Coefficient Bounds for a Certain Class of Analytic and Bi-Univalent Functions

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Abstract. In this paper, we introduce and investigate a subclass of analytic and bi-univalent functions in the open unit disk \( U \). By using the Faber polynomial expansions, we obtain upper bounds for the coefficients of functions belonging to this analytic and bi-univalent function class. Some interesting recent developments involving other subclasses of analytic and bi-univalent functions are also briefly mentioned.

1. Introduction

Let \( \mathcal{A} \) denote the class of functions \( f(z) \) which are \textit{analytic} in the open unit disk

\[ U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \]

and normalized by the following Taylor-Maclaurin series expansion:

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \]  \hspace{1cm} (1.1)

Also let \( \mathcal{S} \) denote the subclass of functions in \( \mathcal{A} \) which are univalent in \( U \) (see, for details, [8]). It is well known that every function \( f \in \mathcal{S} \) has an inverse \( f^{-1} \), which is defined by

\[ f^{-1}(f(z)) = z \quad (z \in \mathbb{U}) \]

and

\[ f(f^{-1}(w)) = w \quad \left( |w| < r_0(f); \quad r_0(f) \geq \frac{1}{4} \right), \] \hspace{1cm} (1.2)

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according to the *Koebe One-Quarter Theorem* (see, for example, [8]). In fact, the inverse function \( f^{-1} \) is given by

\[
f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_3^3 - 5a_2a_3 + a_4)w^4 + \cdots.
\]

(1.3)

A function \( f \in A \) is said to be *bi-univalent* in \( \mathbb{U} \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \mathbb{U} \). Let \( \Sigma \) denote the class of analytic and bi-univalent functions in \( \mathbb{U} \) given by the Taylor-Maclaurin series expansion (1.1). Some examples of functions in the class \( \Sigma \) are presented below:

\[
\frac{z}{1 - z}, \quad \log(1 - z), \quad \frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right),
\]

and so on. However, the familiar Koebe function is not a member of the class \( \Sigma \). Other common examples of functions in \( \mathcal{S} \) such as

\[
z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1 - z^2}
\]

are also not members of the class \( \Sigma \).

For a brief history of functions in the class \( \Sigma \), see [22] (see also [4], [14], [18] and [25]). In fact, judging by the remarkable flood of papers on the subject (see, for example, [5–7, 9–12, 15–17, 19–21, 23, 26, 27, 29, 30]), the recent pioneering work of Srivastava et al. [22] appears to have revived the study of analytic and bi-univalent functions in recent years (see also [3], [13] and [24]).

The object of the present paper is to introduce a new subclass of the function class \( \Sigma \) and use the Faber polynomial expansion techniques to derive bounds for the general Taylor-Maclaurin coefficients \( |a_n| \) for the functions in this class. We also obtain estimates for the first two coefficients \( |a_2| \) and \( |a_3| \) of these functions.

2. Bounds Derivable by the Faber Polynomial Expansion Techniques

We begin by introducing the function class \( \mathcal{N}_{\Sigma}^{(\alpha, \lambda)} \) by means of the following definition.

**Definition.** A function \( f(z) \) given by (1.1) is said to be in the class \( \mathcal{N}_{\Sigma}^{(\alpha, \lambda)} \) \((0 \leq \alpha < 1; \ \lambda \geq 0)\) if the following conditions are satisfied:

\[
f \in \Sigma \quad \text{and} \quad \Re \{f'(z) + \lambda z f''(z)\} > \alpha \quad (z \in \mathbb{U}; \ 0 \leq \alpha < 1; \ \lambda \geq 0).
\]

(2.1)

By using the Faber polynomial expansions of functions \( f \in \mathcal{A} \) of the form (1.1), the coefficients of its inverse map \( g = f^{-1} \) may be expressed as follows (see [1] and [2]; see also [12]):

\[
g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n} (a_2, a_3, \ldots, a_n) w^n.
\]

(2.2)

where

\[
K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))(n-3)!} a_2^{n-3} a_3
\]

\[
\quad + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4
\]

\[
\quad + \frac{(-n)!}{(2(-n+2))(n-5)!} a_2^{n-5} \left[ a_5 + (-n+2)a_3 \right]
\]

\[
\quad + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \left[ a_6 + (-2n+5)a_3a_4 \right] + \sum_{j=7} a_j^{-j} V_j,
\]
where such expressions as (for example) \((-n)!\) are to be interpreted symbolically by
\[
(-n)! \equiv \Gamma(1-n) := (-n)(-n-1)(-n-2) \cdots (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \text{ (N := \{1, 2, 3, \ldots\}))} \tag{2.3}
\]
and \(V_j \ (7 \leq j \leq n)\) is a homogeneous polynomial in the variables \(a_2, a_3, \ldots, a_n\) (see, for details, [2]). In particular, the first three terms of \(K_{n-1}^n\) are given below:

\[
K_{1}^2 = -2a_2,
K_{2}^3 = 3(2a_3^2 - a_3),
K_{3}^4 = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).
\]

In general, an expansion of \(K_n^p\) is given by (see, for details, [1])
\[
K_n^p = pa_n + \frac{p(p-1)}{2}D_n^2 + \frac{p!}{(p-3)!5!}D_n^3 + \cdots + \frac{p!}{(p-n)!n!}D_n^n \quad (p \in \mathbb{Z})
\]

where
\[
Z := \{0, \pm 1, \pm 2, \ldots\} \quad \text{and} \quad D_n^p = D_n^p(a_2, a_3, \ldots)
\]
and, alternatively, by (see, for details, [28])
\[
D_n^m(a_1, a_2, \ldots, a_n) = \sum \left(\frac{m!}{\mu_1! \cdots \mu_n!}\right)a_1^{\mu_1} \cdots a_n^{\mu_n},
\]
where \(a_1 = 1\) and the sum is taken over all nonnegative integers \(\mu_1, \ldots, \mu_n\) satisfying the following conditions:

\[
\begin{cases}
\mu_1 + \mu_2 + \cdots + \mu_n = m \\
\mu_1 + 2\mu_2 + \cdots + n\mu_n = n.
\end{cases}
\]

It is clear that
\[
D_n^p(a_1, a_2, \ldots, a_n) = a_1^p.
\]

Our first main result is given by Theorem 1 below.

**Theorem 1.** Let \(f\) given by (1.1) be in the class \(\mathcal{N}_\lambda^{\alpha, \lambda}\) \((0 \leq \alpha < 1 \text{ and } \lambda \geq 0)\). If \(a_k = 0\) for \(2 \leq k \leq n-1\), then
\[
|a_n| \leq \frac{2(1-\alpha)}{n[1+\lambda(n-1)]} \quad (n \in \mathbb{N} \setminus \{1, 2\}). \tag{2.4}
\]

**Proof.** For analytic functions \(f\) of the form (1.1), we have
\[
f'(z) + \lambda z f''(z) = 1 + \sum_{n=2}^{\infty} [1 + \lambda(n-1)]n!a_n z^{n-1} \tag{2.5}
\]
and, for its inverse map $g = f^{-1}$, it is seen that
\[
g'(w) + \lambda wg''(w) = 1 + \sum_{n=2}^{\infty} [1 + \lambda(n-1)]nb_nw^{n-1}
\]
\[
= 1 + \sum_{n=2}^{\infty} [1 + \lambda(n-1)]K_{n-1}^{-n}(a_2, a_3, \cdots, a_n)w^{n-1}.
\] (2.6)

On the other hand, since \(f \in \mathcal{N}_{\Sigma}^{\alpha, \lambda}\) and \(g = f^{-1} \in \mathcal{N}_{\Sigma}^{\alpha, \lambda}\), by definition, there exist two positive real-part functions
\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n
\]
and
\[
q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n,
\]
where
\[
\Re\left(p(z)\right) > 0 \quad \text{and} \quad \Re\left(q(w)\right) > 0 \quad (z, w \in U),
\]
so that
\[
f'(z) + \lambda zf''(z) = \alpha + (1 - \alpha)p(z)
\]
\[
= 1 + (1 - \alpha) \sum_{n=1}^{\infty} K_n^1 (c_1, c_2, \cdots, c_n) z^n
\] (2.7)

and
\[
g'(w) + \lambda wg''(w) = \alpha + (1 - \alpha)q(w)
\]
\[
= 1 + (1 - \alpha) \sum_{n=1}^{\infty} K_n^1 (d_1, d_2, \cdots, d_n) w^n.
\] (2.8)

Thus, upon comparing the corresponding coefficients in (2.5) and (2.7), we get
\[
[1 + \lambda(n-1)]na_n = (1 - \alpha) c_{n-1}.
\] (2.9)

Similarly, by using (2.6) and (2.8), we find that
\[
[1 + \lambda(n-1)]K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) = (1 - \alpha) d_{n-1}.
\] (2.10)

We note that, for \(a_k = 0 \ (2 \leq k \leq n - 1)\), we have
\[
b_n = -a_n
\]
and so
\[
[1 + \lambda(n-1)]na_n = (1 - \alpha)c_{n-1}
\] (2.11)

and
\[
-[1 + \lambda(n-1)]na_n = (1 - \alpha)d_{n-1}.
\] (2.12)
Thus, according to the Carathéodory Lemma (see [8]), we also observe that
\[ |c_n| \leq 2 \quad \text{and} \quad |d_n| \leq 2 \quad (n \in \mathbb{N}). \]

Now, taking the moduli in (2.11) and (2.12) and applying the Carathéodory Lemma, we obtain
\[
|a_n| \leq \frac{(1 - \alpha) |c_{n-1}|}{n[1 + \lambda(n - 1)]} = \frac{(1 - \alpha) |d_{n-1}|}{n[1 + \lambda(n - 1)]} \leq \frac{2(1 - \alpha)}{n[1 + \lambda(n - 1)]},
\]
which evidently completes the proof of Theorem 1. \[\square\]

3. Estimates for the Initial Coefficients \(a_2\) and \(a_3\)

In this section, we choose to relax the coefficient restrictions imposed in Theorem 1 and derive the resulting estimates for the initial coefficients \(a_2\) and \(a_3\) of functions \(f \in \mathcal{N}_{\Sigma}^{\alpha,\lambda}\) given by the Taylor-Maclaurin series expansion (1.1). We first state the following theorem.

**Theorem 2.** Let \(f\) given by (1.1) be in the class \(\mathcal{N}_{\Sigma}^{\alpha,\lambda}\) \((0 \leq \alpha < 1 \text{ and } \lambda \geq 0)\). Then

\[
|a_2| \leq \begin{cases} \sqrt{\frac{2(1 - \alpha)}{3(1 + 2\lambda)}}, & 0 \leq \alpha < \frac{1 + 2\lambda - 2\lambda^2}{3(1 + 2\lambda)} \\ \frac{1 - \alpha}{1 + \lambda'}, & \frac{1 + 2\lambda - 2\lambda^2}{3(1 + 2\lambda)} \leq \alpha < 1 \end{cases}
\]

and

\[
|a_3| \leq \frac{2(1 - \alpha)}{3(1 + 2\lambda)}.
\]

**Proof.** If we set \(n = 2\) by and \(n = 3\) in (2.9) and (2.10), respectively, we obtain

\[
2(1 + \lambda)a_2 = (1 - \alpha)c_1,
\]
\[
3(1 + 2\lambda)a_3 = (1 - \alpha)c_2,
\]
\[
-2(1 + \lambda)a_2 = (1 - \alpha)d_1
\]
and

\[
3(1 + 2\lambda)(2a_2^2 - a_3) = (1 - \alpha)d_2.
\]

Upon dividing both sides of (3.3) or (3.5) by \(2(1 + \lambda)\), if we take their moduli and apply the Carathéodory Lemma, we find that

\[
|a_2| \leq \frac{(1 - \alpha) |c_1|}{2(1 + \lambda)} = \frac{(1 - \alpha) |d_1|}{2(1 + \lambda)} \leq \frac{1 - \alpha}{1 + \lambda'}.
\]

Now, by adding (3.4) to (3.6), we have

\[
6(1 + 2\lambda)a_2^2 = (1 - \alpha)(c_2 + d_2),
\]
that is,
\[ a_2^2 = \frac{(1 - \alpha)(c_2 + d_2)}{6(1 + 2\lambda)}. \] (3.9)

Another application of the Carathéodory Lemma followed by taking the square roots in this last equation (3.9) yields
\[ |a_2| \leq \sqrt{\frac{2(1 - \alpha)}{3(1 + 2\lambda)}}. \] (3.10)

which proves the first assertion (3.1) of Theorem 2.

Next, for
\[ \frac{1 + 2\lambda - 2\lambda^2}{3(1 + 2\lambda)} \leq \alpha < 1, \]

we note that
\[ \frac{1 - \alpha}{1 + \lambda} \leq \sqrt{\frac{2(1 - \alpha)}{3(1 + 2\lambda)}}. \] (3.11)

Thus, upon dividing both sides of (3.4) by 3(1 + 2\lambda), if we take the modulus of each side and apply the Carathéodory Lemma once again, we get
\[ |a_3| \leq \frac{2(1 - \alpha)}{3(1 + 2\lambda)} \] (3.12)

which completes the proof of the second assertion (3.2) of Theorem 2. \( \square \)

References


