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HARMONIC UNIVALENT FUNCTIONS BASED ON A FRACTIONAL DIFFERENTIAL OPERATOR

R. EZHILARASI, T. V. SUDHARSAN, K. G. SUBRAMANIAN

ABSTRACT. Harmonic functions have been of importance in the areas of applied mathematics, engineering, and many others. In this paper, a class of harmonic functions $f(z) = h(z) + \overline{g(z)}$ that are harmonic univalent and sense-preserving in the unit disc, is introduced. The defining property of the class is based on a generalized fractional differential operator. Sufficient coefficient conditions for this class are obtained, which are also found to be necessary when the coefficients are negative. Properties such as extreme points, distortion bounds, covering theorem for functions in this class are also investigated.

1. INTRODUCTION

Harmonic mappings have found several applications in many diverse fields such as engineering, operations research and other allied branches of applied mathematics. Harmonic mappings have drawn the attention of function theorists, following the pioneering work of Clunie and Sheil-Small [3]. For more details related to this theory, one may refer the compact and comprehensive article by Ahuja [1]. A complex valued continuous function $w = f(z) = u(z) + iv(z)$ defined in a simply connected convex domain $D \subset C$ is harmonic in D if both u and v are real-valued harmonic functions on D. In a simply connected domain, we can write $f = h + \overline{g}$, where h and g are analytic in D. The function h is called the analytic part of f and g the co-analytic part of f. Clunie and Sheil-Small [3] observed that a necessary and sufficient condition for the harmonic function $f = h + \overline{g}$ to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ $(z \in D)$.

Let S_H denote the family of functions $f = h + \overline{g}$ that are harmonic, orientation preserving, and univalent in the open unit disc $U = \{z : |z| < 1\}$ with the normalization

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1}$$

An interesting class of harmonic functions $f = h + \overline{g}$ with negative coefficients which are starlike or convex was investigated by Silverman [8]. Different subclasses

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of the class S_H of harmonic functions have been studied by several authors (see for example [3, 5, 6, 7, 8, 9, 10, 11, 12]).

Al-Oboudi and Al-Amoudi [2] defined a linear multiplier fractional differential operator $D_\lambda^{m,\alpha}$ for $f(z) \in A$, in terms of the Gamma function, where A is the class of all analytic functions defined in the open unit disc U. The functions $f(z)$ are of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and the operator $D_\lambda^{m,\alpha}$ is defined as follows

$$D_\lambda^{m,\alpha} f(z) = z + \sum_{k=2}^{\infty} \phi_{k,m}(\alpha,\lambda) a_k z^k, \quad 0 \le \alpha < 1;\ m \in N_0 = \{0, 1, 2, \dots\}; \tag{2}$$

where,

$$\phi_{k,m}(\alpha,\lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} (1 + \lambda(k-1)) \right]^m .$$

Noor et al. [4] defined a generalized linear multiplier fractional differential operator $\tilde{D}_\lambda^{m,\alpha}$ for the harmonic function $f(z) = h(z) + \overline{g(z)}$ as

$$\tilde{D}_\lambda^{m,\alpha} f(z) = \sum_{k=1}^{\infty} \phi_{k,m}(\alpha,\lambda)(a_k z^k + \overline{b_k z^k}), \quad m \in N_0, \tag{3}$$

where

$$\phi_{k,m}(\alpha,\lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} (1 + \lambda(k-1)) \right]^m .$$

If the co-analytic part $g \equiv 0$, then we obtain the linear multiplier fractional differential operator $D_\lambda^{m,\alpha}$ (2).

In this paper, motivated by study in [4, 7], a new class $RH_\gamma(m, \alpha, \beta, \lambda)$ $(0 \le \alpha < 1, \beta \ge 0, 0 \le \gamma < 1, \lambda \ge 0)$ of harmonic univalent functions in $U = \{z : |z| < 1\}$ is introduced and studied.

2. The Class $RH_\gamma(m, \alpha, \beta, \lambda)$

Definition 2.1. *Let $f(z) = h(z) + \overline{g(z)}$ be a harmonic function, where $h(z)$ and $g(z)$ are given by (1). Then $f(z) \in RH_\gamma(m, \alpha, \beta, \lambda)$ if it satisfies*

$$Re\left\{ \beta \tilde{D}_\lambda^{m,\alpha} \left[\frac{h(z)}{z} + \overline{\left(\frac{g(z)}{z} \right)} \right] + \tilde{D}_\lambda^{m,\alpha} \left[h'(z) + \overline{g'(z)} \right] - \beta \right\} > \gamma, \tag{4}$$

for $0 \le \alpha < 1$, $\beta \ge 0$, $0 \le \gamma < 1$, $\lambda \ge 0$, $z \in D$.

$$\tilde{D}_\lambda^{m,\alpha} f(z) = \sum_{k=1}^{\infty} \phi_{k,m}(\alpha,\lambda)(a_k z^k + \overline{b_k z^k}), \quad m \in N_0$$

where

$$\phi_{k,m}(\alpha,\lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} (1 + \lambda(k-1)) \right]^m .$$

Definition 2.2. *Let $f(z) = h(z) + \overline{g(z)}$ be a harmonic function, where*

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = -\sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1, z \in D. \tag{5}$$

Then $f(z) \in RF_\gamma(m, \alpha, \beta, \lambda)$, *if it satisfies*

$$Re\left\{\beta\tilde{D}_\lambda^{m,\alpha}\left[\frac{h(z)}{z}+\overline{\left(\frac{g(z)}{z}\right)}\right]+\tilde{D}_\lambda^{m,\alpha}\left[h'(z)+\overline{g'(z)}\right]-\beta\right\}>\gamma,$$

for $0 \le \alpha < 1$, $\beta \ge 0$, $0 \le \gamma < 1$, $\lambda \ge 0$, $z \in D$.

Remark 2.1. *We note that* $RH_\gamma(m, \alpha, 0, \lambda) = H_\beta(m, \alpha, \lambda)$ [4].

Theorem 2.1. *Let* $f(z) = h(z) + \overline{g(z)}$ *where* $h(z)$ *and* $g(z)$ *are given by (1). Furthermore let*

$$\sum_{k=1}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)[|a_k|+|b_k|] \le 2+\beta-\gamma, \quad |b_1| < \frac{1-\gamma}{1+\beta} < 1 \tag{6}$$

where $a_1 = 1$, $\beta \ge 0$, $0 \le \alpha < 1$, $0 \le \gamma < 1$, $\lambda \ge 0$, $z \in D$.
Then $f(z)$ *is harmonic univalent and sense preserving in* U *and*
$f \in RH_\gamma(m, \alpha, \beta, \lambda)$.

Proof. Based on the technique used in ([5], Theorem 2.1), we prove that f is harmonic, sense preserving and univalent. For $|z_1| \le |z_2| < 1$, we have by using (6),

$$|f(z_1) - f(z_2)| \ge |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|$$

$$= \left|(z_1 - z_2) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)\right| - \left|\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)\right|$$

$$\ge |z_1 - z_2|\left[1 - \gamma - (1+\beta)|b_1| - |z_2|\sum_{k=2}^{\infty}[k+\beta]\phi_{k,m}(\alpha,\lambda)(|a_k| + |b_k|)\right]$$

$$\ge |z_1 - z_2|[1 - \gamma - (1+\beta)|b_1|](1 - |z_2|) > 0.$$

Consequently, f is univalent in U. We note that f is sense preserving in U, since by using (6), $|z| < 1$,

$$\begin{aligned}|h'(z)| &\ge 1 - \sum_{k=2}^{\infty} k|a_k| \ge 1 - \gamma - \sum_{k=2}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)|a_k| \\ &\ge \sum_{k=1}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)|b_k| \ge |g'(z)|.\end{aligned}$$

Now, we show that $f \in RH_\gamma(m, \alpha, \beta, \lambda)$, using the fact $Re\ w > \gamma$ iff $|1 + w - \gamma| > |1 - w + \gamma|$.

On substituting for $h(z)$ and $g(z)$ from (1) and on using (6), we obtain

$$\left|1+\beta\tilde{D}_{\lambda}^{m,\alpha}\left[\frac{h(z)}{z}+\overline{\left(\frac{g(z)}{z}\right)}\right]+\tilde{D}_{\lambda}^{m,\alpha}\left[h'(z)+\overline{g'(z)}\right]-\beta-\gamma\right|$$
$$-\left|1-\beta\tilde{D}_{\lambda}^{m,\alpha}\left[\frac{h(z)}{z}+\overline{\left(\frac{g(z)}{z}\right)}\right]-\tilde{D}_{\lambda}^{m,\alpha}\left[h'(z)+\overline{g'(z)}\right]+\beta+\gamma\right|$$
$$=\left|2-\gamma+(1+\beta)\bar{b}_1+\sum_{k=2}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)\left[a_k z^{k-1}+\overline{b_k z^{k-1}}\right]\right|$$
$$-\left|\gamma-(1+\beta)\bar{b}_1-\sum_{k=2}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)\left[a_k z^{k-1}+\overline{b_k z^{k-1}}\right]\right|$$
$$\geq 2-2\gamma-2(1+\beta)|b_1|-2\sum_{k=2}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)[|a_k|+|b_k|]|z|^{k-1}$$
$$>2\left[2+\beta-\gamma-\sum_{k=1}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)[|a_k|+|b_k|]\right]\geq 0.$$

The harmonic mapping

$$f(z)=z+\sum_{k=2}^{\infty}\frac{(1-\gamma)x_k}{k+\beta}z^k+\sum_{k=1}^{\infty}\frac{(1-\gamma)\overline{y}_k}{k+\beta}\overline{z}^k \tag{7}$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, $\beta\geq 0$ and $0\leq\gamma<1$, shows that the coefficient bound given by (6) is sharp. □

In the next theorem, we will prove a necessary and sufficient condition for functions f to belong to $RF_{\gamma}(m,\alpha,\beta,\lambda)$.

Theorem 2.2. *Let $f(z)=h(z)+\overline{g(z)}$, where $h(z)$ and $g(z)$ are given by (5) then $f\in RF_{\gamma}(m,\alpha,\beta,\lambda)$ if and only if*

$$\sum_{k=1}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)[|a_k|+|b_k|]\leq 2+\beta-\gamma \tag{8}$$

where $a_1=1$, $0\leq\alpha<1$, $\beta\geq 0$, $0\leq\gamma<1$ and $\lambda\geq 0$, $z\in D$.

Proof. Assume that $f\in RF_{\gamma}(m,\alpha,\beta,\lambda)$. Then we find from (4) with $h(z)$ and $g(z)$ given by (5) that

$$Re\Big[1-(1+\beta)|b_1|-\sum_{k=2}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)|a_k|z^{k-1}$$
$$-\sum_{k=2}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)|b_k|\overline{z}^{k-1}\Big]>\gamma$$

$(0\leq\gamma<1, 0\leq\alpha<1, \beta\geq 0$ and $|b_1|<1)$.
If we choose z to be real and let $z\to 1^-$, we get,

$$1-(1+\beta)|b_1|-\sum_{k=2}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)[|a_k|+|b_k|]\geq\gamma$$

which yield the assertion (8) of Theorem 2.2.

Conversely, assume that (8) holds true.

Then we find from (4) with $h(z)$ and $g(z)$ given by (5) that on using (8), for $|z| < 1$,

$$\begin{aligned} &Re\left\{\beta\tilde{D}_\lambda^{m,\alpha}\left[\frac{h(z)}{z}+\overline{\left(\frac{g(z)}{z}\right)}\right]+\tilde{D}_\lambda^{m,\alpha}\left[h'(z)+\overline{g'(z)}\right]-\beta\right\} \\ &= Re\Big[1-(1+\beta)|b_1|-\sum_{k=2}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)|a_k|z^{k-1} \\ &\qquad -\sum_{k=2}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)|b_k|\overline{z}^{k-1}\Big] \\ &\geq 2+\beta-\sum_{k=1}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)[|a_k|+|b_k|]|z|^{k-1} > \gamma \end{aligned}$$

This shows that $f \in RF_\gamma(m,\alpha,\beta,\lambda)$. □

Theorem 2.3. *Let $f \in RF_\gamma(m,\alpha,\beta,\lambda)$. Then for $|z| = r < 1$, we have*

$$|f(z)| \leq (1+|b_1|)r+\frac{(2-\alpha)^m}{(2+\beta)2^m(1+\lambda)^m}(1-(1+\beta)|b_1|-\gamma)r^2$$

and

$$|f(z)| \geq (1-|b_1|)r-\frac{(2-\alpha)^m}{(2+\beta)2^m(1+\lambda)^m}(1-(1+\beta)|b_1|-\gamma)r^2.$$

These bounds are sharp.

Proof. Let $f(z) = h(z)+\overline{g(z)}$, where $h(z)$ and $g(z)$ are defined in (5). Then

$$\begin{aligned} |f(z)| &= \left|z-\sum_{k=2}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\overline{z}^k\right| \\ &\geq (1-|b_1|)r-\frac{(2-\alpha)^m}{(2+\beta)2^m(1+\lambda)^m}\sum_{k=1}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)(|a_k|+|b_k|)r^2 \\ &\geq (1-|b_1|)r-\frac{(2-\alpha)^m}{(2+\beta)2^m(1+\lambda)^m}(1-\gamma-(1+\beta)|b_1|)r^2 \end{aligned}$$

The proof of upper bounds of $|f(z)|$ is similar.

Hence, these bounds are sharp and equalities occur if

$$f(z) = z+|b_1|\overline{z}+\frac{(2-\alpha)^m}{(2+\beta)2^m(1+\lambda)^m}(1-(1+\beta)|b_1|-\gamma)\overline{z}^2$$

and

$$f(z) = (1-|b_1|)z-\frac{(2-\alpha)^m}{(2+\beta)2^m(1+\lambda)^m}(1-(1+\beta)|b_1|-\gamma)z^2, \quad |b_1|<1.$$

□

The following result is due to the left hand side inequality in Theorem 2.3.

Corollary 2.1. *Let $f \in RF_\gamma(m,\alpha,\beta,\lambda)$. Then*

$$\left\{w : |w| < \frac{(1-|b_1|)(2+\beta)2^m(1+\lambda)^m-(2-\alpha)^m(1-(1+\beta)|b_1|-\gamma)}{(2+\beta)2^m(1+\lambda)^m}\right\} \subset f(U).$$

Here we determine a representation theorem for functions in $f \in RF_\gamma(m,\alpha,\beta,\lambda)$.

Theorem 2.4. *Let $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are defined in (5). Then $f \in RF_\gamma(m, \alpha, \beta, \lambda)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty} (\lambda_k h_k(z) + \mu_k g_k(z)), \quad z \in U$$

where

$$h_1(z) = z, \quad h_k(z) = z - \frac{(1-\gamma)}{(k+\beta)\phi_{k,m}(\alpha, \lambda)} z^k, \quad k = 2, 3, 4, \ldots.$$

$$g_k(z) = z - \frac{(1-\gamma)}{(k+\beta)\phi_{k,m}(\alpha, \lambda)} \overline{z}^k, \quad k = 1, 2, 3, \ldots.$$

and $\sum_{k=1}^{\infty} (\lambda_k + \mu_k) = 1$, $\lambda_k \geq 0$, $\mu_k \geq 0$.
In particular, the extreme points of $RF_\gamma(m, \alpha, \beta, \lambda)$ are $\{h_k\}$, $\{g_k\}$.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} (\lambda_k h_k(z) + \mu_k g_k(z)) \\ &= \sum_{k=1}^{\infty} (\lambda_k + \mu_k) z - \sum_{k=2}^{\infty} \frac{(1-\gamma)}{(k+\beta)\phi_{k,m}(\alpha, \lambda)} \lambda_k z^k - \sum_{k=1}^{\infty} \frac{(1-\gamma)}{(k+\beta)\phi_{k,m}(\alpha, \lambda)} \mu_k \overline{z}^k \\ &= z - \sum_{k=2}^{\infty} a_k z^k - \sum_{k=1}^{\infty} b_k \overline{z}^k \end{aligned}$$

where $a_1 = 1$ and

$$a_k = \frac{(1-\gamma)}{(k+\beta)\phi_{k,m}(\alpha, \lambda)} \lambda_k \, (k \geq 2),$$

$$b_k = \frac{(1-\gamma)}{(k+\beta)\phi_{k,m}(\alpha, \lambda)} \mu_k \, (k \geq 1).$$

Since

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{(k+\beta)\phi_{k,m}(\alpha, \lambda)}{(1-\gamma)} |a_k| + \sum_{k=1}^{\infty} \frac{(k+\beta)\phi_{k,m}(\alpha, \lambda)}{(1-\gamma)} |b_k| \\ &= \sum_{k=2}^{\infty} \frac{(k+\beta)\phi_{k,m}(\alpha, \lambda)}{(1-\gamma)} \left(\frac{(1-\gamma)}{(k+\beta)\phi_{k,m}(\alpha, \lambda)} \right) \lambda_k \\ &\quad + \sum_{k=1}^{\infty} \frac{(k+\beta)\phi_{k,m}(\alpha, \lambda)}{(1-\gamma)} \left(\frac{(1-\gamma)}{(k+\beta)\phi_{k,m}(\alpha, \lambda)} \right) \mu_k \\ &= \sum_{k=1}^{\infty} (\lambda_k + \mu_k) - \lambda_1 = 1 - \lambda_1 \leq 1 \end{aligned}$$

we have $\sum_{k=1}^{\infty} (k+\beta)\phi_{k,m}(\alpha, \lambda)(|a_k| + |b_k|) \leq 1 - \gamma + 1 + \beta = 2 + \beta - \gamma$ and so $f \in RF_\gamma(m, \alpha, \beta, \lambda)$.

Conversely, suppose that $f \in RF_\gamma(m,\alpha,\beta,\lambda)$.
Let

$$\lambda_k = \frac{(k+\beta)\phi_{k,m}(\alpha,\lambda)}{(1-\gamma)}|a_k|, \quad (k=2,3,\dots)$$
$$\mu_k = \frac{(k+\beta)\phi_{k,m}(\alpha,\lambda)}{(1-\gamma)}|b_k|, \quad (k=1,2,\dots)$$
$$\lambda_1 = 1-\sum_{k=2}^{\infty}\lambda_k-\sum_{k=1}^{\infty}\mu_k.$$

Then note that by Theorem 2.2, $0 \le \lambda_k \le 1$ $(k=2,3,\dots)$, $0 \le \mu_k \le 1$ $(k = 1,2,3,\dots)$ and $\lambda_1 \ge 0$. Consequently we obtain $f(z)=\sum_{k=1}^{\infty}(\lambda_k h_k(z)+\mu_k g_k(z))$ as required.

Now we study the invariance property of the class $RF_\gamma(m,\alpha,\beta,\lambda)$ under convolution and convex combination of its elements.
Let

$$f(z) = z-\sum_{k=2}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\overline{z}^k$$

and

$$F(z) = z-\sum_{k=2}^{\infty}|A_k|z^k-\sum_{k=1}^{\infty}|B_k|\overline{z}^k, \quad z\in U.$$

Then the convolution of f and F is given by

$$(f*F)(z) = f(z)*F(z).$$

This can be written as

$$(f*F)(z) = z-\sum_{k=2}^{\infty}|a_k||A_k|z^k-\sum_{k=1}^{\infty}|b_k||B_k|\overline{z}^k. \tag{9}$$

□

Theorem 2.5. *Let $f \in RF_\gamma(m,\alpha,\beta,\lambda)$ and $F \in RF_\delta(m,\alpha,\beta,\lambda)$ for $0 \le \delta \le \gamma < 1$. Then*

$$(f*F)(z) \in RF_\gamma(m,\alpha,\beta,\lambda) \subseteq RF_\delta(m,\alpha,\beta,\lambda).$$

Proof. The convolution of $f*F$ is defined by (9). We want to show that the coefficients of $f*F$ satisfy the condition given in (6).

For $F \in RF_\delta(m,\alpha,\beta,\lambda)$, we have $|A_k| \le 1$, $|B_k| \le 1$.

Now for the convolution $f*F$, we obtain

$$\begin{aligned}
&1+\sum_{k=2}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)|a_k||A_k|+\sum_{k=1}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)|b_k||B_k| \\
&\le 1+\sum_{k=2}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)|a_k|+\sum_{k=1}^{\infty}(k+\beta)\phi_{k,m}(\alpha,\lambda)|b_k| \\
&\le 2+\beta-\gamma \le 2+\beta-\delta.
\end{aligned}$$

Hence, we have the desired result. □

Theorem 2.6. *The family $RF_\gamma(m,\alpha,\beta,\lambda)$ is closed under convex combination.*

Proof. Let $f_i(z) \in RF_\gamma(m, \alpha, \beta, \lambda)$, where

$$f_i(z) = z - \sum_{k=2}^{\infty} |a_{k,i}| z^k - \sum_{k=1}^{\infty} |b_{k,i}| \overline{z}^k, \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Then for $\sum_{i=1}^{n} \mu_i = 1$, $0 \leq \mu \leq 1$, the convex combination of f_i may be written as:

$$\sum_{i=1}^{n} \mu_i f_i(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{n} \mu_i |a_{k,i}| \right) z^k - \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} \mu_i |b_{k,i}| \right) \overline{z}^k.$$

Using (6), we have

$$\begin{aligned} & 1 + \sum_{k=2}^{\infty} (k+\beta) \phi_{k,m}(\alpha, \lambda) \left(\sum_{i=1}^{n} \mu_i |a_{k,i}| \right) + \sum_{k=1}^{\infty} (k+\beta) \phi_{k,m}(\alpha, \lambda) \left(\sum_{i=1}^{n} \mu_i |b_{k,i}| \right) \\ & = \sum_{i=1}^{n} \mu_i \left(\sum_{k=1}^{\infty} (k+\beta) \phi_{k,m}(\alpha, \lambda) |a_{k,i}| + \sum_{k=1}^{\infty} (k+\beta) \phi_{k,m}(\alpha, \lambda) |b_{k,i}| \right) \\ & \leq 2 + \beta - \gamma. \end{aligned}$$

The proof of the theorem is completed. □

3. Conclusion

In this paper we have introduced a new class $RH_\gamma(m, \alpha, \beta, \lambda)$ of harmonic univalent functions using a generalized fractional differential operator. Various properties relating to the functions in this class have been obtained. Other properties such as connections of the harmonic functions in the class $RH_\gamma(m, \alpha, \beta, \lambda)$ with hypergeometric functions, can be explored.

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R. Ezhilarasi
Department of Mathematics, SIVET College, Chennai, 600 073 India
E-mail address: ezhilarasi2008@ymail.com

T. V. Sudharsan
Department of Mathematics, SIVET College, Chennai, 600 073 India
E-mail address: tvsudharsan@rediffmail.com

K. G. Subramanian
School of Computer Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malysia
E-mail address: kgsmani1948@yahoo.com