1$.

Coefficient inequalities, extreme points, distortion bounds, inclusion results and closure under an integral operator for functions in the class $\overline{H_p}(n, \beta, t)$ are obtained.

2 Main Results

A sufficient coefficient condition for harmonic functions belonging to the class $H_p(n, \beta, t)$ is now derived.

Theorem 2.1 *Let* $f = h + \overline{g}$ *be given by* (1). *If*

$$\sum_{k=2}^{\infty} \phi(n,p,k,\beta,t)|a_{k+p-1}| + \sum_{k=1}^{\infty} \psi(n,p,k,\beta,t)|b_{k+p-1}| \leq 1 \qquad (5)$$

where

$$\phi(n,p,k,\beta,t) = \frac{\left(\frac{p}{k+p-1}\right)^n \left[1 - \beta t \left(\frac{p}{k+p-1}\right)\right]}{1-\beta}$$

$$\psi(n,p,k,\beta,t) = \frac{\left(\frac{p}{k+p-1}\right)^n \left[1 + \beta t \left(\frac{p}{k+p-1}\right)\right]}{1-\beta},$$

$0 \leq \beta < 1$, $0 \leq t \leq 1$, $n \in N$. *Then* $f \in H_p(n, \beta, t)$.

Proof. To show that $f \in H_p(n, \beta, t)$ according to the condition (3), we only need to show that if (5) holds, then

$$Re \left\{ \frac{I^n f(z)}{(1-t)z^p + tI^{n+1} f(z)} \right\} = Re \frac{A(z)}{B(z)} \geq \beta$$

where $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, $0 \leq r < 1$ and $0 \leq \beta < 1$.
Note that $A(z) = I^n f(z)$ and
$B(z) = (1-t)z^p + tI^{n+1} f(z)$.
Using the fact that $Re\ w \geq \beta$ if and only if $|1 - \beta + w| \geq |1 + \beta - w|$, it suffices to show that

$$|A(z) + (1-\beta)B(z)| - |A(z) - (1+\beta)B(z)| \geq 0 \qquad (6)$$

Substituting $A(z)$ and $B(z)$ in (6) we obtain

$$
\begin{aligned}
&|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)|\\
&=\left|I^n f(z)+(1-\beta)[(1-t)z^p+tI^{n+1}f(z)]\right|\\
&\qquad-\left|I^n f(z)-(1+\beta)[(1-t)z^p+tI^{n+1}f(z)]\right|\\
&=\Bigg|z^p+\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n a_{k+p-1}z^{k+p-1}+(-1)^n\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^n\overline{b_{k+p-1}z^{k+p-1}}\\
&\qquad+(1-\beta)\Bigg[(1-t)z^p+tz^p+t\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}z^{k+p-1}\\
&\qquad\qquad+t(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}\overline{b_{k+p-1}z^{k+p-1}}\Bigg]\Bigg|\\
&-\Bigg|z^p+\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n a_{k+p-1}z^{k+p-1}+(-1)^n\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^n\overline{b_{k+p-1}z^{k+p-1}}\\
&\qquad-(1+\beta)\Bigg[(1-t)z^p+tz^p+t\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}z^{k+p-1}\\
&\qquad\qquad+t(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}\overline{b_{k+p-1}z^{k+p-1}}\Bigg]\Bigg|\\
&=\Bigg|(2-\beta)z^p+\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n\left[1+(1-\beta)t\left(\frac{p}{k+p-1}\right)\right]a_{k+p-1}z^{k+p-1}\\
&\qquad-(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^n\left[1-(1-\beta)t\left(\frac{p}{k+p-1}\right)\right]\overline{b_{k+p-1}z^{k+p-1}}\Bigg|\\
&\qquad-\Bigg|-\beta z^p+\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n\left[1-(1+\beta)t\left(\frac{p}{k+p-1}\right)\right]a_{k+p-1}z^{k+p-1}\\
&\qquad-(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^n\left[1+(1+\beta)t\left(\frac{p}{k+p-1}\right)\right]\overline{b_{k+p-1}z^{k+p-1}}\Bigg|\\
&\geq(2-\beta)|z|^p-\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n\left[1+(1-\beta)t\left(\frac{p}{k+p-1}\right)\right]|a_{k+p-1}||z|^{k+p-1}\\
&\qquad-\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^n\left[1-(1-\beta)t\left(\frac{p}{k+p-1}\right)\right]|b_{k+p-1}||z|^{k+p-1}\\
&\qquad-\beta|z|^p-\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n\left[1-(1+\beta)t\left(\frac{p}{k+p-1}\right)\right]|a_{k+p-1}||z|^{k+p-1}\\
&\qquad-\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^n\left[1+(1+\beta)t\left(\frac{p}{k+p-1}\right)\right]|b_{k+p-1}||z|^{k+p-1}
\end{aligned}
$$

$$\geq 2(1-\beta)|z|^p - \sum_{k=2}^{\infty} \left(\frac{p}{k+p-1}\right)^n \left[1 + (1-\beta)t\left(\frac{p}{k+p-1}\right)\right.$$
$$\left. +1 - (1+\beta)t\left(\frac{p}{k+p-1}\right)\right] |a_{k+p-1}||z|^{k+p-1}$$
$$- \sum_{k=1}^{\infty} \left(\frac{p}{k+p-1}\right)^n \left[1 - (1-\beta)t\left(\frac{p}{k+p-1}\right)\right.$$
$$\left. +1 + (1+\beta)t\left(\frac{p}{k+p-1}\right)\right] |b_{k+p-1}||z|^{k+p-1}$$
$$\geq 2(1-\beta)|z|^p - \sum_{k=2}^{\infty} 2\left(\frac{p}{k+p-1}\right)^n \left[1 - \beta t\left(\frac{p}{k+p-1}\right)\right] |a_{k+p-1}||z|^{k+p-1}$$
$$- \sum_{k=1}^{\infty} 2\left(\frac{p}{k+p-1}\right)^n \left[1 + \beta t\left(\frac{p}{k+p-1}\right)\right] |b_{k+p-1}||z|^{k+p-1}$$

$$= 2(1-\beta)|z|^p \left[1 - \left\{\sum_{k=2}^{\infty} \frac{\left(\frac{p}{k+p-1}\right)^n \left[1 - \beta t\left(\frac{p}{k+p-1}\right)\right]}{1-\beta} |a_{k+p-1}||z|^{k-1}\right.\right.$$
$$\left.\left. + \sum_{k=1}^{\infty} \frac{\left(\frac{p}{k+p-1}\right)^n \left[1 + \beta t\left(\frac{p}{k+p-1}\right)\right]}{1-\beta} |b_{k+p-1}||z|^{k-1}\right\}\right]$$
$$\geq 2(1-\beta) \left[1 - \left\{\sum_{k=2}^{\infty} \frac{\left(\frac{p}{k+p-1}\right)^n \left[1 - \beta t\left(\frac{p}{k+p-1}\right)\right]}{1-\beta} |a_{k+p-1}|\right.\right.$$
$$\left.\left. + \sum_{k=1}^{\infty} \frac{\left(\frac{p}{k+p-1}\right)^n \left[1 + \beta t\left(\frac{p}{k+p-1}\right)\right]}{1-\beta} |b_{k+p-1}|\right\}\right]$$
$$\geq 0, \quad \text{by (5).}$$

This completes the proof.

The harmonic univalent functions

$$f(z) = z^p + \sum_{k=2}^{\infty} \frac{1}{\phi(n,p,k,\beta,t)} x_k z^{k+p-1} + \sum_{k=1}^{\infty} \frac{1}{\psi(n,p,k,\beta,t)} \overline{y_k z^{k+p-1}} \tag{7}$$

where $n \in N$ and $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, shows that the coefficient bound given by (5) is sharp.

This is because

$$\begin{aligned}
&\sum_{k=2}^{\infty}\phi(n,p,k,\beta,t)|a_{k+p-1}|+\sum_{k=1}^{\infty}\psi(n,p,k,\beta,t)|b_{k+p-1}|\\
&=\sum_{k=2}^{\infty}\phi(n,p,k,\beta,t)\frac{1}{\phi(n,p,k,\beta,t)}|X_k|+\sum_{k=1}^{\infty}\psi(n,p,k,\beta,t)\frac{1}{\psi(n,p,k,\beta,t)}|Y_k|\\
&=\sum_{k=2}^{\infty}|X_k|+\sum_{k=1}^{\infty}|Y_k|=1.
\end{aligned}$$

We now show that the condition (5) is also necessary for functions $f_n = h+\overline{g_n}$, where h and g_n are of the form (4).

Theorem 2.2 *Let $f_n = h+\overline{g_n}$ be given by (4). Then $f_n \in \overline{H_p}(n,\beta,t)$ if and only if*

$$\sum_{k=2}^{\infty}\phi(n,p,k,\beta,t)a_{k+p-1}+\sum_{k=1}^{\infty}\psi(n,p,k,\beta,t)b_{k+p-1}\leq 1. \tag{8}$$

where $0\leq\beta<1$, $0\leq t\leq 1$, $n\in N$, with $b_{k+p-1}>a_{k+p-1}$, for every $k\geq 2$.

Proof. We only need to prove the "only if" part of the theorem because $\overline{H_p}(n,\beta,t)\subset H_p(n,\beta,t)$. To this end, for functions f_n of the form (4), we notice that the condition

$$Re\left\{\frac{I^n f(z)}{(1-t)z^p+tI^{n+1}f(z)}\right\}>\beta$$

is equivalent to

$$Re\left\{\frac{\begin{array}{c}\left[(1-\beta)z^p-\displaystyle\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n\left[1-\beta t\left(\frac{p}{k+p-1}\right)\right]a_{k+p-1}z^{k+p-1}\right.\\ \left.+(-1)^{2n-1}\displaystyle\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^n\left[1+\beta t\left(\frac{p}{k+p-1}\right)\right]b_{k+p-1}\overline{z}^{k+p-1}\right]\end{array}}{\begin{array}{c}\left[z^p-t\displaystyle\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}z^{k+p-1}\right.\\ \left.+t(-1)^{2n}\displaystyle\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}b_{k+p-1}\overline{z}^{k+p-1}\right]\end{array}}\right\}$$
$$\geq 0 \tag{9}$$

We observe that the above required condition (9) must hold for all values of z in Δ. Choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we have for $b_{k+p-1} > a_{k+p-1}$, for every $k \geq 2$,

$$\frac{\begin{array}{l}\left[(1-\beta) - \sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n}\left[1-\beta t\left(\frac{p}{k+p-1}\right)\right] a_{k+p-1}r^{k-1}\right. \\ \left. - \sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n}\left[1+\beta t\left(\frac{p}{k+p-1}\right)\right] b_{k+p-1}r^{k-1}\right]\end{array}}{\begin{array}{l}\left[1 - t\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1} a_{k+p-1}r^{k-1}\right. \\ \left. +t\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1} b_{k+p-1}r^{k-1}\right]\end{array}} \geq 0 \quad (10)$$

If the condition (8) does not hold, then the expression in (10) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in (0, 1) for which the quotient in (10) is negative. This contradicts the required condition for $f_n \in \overline{H_p}(n,\beta,t)$ and this completes the proof.

The extreme points of closed convex hull of $\overline{H_p}(n,\beta,t)$, denoted by *clco* $\overline{H_p}(n,\beta,t)$ is now determined.

Theorem 2.3 *Let f_n be given by (4). Then $f_n \in \overline{H_p}(n,\beta,t)$ if and only if*

$$f_n(z) = \sum_{k=1}^{\infty}[x_{k+p-1}h_{k+p-1}(z) + y_{k+p-1}g_{n_{k+p-1}}(z)],$$

where $h_p(z) = z^p$, $h_{k+p-1}(z) = z^p - \dfrac{1}{\phi(n,p,k,\beta,t)}z^{k+p-1}$, $k = 2,3,\ldots,$
and $g_{n_{k+p-1}}(z) = z^p + (-1)^{n-1}\dfrac{1}{\psi(n,p,k,\beta,t)}\overline{z}^{k+p-1}$, $k = 1,2,3,\ldots.$
$x_{k+p-1} \geq 0$, $y_{k+p-1} \geq 0$, $x_p = 1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}$.
In particular, the extreme point of $\overline{H_p}(n,\beta,t)$ are $\{h_{k+p-1}\}$ and $\{g_{n_{k+p-1}}\}$.

Proof. Suppose

$$
\begin{aligned}
f_n(z) &= \sum_{k=1}^{\infty}[x_{k+p-1}h_{k+p-1}(z) + y_{k+p-1}g_{n_{k+p-1}}(z)] \\
&= \sum_{k=1}^{\infty}(x_{k+p-1} + y_{k+p-1})z^p - \sum_{k=2}^{\infty}\frac{1}{\phi(n,p,k,\beta,t)}x_{k+p-1}z^{k+p-1} \\
&\qquad + (-1)^{n-1}\sum_{k=1}^{\infty}\frac{1}{\psi(n,p,k,\beta,t)}y_{k+p-1}\overline{z}^{k+p-1} \\
&= z^p - \sum_{k=2}^{\infty}\frac{1}{\phi(n,p,k,\beta,t)}x_{k+p-1}z^{k+p-1} \\
&\qquad + (-1)^{n-1}\sum_{k=1}^{\infty}\frac{1}{\psi(n,p,k,\beta,t)}y_{k+p-1}\overline{z}^{k+p-1}
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{k=2}^{\infty}\phi(n,p,k,\beta,t)|a_{k+p-1}| + \sum_{k=1}^{\infty}\psi(n,p,k,\beta,t)|b_{k+p-1}| \\
&= \sum_{k=2}^{\infty}\phi(n,p,k,\beta,t)\left(\frac{1}{\phi(n,p,k,\beta,t)}x_{k+p-1}\right) \\
&\qquad + \sum_{k=1}^{\infty}\psi(n,p,k,\beta,t)\left(\frac{1}{\psi(n,p,k,\beta,t)}y_{k+p-1}\right) \\
&= \sum_{k=2}^{\infty}x_{k+p-1} + \sum_{k=1}^{\infty}y_{k+p-1} = 1 - x_p \leq 1.
\end{aligned}
$$

and so $f_n(z) \in clco\,\overline{H_p}(n,\beta,t)$.

Conversely, if $f_n(z) \in clco\,\overline{H_p}(n,\beta,t)$. Letting

$$
x_p = 1 - \sum_{k=2}^{\infty}x_{k+p-1} - \sum_{k=1}^{\infty}y_{k+p-1}.
$$

Set

$$
\begin{aligned}
x_{k+p-1} &= \phi(n,p,k,\beta,t)a_{k+p-1}, \quad k = 2,3,\ldots \text{ and} \\
y_{k+p-1} &= \psi(n,p,k,\beta,t)\overline{b_{k+p-1}}, \quad k = 1,2,\ldots.
\end{aligned}
$$

The required representations is obtained as

$$
\begin{aligned}
f_n(z) &= z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} \overline{b_{k+p-1}} \overline{z}^{k+p-1} \\
&= z^p - \sum_{k=2}^{\infty} \frac{1}{\phi(n,p,k,\beta,t)} x_{k+p-1} z^{k+p-1} \\
&+ (-1)^{n-1} \sum_{k=1}^{\infty} \frac{1}{\psi(n,p,k,\beta,t)} y_{k+p-1} \overline{z}^{k+p-1} \\
&= z^p - \sum_{k=2}^{\infty} [z^p - h_{k+p-1}(z)] x_{k+p-1} - \sum_{k=1}^{\infty} [z^p - g_{k+p-1}(z)] y_{k+p-1} \\
&= \left[1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}\right] z^p \\
&+ \sum_{k=2}^{\infty} x_{k+p-1} h_{k+p-1}(z) + \sum_{k=1}^{\infty} y_{k+p-1} g_{n_{k+p-1}}(z) \\
&= \sum_{k=1}^{\infty} [x_{k+p-1} h_{k+p-1}(z) + y_{k+p-1} g_{n_{k+p-1}}(z)]
\end{aligned}
$$

We now obtain the distortion bounds for functions in $\overline{H_p}(n, \beta, t)$.

Theorem 2.4 *Let $f_n \in \overline{H_p}(n, \beta, t)$. Then for $|z| = r < 1$ we have*

$$|f_n(z)| \leq (1 + b_p) r^p + \{\theta(n,p,k,\beta,t) - \Omega(n,p,k,\beta,t) b_p\} r^{n+p+1}$$

and

$$|f_n(z)| \geq (1 - b_p) r^p - \{\theta(n,p,k,\beta,t) - \Omega(n,p,k,\beta,t) b_p\} r^{n+p+1}$$

where

$$
\begin{aligned}
\theta(n,p,k,\beta,t) &= \frac{1 - \beta t}{\left(\frac{p}{p+1}\right)^n \left[1 - \left(\frac{p}{p+1}\right) \beta t\right]} \\
\Omega(n,p,k,\beta,t) &= \frac{1 + \beta t}{\left(\frac{p}{p+1}\right)^n \left[1 - \left(\frac{p}{p+1}\right) \beta t\right]}
\end{aligned}
$$

We prove the right hand side inequality for $|f_n|$. The proof for the left hand inequality is similar. Let $f_n \in \overline{H_p}(n, \beta, t)$ taking the absolute value of f_n then by Theorem 2.2, we obtain:

$$\begin{aligned}
|f_n(z)| &= \left| z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} \overline{b_{k+p-1}} \overline{z}^{k+p-1} \right| \\
&\leq r^p + \sum_{k=2}^{\infty} a_{k+p-1} r^{k+p-1} + \sum_{k=1}^{\infty} b_{k+p-1} r^{k+p-1} \\
&= r^p + b_p r^p + \sum_{k=2}^{\infty} (a_{k+p-1} + b_{k+p-1}) r^{k+p-1} \\
&\leq r^p + b_p r^p + \sum_{k=2}^{\infty} (a_{k+p-1} + b_{k+p-1}) r^{p+1} \\
&= (1 + b_p) r^p + \theta(n, p, k, \beta, t) \sum_{k=2}^{\infty} \frac{1}{\theta(n, p, k, \beta, t)} (a_{k+p-1} + b_{k+p-1}) r^{p+1} \\
&\leq (1 + b_p) r^p + \theta(n, p, k, \beta, t) r^{n+p+1} \\
&\qquad \left[\sum_{k=2}^{\infty} \phi(n, p, k, \beta, t) a_{k+p-1} + \psi(n, p, k, \beta, t) b_{k+p-1} \right] \\
&\leq (1 + b_p) r^p + \{\theta(n, p, k, \beta, t) - \Omega(n, p, k, \beta, t) b_p\} r^{n+p+1}.
\end{aligned}$$

3 Closure Property of the Class $\overline{H_p}(n, \beta, t)$

In the next two theorems, we prove that the class $\overline{H_p}(n, \beta, t)$ is invariant under convolution and convex combinations of its members.

The convolution of two harmonic functions,

$$f_n(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} \overline{z}^{k+p-1} \tag{11}$$

and

$$F_n(z) = z^p - \sum_{k=2}^{\infty} A_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} B_{k+p-1} \overline{z}^{k+p-1} \tag{12}$$

is defined as

$$\begin{aligned}
(f_n * F_n)(z) &= f_n(z) * F_n(z) \\
&= z^p - \sum_{k=2}^{\infty} a_{k+p-1} A_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} B_{k+p-1} \overline{z}^{k+p-1}
\end{aligned} \tag{13}$$

Using this definition, we first show that the class $\overline{H_p}(n, \beta, t)$ is closed under convolution.

Theorem 3.1 *For $0 \leq \alpha \leq \beta < 1$, $0 \leq t \leq 1$, let $f_n \in \overline{H_p}(n,\beta,t)$ and $F_n \in \overline{H_p}(n,\alpha,t)$. Then*

$$f_n * F_n \in \overline{H_p}(n,\beta,t) \subset \overline{H_p}(n,\alpha,t).$$

Proof. Let $f_n(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} \overline{z}^{k+p-1}$ be in $\overline{H_p}(n,\beta,t)$ and $F_n(z) = z^p - \sum_{k=2}^{\infty} A_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} B_{k+p-1} \overline{z}^{k+p-1}$ be in $\overline{H_p}(n,\alpha,t)$.

Then the convolution $f_n * F_n$ is given by (13). We wish to show that the coefficients of $f_n * F_n$ satisfy the required condition given in Theorem 2.2. For $F_n \in \overline{H_p}(n,\alpha,t)$, we note that $A_{k+p-1} < 1$ and $B_{k+p-1} < 1$. Now, for the convolution function $f_n * F_n$, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \phi(n,p,k,\alpha,t) a_{k+p-1} A_{k+p-1} + \sum_{k=1}^{\infty} \psi(n,p,k,\alpha,t) b_{k+p-1} B_{k+p-1} \\
&\leq \sum_{k=2}^{\infty} \phi(n,p,k,\alpha,t) a_{k+p-1} + \sum_{k=1}^{\infty} \psi(n,p,k,\alpha,t) b_{k+p-1} \\
&\leq \sum_{k=2}^{\infty} \phi(n,p,k,\beta,t) a_{k+p-1} + \sum_{k=1}^{\infty} \psi(n,p,k,\beta,t) b_{k+p-1} \\
&\leq 1,
\end{aligned}$$

since $0 \leq \alpha \leq \beta < 1$ and $f_n \in \overline{H_p}(n,\beta,t)$.

Now, we show that $\overline{H_p}(n,\beta,t)$ is closed under convex combination of its members.

Theorem 3.2 *The class $\overline{H_p}(n,\beta,t)$ is closed under convex combination.*

Proof. For $i = 1,2,3,\ldots$. Suppose $f_{n_i} \in \overline{H_p}(n,\beta,t)$, where f_{n_i} is given by

$$f_{n_i}(z) = z^p - \sum_{k=2}^{\infty} a_{i,k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} b_{i,k+p-1} \overline{z}^{k+p-1}.$$

Then by (8)

$$\sum_{k=2}^{\infty} \phi(n,p,k,\beta,t) a_{i,k+p-1} + \sum_{k=1}^{\infty} \psi(n,p,k,\beta,t) b_{i,k+p-1} \leq 1 \tag{14}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \le t_i \le 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z^p - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{i,k+p-1}\right) z^{k+p-1} + (-1)^{n-1}\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{i,k+p-1}\right)\overline{z}^{k+p-1}$$

Using the inequality (14), we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty}\phi(n,p,k,\beta,t)\left(\sum_{i=1}^{\infty} t_i a_{i,k+p-1}\right) + \sum_{k=1}^{\infty}\psi(n,p,k,\beta,t)\left(\sum_{i=1}^{\infty} t_i b_{i,k+p-1}\right)\\
&= \sum_{i=1}^{\infty} t_i\left(\sum_{k=2}^{\infty}\phi(n,p,k,\beta,t)a_{i,k+p-1} + \sum_{k=1}^{\infty}\psi(n,p,k,\beta,t)b_{i,k+p-1}\right)\\
&\le \sum_{i=1}^{\infty} t_i = 1,
\end{aligned}$$

which is the required coefficient condition.

Finally, we examine the closure property of the class $\overline{H_p}(n,\beta,t)$ under the generalized Bernardi-Libera-Livingston integral operator (see [4, 13]) $\mathcal{L}_c(f)$ which is defined by,

$$\mathcal{L}_c(f) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt, \quad c > -1.$$

Theorem 3.3 *Let $f_n(z) \in \overline{H_p}(n,\beta,t)$. Then*

$$\mathcal{L}_c(f(z)) \in \overline{H_p}(n,\beta,t).$$

Proof. From the representation of $\mathcal{L}_c(f_n(z))$, it follows that

$$\begin{aligned}
\mathcal{L}_c(f_n(z)) &= \frac{c+p}{z^c}\int_0^z t^{c-1}\left[t^p - \sum_{k=2}^{\infty} a_{k+p-1}t^{k+p-1} + \overline{(-1)^{n-1}\sum_{k=1}^{\infty} b_{k+p-1}t^{k+p-1}}\right]dt\\
&= z^p - \sum_{k=2}^{\infty}\frac{c+p}{c+p+k-1}a_{k+p-1}z^{k+p-1}\\
&+ (-1)^{n-1}\sum_{k=1}^{\infty}\frac{c+p}{c+p+k-1}b_{k+p-1}z^{k+p-1}\\
&= z^p - \sum_{k=2}^{\infty} X_{k+p-1}z^{k+p-1} + (-1)^{n-1}\sum_{k=1}^{\infty} Y_{k+p-1}z^{k+p-1}
\end{aligned}$$

where

$$X_{k+p-1} = \frac{c+p}{c+p+k-1} a_{k+p-1} \quad \text{and}$$
$$Y_{k+p-1} = \frac{c+p}{c+p+k-1} b_{k+p-1}.$$

Hence

$$\sum_{k=2}^{\infty} \phi(n,p,k,\beta,t) \frac{c+p}{c+p+k-1} a_{k+p-1} + \sum_{k=1}^{\infty} \psi(n,p,k,\beta,t) \frac{c+p}{c+p+k-1} b_{k+p-1}$$
$$\leq \sum_{k=2}^{\infty} \phi(n,p,k,\beta,t) a_{k+p-1} + \sum_{k=1}^{\infty} \psi(n,p,k,\beta,t) b_{k+p-1}$$
$$\leq 1 \quad \text{by (8)}.$$

Hence by Theorem 2.2, $\mathcal{L}_c(f_n(z)) \in \overline{H_p}(n,\beta,t)$.

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References

[1] O.P. Ahuja, Planar harmonic univalent and related mappings, *JIPAM*, 6(4)(Article 122) (2005), 18.

[2] M.K. Aouf, A.O. Mostafa, A.A. Shamandy and A.K. Wagdy, A study on certain class of harmonic functions of complex order associated with convolution, *Le Matematiche*, LXVI(Fist. 11) (2012), 169-183.

[3] W.G. Atshan and S.R. Kulkarni, New classes of multivalently harmonic functions, *Int. Journal of Math. Analysis*, 2(3) (2008), 111-121.

[4] S.D. Bernardi, Convex and starlike univalent function, *Trans. Amer. Math. Soc.*, 135(1969), 429-446.

[5] R. Chandrashekar, G. Murugusundaramoorthy, S. K. Lee and K.G. Subramanian, A class of complex-valued harmonic functions defined by Dziok-Srivastava operator, *Chamchuri Journal of Mathematics*, 1(2) (2009), 31-42.

[6] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn., Ser. A.I., Math.*, 9(1984), 3-25.

[7] L.I. Cotîrlă, Harmonic multivalent functions defined by integral operator, *Studia Univ. Babeş - Bolyai Mathematica*, LIV(1) (2009), 65-74.

[8] L.I. Cotîrlă, A new class of Harmonic multivalent functions defined by an integral operator, *Acta Universitatis Apulensis*, 21(2010), 55-63.

[9] J.M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, *Ann. Uni. Mariae Curie-Sklodowska*, 52(1998), 57-66.

[10] J.M. Jahangiri, Harmonic functions starlike in the unit disc, *J. Math. Anal. Appl.*, 235(1999), 470-477.

[11] J.M. Jahangiri and O.P. Ahuja, Multivalent harmonic starlike functions, *Ann. Univ. Marie Curie-Sklodowska (Sect. A)*, LVI(2001), 1-13.

[12] J.M. Jahangiri, Y.C. Kim and H.M. Srivastava, Construction of a certain class of harmonic close-to-convex function associated with the Alexander integral transform, *Int. Trans. and Special Func.*, 14(3) (2003), 237-242.

[13] R.J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.*, 16(1965), 755-758.

[14] G.S. Salagean, Subclass of univalent functions, *Lecture Notes in Math.*, Springer-Verlag, 1013(1983), 362-372.