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# On the Second Hankel Determinant for the $k$th-Root Transform of Analytic Functions

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**Abstract.** Let $f$ be a normalized analytic function in the open unit disk of the complex plane satisfying $zf'(z)/f(z)$ is subordinate to a given analytic function $\varphi$. A sharp bound is obtained for the second Hankel determinant of the $k$th-root transform $z\left[f\left(z^k\right)/z^k\right]^{\frac{1}{k}}$. Best bounds for the Hankel determinant are also derived for the $k$th-root transform of several other classes, which include the class of $\alpha$-convex functions and $\alpha$-logarithmically convex functions. These bounds are expressed in terms of the coefficients of the given function $\varphi$, and thus connect with earlier known results for particular choices of $\varphi$.

## 1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For $n = 1, 2, \ldots$ and $q = 1, 2, \ldots$, the *Hankel determinants* $H_q(n)$ of the function $f \in \mathcal{A}$ are defined by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

where $a_1 = 1$. It is evident that $H_2(1) = a_3 - a_2^2$ is the Fekete-Szegö coefficient functional of $f$. Interestingly the determinant also satisfies $H_2(1) = S_f(0)/6$, where $S_f$ is the Schwarzian derivative of $f$ defined by

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$S_f = (f''/f')' - (f''/f')^2/2$. The Hankel determinants play an importaant role in the study of singularities as well as in the study of power series with integral coefficients [8, 45, 46]. Several earlier investigations include those of [12–14, 21, 24, 33–43, 45, 46], while several recent works are those of [6, 7, 18–20, 22, 23, 25, 29, 31, 32]. In [25], Lee *et al.* provided a brief survey on the Hankel determinants and obtained bounds for $H_2(2)$ for functions belonging to several classes defined by subordination. In recent years, various interesting properties and characteristics including coefficient bounds and coefficient inequalities of many different subclasses of univalent and bi-univalent functions have been investigated. The technique used by Ma and Minda for the Fekete-Szego problem for subclasses of convex and starlike functions were used by many authors to solve the same problem for other classes. This technique was also used to solve Hankel determinant problem as well as the coefficient problem for bi-univalent functions [3, 9, 50–55]. The recent works on bi-univalent functions are motivated by the estimates of initial coefficients of certain subclasses of bi-univalent functions by Srivatava *et al.* [50].

For a univalent function $f$ of the form (1), the *kth-root transform* is defined by

$$F(z) = \left[f\left(z^k\right)\right]^{\frac{1}{k}} := z\left[f\left(z^k\right)/z^k\right]^{\frac{1}{k}}. \tag{2}$$

The $k$th-root transform has been widely used in a variety of ways in complex function theory. Since $f$ is univalent, whence $f\left(z^k\right)/z^k$ is non-vanishing in $\mathbb{D}$, the $k$th-root is an analytic function in $\mathbb{D}$. Not only does the $k$th-root transform preserves univalence, it is also known [16] to preserve boundedness and starlikeness. While the convexity of the $k$th-root transform of $f$ implies convexity of $f$, the converse however is false [16]. In [2], Ali *et al.* investigated the Fekete-Szegö coefficient functional for the $k$th-root transform of functions belonging to several classes defined via subordination. It follows from (2) that

$$F(z) = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1},$$

where the initial coefficients are

$$\begin{aligned} b_{k+1} &= \frac{1}{k}a_2, \quad b_{2k+1} = \frac{1}{k}a_3 + \frac{(1-k)}{2k^2}a_2^2 \\ b_{3k+1} &= \frac{1}{k}a_4 + \frac{(1-k)}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{3!k^3}a_2^3. \end{aligned} \tag{3}$$

The second Hankel determinant of the associated function $\tilde{F}(z) = z + \sum_{n=1}^{\infty} b_{kn+1} z^{n+1}$ is given by $H_2(2) = |b_{k+1}b_{3k+1} - b_{2k+1}^2|$, and this quantity is also known as the second Hankel determinant of the $k$th-root transform $F$.

Recall that an analytic function $f$ is *subordinate* to an analytic function $g$, written $f(z) \prec g(z)$, if there exists an analytic self-map $w$ of $\mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. In this paper, the best bounds for the second Hankel determinant of the $k$th-root transform are obtained for several classes of functions defined via subordination. These classes can be seen as belonging to the genre of Ma-Minda starlike functions, which will be made apparent in the next section. The results in this paper are derived through several meticulous lengthy computations, and thus in several instances, these computations were validated by use of Mathematica.

Closely related to the classes of functions treated in this paper is the class $\mathcal{P}$ consisting of analytic functions with $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$ and normalized by $p(0) = 1$. The following results will be required.

**Lemma 1.1.** [10] *If* $p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots \in \mathcal{P}$, *then*

$$|c_n| \leq 2.$$

*Further this bound is sharp.*

**Lemma 1.2.** [15, p.152] *If* $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \mathcal{P}$, *then*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \tag{4}$$

$$4c_3 = c_1^3 + 2c_1\left(4 - c_1^2\right)x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)y, \tag{5}$$

*for some* $x, y \in \overline{\mathbb{D}}$.

Another result that will be required is the optimal value of a quadratic expression. Standard computations show that

$$\max_{0\leqslant t\leqslant 4}\left(Lt^2 + Mt + N\right) = \begin{cases} \frac{4LN-M^2}{4L}, & M > 0,\ L \leqslant -\frac{M}{8}, \\ N, & M \leqslant 0,\ L \leqslant -\frac{M}{4}, \\ 16L + 4M + N, & M \geqslant 0,\ L \geqslant -\frac{M}{8} \text{ or} \\ & M \leqslant 0,\ L \geqslant -\frac{M}{4}. \end{cases} \tag{6}$$

## 2. The Second Hankel Determinant of the $k$th-Root Transform

This section introduces several classes of normalized analytic functions. For each class, a sharp bound is obtained for its second Hankel determinant.

**Definition 2.1.** *[27] Let* $\varphi \in \mathcal{P}$ *be given by*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0, z \in \mathbb{D}). \tag{7}$$

*Further assume that* $\varphi$ *maps* $\mathbb{D}$ *onto a region starlike with respect to* 1, *and* $\varphi(\mathbb{D})$ *is symmetric with respect to the real axis. The class* $\mathcal{S}^*(\varphi)$ *consists of functions* $f \in \mathcal{A}$ *satisfying* $zf'(z)/f(z) \prec \varphi(z)$.

In the literature, this class $\mathcal{S}^*(\varphi)$ is widely called the *Ma-Minda starlike functions with respect to* $\varphi$. For the particular case when $\varphi$ is given by

$$\varphi_\alpha(z) := \frac{1 + (1 - 2\alpha)z}{1 - z} = 1 + 2(1 - \alpha)z + 2(1 - \alpha)z^2 + 2(1 - \alpha)z^3 + \cdots, \quad 0 \le \alpha < 1,$$

the class $\mathcal{S}^*(\alpha) := \mathcal{S}^*(\varphi_\alpha)$ is the well-known class of *starlike functions of order* $\alpha$. For the function

$$\varphi_{PAR}(z) := 1 + \frac{2}{\pi^2}\left(\log\frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)^2 = 1 + \frac{8}{\pi^2}z + \frac{16}{3\pi^2}z^2 + \frac{184}{45\pi^2}z^3 + \cdots,$$

$\mathcal{S}^*(\varphi_{PAR})$ is the class $\mathcal{S}_p^*$ of *parabolic starlike* functions introduced by Rønning [48]:

$$\mathcal{S}_p^* := \left\{f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'z}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|\right\}.$$

Ali and Ravichandran [4] gave a survey on parabolic starlike functions and its related class of uniformly convex functions. When

$$\varphi_\beta(z) := \left(\frac{1 + z}{1 - z}\right)^\beta = 1 + 2\beta z + 2\beta^2 z^2 + \frac{2}{3}\beta(1 + 2\beta^2)z^3 + \cdots, \quad 0 < \beta \le 1,$$

the class $\mathcal{S}^*\left(\varphi_\beta\right)$ is the familiar class $\mathcal{S}_\beta^*$ of *strongly starlike functions of order* $\beta$:

$$\mathcal{S}_\beta^* := \left\{f \in \mathcal{A} : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2}\right\}.$$

The class $\mathcal{S}^*(\sqrt{1 + z})$ is the class of *lemniscate of Bernoulli starlike* functions studied in [49]:

$$\mathcal{S}_L^* := \left\{f \in \mathcal{A} : \left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1\right\}.$$

**Theorem 2.2.** *Let $\varphi$ be given by (7), $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(\varphi)$, and $F(z) = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}$ be its kth-root transform. Further let $\delta = 1/k^2$ .*

1. *If $B_1, B_2$ and $B_3$ satisfy the conditions*
$$|B_2| \leqslant B_1, \quad \textit{and} \quad |4B_1B_3 - \delta B_1^4 - 3B_2^2| - 3B_1^2 \leqslant 0,$$
*then the second Hankel determinant satisfies*
$$|H_2(2)| = |b_{k+1}b_{3k+1} - b_{2k+1}^2| \leqslant \frac{B_1^2}{4k^2}.$$
2. *If $B_1, B_2$ and $B_3$ satisfy the conditions*
$$|B_2| \geqslant B_1, \quad \textit{and} \quad |4B_1B_3 - \delta B_1^4 - 3B_2^2| - B_1|B_2| - 2B_1^2 \geqslant 0,$$
*or the conditions*
$$|B_2| \leqslant B_1, \quad \textit{and} \quad |4B_1B_3 - \delta B_1^4 - 3B_2^2| - 3B_1^2 \geqslant 0,$$
*then the second Hankel determinant satisfies*
$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leqslant \frac{1}{12k^2}|4B_1B_3 - \delta B_1^4 - 3B_2^2|.$$
3. *If $B_1, B_2$ and $B_3$ satisfy the conditions*
$$|B_2| \geqslant B_1, \quad \textit{and} \quad 2B_1^2 + B_1|B_2| - |4B_1B_3 - \delta B_1^4 - 3B_2^2| \geqslant 0,$$
*then the second Hankel determinant satisfies*
$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leqslant \frac{B_1^2}{12k^2}\left(\frac{3|4B_1B_3 - \delta B_1^4 - 3B_2^2| - 4B_1^2 - B_2^2 - 4B_1|B_2|}{|4B_1B_3 - \delta B_1^4 - 3B_2^2| - 2B_1|B_2| - B_1^2}\right).$$

*Proof.* Since $f \in S^*(\varphi)$, there exists an analytic self-map $w$ of $\mathbb{D}$ with $w(0) = 0$ satisfying
$$\frac{zf'(z)}{f(z)} = \varphi(w(z)). \tag{8}$$
Define the function $P_1 \in \mathcal{P}$ by
$$P_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \cdots,$$
or equivalently,
$$w(z) = \frac{P_1(z) - 1}{P_1(z) + 1} = \frac{1}{2}\left(c_1z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right). \tag{9}$$
By using (9) along with (7) lead to the expansion
$$\varphi(w(z)) = 1 + B_1w(z) + B_2w^2(z) + \cdots = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2$$
$$+ \left(B_1\left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{8}\right) + B_2c_1\left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) + \frac{B_3c_1^3}{8}\right)z^3 + \cdots. \tag{10}$$
Now
$$\frac{zf'(z)}{f(z)} = 1 + a_2z + \left(2a_3 - a_2^2\right)z^2 + \left(3a_4 - 3a_2a_3 + a_2^3\right)z^3 + \cdots,$$

and comparing with (8) and (10), it follows that

$$\begin{aligned} a_2 &= \frac{B_1 c_1}{2}, \\ a_3 &= \frac{1}{8}\Big(\left(B_1^2 - B_1 + B_2\right)c_1^2 + 2B_1c_2\Big), \\ a_4 &= \frac{1}{48}\Big(\left(-4B_2 + 2B_1 + B_1^3 - 3B_1^2 + 3B_1B_2 + 2B_3\right)c_1^3 + 2\left(3B_1^2 - 4B_1 + 4B_2\right)c_1c_2 + 8B_1c_3\Big). \end{aligned} \tag{11}$$

Consequently (11) and (3) yield

$$\begin{aligned} b_{k+1} &= \frac{B_1c_1}{2k}, \\ b_{2k+1} &= \frac{1}{8k}\Big(\left(B_1^2 - B_1 + B_2\right)c_1^2 + 2B_1c_2\Big) + \frac{(1-k)}{8k^2}B_1^2c_1^2, \\ b_{3k+1} &= \frac{1}{48k}\Big(\left(B_1^3 - 3B_1^2 + 2B_1 - 4B_2 + 3B_1B_2 + 2B_3\right)c_1^3 + 2\left(3B_1^2 - 4B_1 + 4B_2\right)c_1c_2 \\ &\quad + 8B_1c_3\Big) + \frac{(1-k)B_1c_1}{16k^2}\left[\left(B_1^2 - B_1 + B_2\right)c_1^2 + 2B_1c_2\right] + \frac{(1-k)(1-2k)}{48k^3}B_1^3c_1^3. \end{aligned}$$

A lengthy computations, validated by Mathematica, show that

$$b_{k+1}b_{3k+1} - b_{2k+1}^2 = \frac{B_1}{192k^2}\Big(4\big(B_2 - B_1\big)c_1^2c_2 + \big(B_1 - \frac{1}{k^2}B_1^3 - 2B_2 + 4B_3 - 3\frac{B_2^2}{B1}\big)c_1^4 + 16B_1c_1c_3 - 12B_1c_2^2\Big).$$

Next for ease in computations, let

$$\begin{aligned} &d_1 = 16B_1, \quad d_2 = 4\left(B_2 - B_1\right), \quad d_3 = -12B_1, \\ &d_4 = B_1 - \delta B_1^3 - 2B_2 + 4B_3 - 3\frac{B_2^2}{B_1}, \quad \text{and} \quad T = \frac{B_1}{192k^2}. \end{aligned} \tag{12}$$

Then

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| = T\left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right|. \tag{13}$$

Since the function $p\left(e^{i\theta}z\right)(\theta \in \mathbb{R})$ is in the class $\mathcal{P}$ for any $p \in \mathcal{P}$, there is no loss of generality in assuming $c_1 = c > 0$, $c \in [0,2]$. Substituting the values of $c_2$ and $c_3$ respectively from (4) and (5) in (13), it follows that

$$\begin{aligned} \left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| = \frac{T}{4}\big|&c^4\left(d_1 + 2d_2 + d_3 + 4d_4\right) + 2xc^2(4-c^2)(d_1 + d_2 + d_3) \\ &+ (4-c^2)x^2(-d_1c^2 + d_3(4-c^2)) + 2d_1(4-c^2)(1-|x|^2)y\big| \end{aligned}$$

for some $x, y \in \overline{\mathbb{D}}$. With $s = |x|$, (12) yields

$$\begin{aligned} \left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| &\leqslant \frac{T}{4}\Big(c^4\left|16B_3 - 4\delta B_1^3 - 12\frac{B_2^2}{B_1}\right| + 8sc^2(4-c^2)\left|B_2\right| \\ &\quad + s^2\left(4-c^2\right)\left(4B_1c^2 + 48B_1\right) + 32B_1c\left(4-c^2\right)\left(1-s^2\right)\Big) \\ &= T\Big(c^4\left|4B_3 - \delta B_1^3 - 3\frac{B_2^2}{B_1}\right| + 8B_1c\left(4-c^2\right) + 2\left|B_2\right|sc^2\left(4-c^2\right) \\ &\quad + B_1s^2\left(4-c^2\right)(c-2)(c-6)\Big) \\ &:= F(c,s), \end{aligned}$$

$(c, s) \in [0, 2] \times [0, 1]$. Now

$$\frac{\partial F}{\partial s} = T\left(2\,|B_2|\,c^2\left(4-c^2\right) + 2B_1 s\left(4-c^2\right)(c-2)(c-6)\right),$$

and so $\partial F/\partial s > 0$; that is, $F(c, s)$ is an increasing function of $s$. Hence

$$\max_{0\leqslant s\leqslant 1} F(c, s) = F(c, 1) := G(c).$$

Upon simplification, we find that

$$G(c) = \frac{B_1}{192k^2}\Big(c^4\Big(\left|4B_3 - \delta B_1^3 - \frac{3B_2^2}{B_1}\right| - 2\,|B_2| - B_1\Big) + 8c^2\Big(|B_2| - B_1\Big) + 48B_1\Big).$$

Writing $c^2 = t$ and

$$L = \left|4B_3 - \delta B_1^3 - \frac{3B_2^2}{B_1}\right| - 2\,|B_2| - B_1, \quad M = 8\left(|B_2| - B_1\right), \quad N = 48B_1,$$

it follows from (6) that

$$\frac{192k^2}{B_1}\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leqslant \begin{cases} \frac{4LN-M^2}{4L}, & M > 0,\ L \leqslant -\frac{M}{8}, \\ N, & M \leqslant 0,\ L \leqslant -\frac{M}{4}, \\ 16L + 4M + N, & M \geqslant 0,\ L \geqslant -\frac{M}{8} \text{ or } M \leqslant 0,\ L \geqslant -\frac{M}{4}. \end{cases}$$

These conditions now lead to the desired bounds for the second Hankel determinant . □

**Remark 2.3.**

1. *The special case $k = 1$ in Theorem 2.2 reduces to [25, Theorem 1].*
2. *If $k = 1$ and $B_1 = B_2 = B_3 = 2$, then Theorem 2.2 reduces to [23, Theorem 3.1].*

Judicious choices of $\varphi$ in Theorem 2.2 lead to the following results for the special cases.

**Corollary 2.4.**

1. *If $f \in S^*(\alpha)$, then $|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leqslant (1-\alpha)^2/k^2$.*
2. *If $f \in S_L^*$, then $|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leqslant 1/(16k^2)$.*
3. *If $f \in S_P^*$, then $|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leqslant 16/(\pi^4 k^2)$.*
4. *If $f \in S_\beta^*$, then $|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leqslant \beta^2/k^2$.*

**Definition 2.5.** *Let $\varphi \in \mathcal{P}$ be given by Definition 2.1, and $b$ be a non-zero complex number. The class $R_b(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the subordination*

$$1 + \frac{1}{b}\left(f'(z) - 1\right) \prec \varphi(z).$$

This class was considered in [5] for the more general case of $p$-valent functions. The case $b = 1$ and $\varphi(z) = (1+z)/(1-z)$ gives the subclass of close-to-convex functions, studied by MacGregor [28], consisting of functions whose derivative has positive real part. Al Amiri *et al.* [1] introduced the general class of analytic functions satisfying $\operatorname{Re}\{1 + 1/b((zf'(z)/g(z)) - 1)\} > 0$, for some starlike function $g$. It is evident that $R_b(\varphi)$ coincides with this class for $g(z) = z$ and $\varphi(z) = (1+z)/(1-z)$.

**Theorem 2.6.** *Let $\varphi$ be given by (7), $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R_b(\varphi)$, and $F(z) = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}$ be its kth-root transform. Further let*

$$\lambda = \left|9B_1B_3 - 8B_2^2 + \delta B_1^4\right|, \quad \text{and} \quad \delta = \frac{3\left(k^2-1\right)}{2^3k^2}b^2.$$

1. *If $B_1, B_2$ and $B_3$ satisfy the conditions*
$$|B_2| \geqslant \frac{7}{2}B_1, \quad \textit{and} \quad 8B_1^2 - \lambda \geqslant 0,$$
*then the second Hankel determinant satisfies*
$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leqslant \frac{B_1^2 b^2}{9k^2}.$$
2. *If $B_1, B_2$ and $B_3$ satisfy the conditions*
$$|B_2| \leqslant \frac{7}{2}B_1, \quad \textit{and} \quad \frac{9}{2}B_1^2 + B_1|B_2| - \lambda \leqslant 0,$$
*or the conditions*
$$|B_2| \geqslant \frac{7}{2}B_1, \quad \textit{and} \quad 8B_1^2 - \lambda \leqslant 0,$$
*then the second Hankel determinant satisfies*
$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leqslant \frac{b^2\lambda}{2^3 3^2 k^2}.$$
3. *If $B_1, B_2$ and $B_3$ satisfy the conditions*
$$|B_2| \leqslant \frac{7}{2}B_1, \quad \textit{and} \quad \frac{9}{2}B_1^2 + B_1|B_2| - \lambda \geqslant 0,$$
*then the second Hankel determinant satisfies*
$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leqslant \frac{b^2B_1^2}{2^5 3^2 k^2}\left(\frac{32\lambda - 36B_1|B_2| - 81B_1^2 - 4B_2^2}{\lambda - 2B_1|B_2| - B_1^2}\right).$$

*Proof.* The proof is similar to Theorem 2.2. There exists an analytic self-map $w$ of $\mathbb{D}$ satisfying
$$1 + \frac{1}{b}\left(f'(z) - 1\right) = \varphi\left(w(z)\right). \tag{14}$$
Now
$$1 + \frac{1}{b}\left(f'(z) - 1\right) = 1 + \frac{2}{b}a_2 z + \frac{3}{b}a_3 z^2 + \frac{4}{b}a_4 z^3 + \cdots,$$
and comparing with (10) and (14), we find that
$$\begin{aligned} a_2 &= b\frac{B_1c_1}{4}, \\ a_3 &= b\frac{\left(2B_1c_2 - B_1c_1^2 + B_2c_1^2\right)}{12}, \\ a_4 &= b\frac{\left(4B_1c_3 - 4B_1c_1c_2 + B_1c_1^3 + 4B_2c_1c_2 - 2B_2c_1^3 + B_3c_1^3\right)}{32}. \end{aligned} \tag{15}$$
Consequently (15) and (3) yield
$$\begin{aligned} b_{k+1} &= b\frac{B_1c_1}{4k}, \\ b_{2k+1} &= \frac{b}{12k}\left(2B_1c_2 - B_1c_1^2 + B_2c_1^2\right) + \frac{(1-k)b^2}{32k^2}B_1^2c_1^2, \\ b_{3k+1} &= \frac{b}{32k}\left(4B_1c_3 - 4B_1c_1c_2 + B_1c_1^3 + 4B_2c_1c_2 - 2B_2c_1^3 + B_3c_1^3\right) \\ &\quad + \frac{(1-k)b^2}{48k^2}\left(2B_1^2c_1c_2 - B_1^2c_1^3 + B_1B_2c_1^3\right) + \frac{(k-1)(2k-1)}{384}b^3B_1^3c_1^3. \end{aligned}$$

Thus

$$b_{k+1}b_{3k+1}-b_{2k+1}^2=\frac{b^2}{2^7 3^2 k^2}\Bigg(\Big(B_1^2-2B_1B_2+3^2B_1B_3-2^3B_2^2+\frac{3\left(k^2-1\right)}{2^3k^2}b^2B_1^4\Big)c_1^4$$
$$+2^2\left(B_1B_2-B_1^2\right)c_1^2c_2+2^23^2B_1^2c_1c_3-2^5B_1^2c_2^2\Bigg).$$

Writing

$$d_1=36B_1^2,\quad d_2=4B_1\left(B_2-B_1\right),\quad d_3=-32B_1^2,$$
$$d_4=B_1^2-2B_1B_2+9B_1B_3-8B_2^2+\delta B_1^4,\quad \text{and}\quad T=\frac{b^2}{2^73^2k^2}, \tag{16}$$

then

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|=T\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|.$$

Equations (4) and (5) show that

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|=\frac{T}{4}\Big|c^4\left(d_1+2d_2+d_3+4d_4\right)+2xc^2\left(4-c^2\right)\left(d_1+d_2+d_3\right)$$
$$+\left(4-c^2\right)x^2\left(-d_1c^2+d_3\left(4-c^2\right)\right)+2d_1c\left(4-c^2\right)\left(1-|x|^2\right)y\Big|$$

for some $x,y\in\overline{\mathbb{D}}$. With $s=|x|$, (16) yields

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|\leqslant T\Big(c^4\lambda+2sc^2\left(4-c^2\right)B_1\left|B_2\right|+\left(4-c^2\right)s^2B_1^2\left(9c^2+8\left(4-c^2\right)-18c\right)$$
$$+18B_1^2c\left(4-c^2\right)\left(1-s^2\right)\Big)$$
$$=T\Big(c^4\lambda+2sc^2\left(4-c^2\right)B_1\left|B_2\right|+18B_1^2c\left(4-c^2\right)+\left(4-c^2\right)s^2B_1^2\left(c-2\right)\left(c-16\right)\Big)$$
$$:=F(c,s),$$

$(c,s)\in[0,2]\times[0,1]$. Now

$$\frac{\partial F}{\partial s}=T\Big(2c^2\left(4-c^2\right)B_1\left|B_2\right|+2s\left(4-c^2\right)B_1^2\left(c-2\right)\left(c-16\right)\Big),$$

and so $\partial F/\partial s>0$; that is,

$$\max_{0\leqslant s\leqslant 1}F(c,s)=F(c,1):=G(c).$$

Routine simplifications yield

$$G(c)=T\Big(c^4(\lambda-2B_1\left|B_2\right|-B_1^2)+4c^2B_1\left(2\left|B_2\right|-7B_1\right)+128B_1^2\Big).$$

With $c^2=t$ and

$$L=\lambda-2B_1\left|B_2\right|-B_1^2,\quad M=4B_1\left(2\left|B_2\right|-7B_1\right),\quad N=128B_1^2,$$

it follows from (6) that

$$\frac{2^73^2k^2}{b^2}\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|\leqslant\begin{cases}\frac{4LN-M^2}{4L}, & M>0,\ L\leqslant-\frac{M}{8},\\ N, & M\leqslant 0,\ L\leqslant-\frac{M}{4},\\ 16L+4M+N, & M\geqslant 0,\ L\geqslant-\frac{M}{8}\ \text{ or }\ M\leqslant 0,\ L\geqslant-\frac{M}{4}.\end{cases}$$

Inserting the values of the parameters $L,M$ and $N$ yield the desired conditions and bounds. $\square$

Theorem 2.6 yields the following special cases.

**Remark 2.7.**

1. *With $k = 1$ and $b = \tau$, Theorem 2.6 reduces to [25, Theorem 3] for $\gamma = 0$.*
2. *With $k = 1$ and $b = 1$, Theorem 2.6 reduces to [22, Theorem 2.1] for $\gamma = 0$.*
3. *For the choice $\varphi(z) = (1 + Az)/(1 + Bz)$, $-1 \leqslant B < A \leqslant 1$, $k = 1$ and $b = \tau$, Theorem 2.6 reduces to [7, Theorem 2.1] for $\gamma = 0$.*

We next introduce a third class of functions to be studied.

**Definition 2.8.** *Let $\varphi \in \mathcal{P}$ be given by Definition 2.1, and $\alpha \geqslant 0$. The class $S^*(\alpha, \varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the subordination*

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \varphi(z).$$

Padmanabhan [44] introduced the class $S^*(\alpha, \varphi)$ in 2001 and investigated sufficient conditions for starlikeness. The special case when $\alpha = 1$ and $\varphi = (1 + z)/(1 - z)$ was considered in [47]. It is evident that $S^*(0, \varphi)$ reduces to the class $\mathcal{S}^*(\varphi)$ treated in Definition 2.1.

**Theorem 2.9.** *Let $\varphi$ be given by (7), $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(\alpha, \varphi)$, and $F(z) = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}$ be its kth-root transform. Further let*

$$\lambda = \left|\delta B_1^4 - 2\alpha B_1^2 B_2 + 4(1 + 2\alpha)(1 + 3\alpha)^3 B_1 B_3 - 3(1 + 2\alpha)^2(1 + 4\alpha)B_2^2\right|,$$

$$u = (1 + 2\alpha)(12\alpha^2 + 6\alpha + 1), \quad \text{and}$$

$$\delta = \frac{-1 - 10\alpha + \left(5k^2 - 33\right)\alpha^2 + 12\left(k^2 - 3\right)\alpha^3}{((1 + 2\alpha)k)^2}.$$

1. *If $B_1, B_2$ and $B_3$ satisfy the conditions*

$$u\,|B_2| - (1 + 2\alpha)\left(1 + 6\alpha + 6\alpha^2\right) B_1 + \alpha B_1^2 \leqslant 0, \quad \text{and} \quad \lambda - 3(1 + 2\alpha)^2(1 + 4\alpha)B_1^2 \leqslant 0,$$

*then the second Hankel determinant satisfies*

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leqslant \frac{B_1^2}{(2(1 + 3\alpha)k)^2}.$$

2. *If $B_1, B_2$ and $B_3$ satisfy the conditions*

$$u\,|B_2| - (1 + 2\alpha)\left(1 + 6\alpha + 6\alpha^2\right) B_1 + \alpha B_1^2 \geqslant 0, \quad \text{and}$$

$$\lambda - B_1\Big(u\,|B_2| + \alpha B_1^2 - 2(1 + 2\alpha)\left(1 + 6\alpha + 9\alpha^2\right) B_1\Big) \geqslant 0,$$

*or the conditions*

$$u\,|B_2| - (1 + 2\alpha)\left(1 + 6\alpha + 6\alpha^2\right) B_1 + \alpha B_1^2 \leqslant 0, \quad \text{and} \quad \lambda - 3(1 + 2\alpha)^2(1 + 4\alpha)B_1^2 \geqslant 0,$$

*then the second Hankel determinant satisfies*

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leqslant \frac{\lambda}{3(1 + 4\alpha)\Big(2(1 + 2\alpha)(1 + 3\alpha)k\Big)^2}.$$

3. *If $B_1, B_2$ and $B_3$ satisfy the conditions*

$$u\,|B_2| - (1+2\alpha)\left(1+6\alpha+6\alpha^2\right)B_1 + \alpha B_1^2 \geqslant 0, \quad \textit{and}$$
$$\lambda - B_1\Big(u\,|B_2| + \alpha B_1^2 - 2(1+2\alpha)\left(1+6\alpha+9\alpha^2\right)B_1\Big) \leqslant 0,$$

*then the second Hankel determinant satisfies*

$$\begin{aligned}&\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right|\\ &\leqslant B_1^2\Big(-3(1+2\alpha)(1+6\alpha+8\alpha^2)\Big(-\lambda + (u+2\alpha B_1)B_1^2 + 2uB_1|B_2|\Big)\\ &\quad - \Big(\alpha B_1^2 - (1+2\alpha)(1+6\alpha+6\alpha^2)B_1 + u|B_2|\Big)^2\Big)\Big/\\ &\quad \Big(12k^2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)\Big(\lambda - B_1^2(u+2\alpha B_1) - 2uB_1|B_2|\Big)\Big).\end{aligned}$$

*Proof.* Let

$$\frac{zf'(z) + \alpha^2 z^2 f''(z)}{f(z)} = \varphi\left(w(z)\right) \tag{17}$$

for an analytic self-map $w$ of $\mathbb{D}$. Since

$$\begin{aligned}\frac{zf'(z) + \alpha^2 z^2 f''(z)}{f(z)} &= 1 + (1+2\alpha)a_2 z + \Big(2(1+3\alpha)a_3 - (1+2\alpha)a_2^2\Big)z^2\\ &\quad + \Big(3(1+4\alpha)a_4 - (3+8\alpha)a_2a_3 + (1+2\alpha)a_2^3\Big)z^3 + \cdots,\end{aligned} \tag{18}$$

it follows from (10), (17) and (18) that

$$\begin{aligned}a_2 &= \frac{1}{2(1+2\alpha)}B_1c_1,\\ a_3 &= \frac{1}{2^3(1+2\alpha)(1+3\alpha)}\Big((1+2\alpha)\left(2B_1c_2 - (B_1-B_2)\,c_1^2\right) + B_1^2c_1^2\Big),\\ a_4 &= \frac{1}{2^4\cdot 3(1+4\alpha)}\Big(8\left(B_1c_3 - B_1c_1c_2 + B_2c_1c_2\right) - 4B_2c_1^3 + 2\left(B_1c_1^3 + B_3c_1^3\right)\\ &\quad + \frac{(3+8\alpha)}{(1+2\alpha)^2(1+3\alpha)}\left((1+2\alpha)\left(2B_1^2c_1c_2 - B_1^2c_1^3 + B_1B_2c_1^3\right) + B_1^3c_1^3\right) - \frac{2B_1^3c_1^3}{(1+2\alpha)^2}\Big).\end{aligned}$$

Consequently (3) yields

$$\begin{aligned}b_{k+1} &= \frac{B_1c_1}{2k(1+2\alpha)},\\ b_{2k+1} &= \frac{1}{2^3k(1+3\alpha)}\left(2B_1c_2 + (B_2-B_1)\,c_1^2\right) + \frac{(3-k)\alpha+1}{8k^2(1+2\alpha)^2(1+3\alpha)}B_1^2c_1^2,\\ b_{3k+1} &= \frac{1}{2^3 3k(1+4\alpha)}\Big(4\left(B_1c_3 - B_1c_1c_2 + B_2c_1c_2\right) - 2B_2c_1^3 + B_1c_1^3 + B_3c_1^3\Big)\\ &\quad + \frac{4\alpha(3-k)+3}{2^4\cdot 3k^2(1+2\alpha)(1+3\alpha)(1+4\alpha)}\left(2B_1^2c_1c_2 - B_1^2c_1^3 + B_1B_2c_1^3\right)\\ &\quad + \frac{\left(12\alpha^2+7\alpha+1\right) - \alpha k\left(4\alpha(3-k)+3\right)}{2^4\cdot 3k^3(1+2\alpha)^3(1+3\alpha)(1+4\alpha)}B_1^3c_1^3.\end{aligned}$$

Routine computations show that

$$\begin{aligned}
b_{k+1}b_{3k+1}-b_{2k+1}^2 &= \frac{\left(12\alpha^3k^2+5\alpha^2k^2-36\alpha^3-33\alpha^2-10\alpha-1\right)}{2^6\cdot 3k^4(1+2\alpha)^4(1+3\alpha)^2(1+4\alpha)}B_1^4c_1^4 \\
&+ \frac{\alpha k}{2^5\cdot 3k^3(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}\left(B_1^2\left(B_1-B_2\right)c_1^4-2B_1^3c_1^2c_2\right) \\
&+ \frac{\left(12\alpha^2+6\alpha+1\right)}{2^6\cdot 3k^2(1+2\alpha)(1+3\alpha)^2(1+4\alpha)}\Big(B_1\left(B_1-2B_2\right)c_1^4 \\
&+ 4B_1\left(B_2-B_1\right)c_1^2c_2\Big)+\frac{1}{2^4\cdot 3k^2(1+2\alpha)(1+4\alpha)}\left(4B_1^2c_1c_3+B_1B_3c_1^4\right) \\
&- \frac{1}{2^6k^2(1+3\alpha)^2}\left(4B_1^2c_2^2+B_2^2c_1^4\right),
\end{aligned}$$

which yields

$$\begin{aligned}
\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| = T\Bigg|&\Bigg(B_1\Big(u(B_1-2B_2)+4(1+2\alpha)(1+3\alpha)^2B_3 \\
&+2\alpha B_1(B_1-B_2)+\delta B_1^3\Big)-3(1+2\alpha)^2(1+4\alpha)B_2^2\Bigg)c^4 \\
&+4B_1\left(u(B_2-B_1)-\alpha B_1^2\right)c_1^2c_2 \\
&+16B_1^2(1+2\alpha)(1+3\alpha)^2c_1c_3-12B_1^2(1+2\alpha)^2(1+4\alpha)c_2^2\Bigg|,
\end{aligned}$$

where

$$T=\frac{1}{2^6 3k^2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}.$$

By writing

$$\begin{aligned}
d_1 &= 16B_1^2(1+2\alpha)(1+3\alpha)^2, \\
d_2 &= 4B_1\left(u(B_2-B_1)-\alpha B_1^2\right), \\
d_3 &= -12B_1^2(1+2\alpha)^2(1+4\alpha), \\
d_4 &= B_1\Big(u(B_1-2B_2)+4(1+2\alpha)(1+3\alpha)^2B_3 \\
&\quad +2\alpha B_1(B_1-B_2)+\delta B_1^3\Big)-3(1+2\alpha)^2(1+4\alpha)B_2^2,
\end{aligned} \tag{19}$$

then

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| = T\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|.$$

Consequently

$$\begin{aligned}
\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| = \frac{T}{4}\Big|&c^4\left(d_1+2d_2+d_3+4d_4\right)+2xc^2\left(4-c^2\right)\left(d_1+d_2+d_3+d_4\right) \\
&+\left(4-c^2\right)x^2\left(-d_1c^2+d_3\left(4-c^2\right)\right)+2d_1c\left(4-c^2\right)\left(1-|x|^2\right)y\Big|
\end{aligned}$$

for some $x, y \in \overline{\mathbb{D}}$. With $s = |x|$, (19) yields

$$\begin{aligned}\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| &\leqslant T\Big(c^4\lambda + 2B_1sc^2\left(4-c^2\right)\left(u\,|B_2| + \alpha B_1^2\right) + s^2(1+2\alpha)\left(4-c^2\right)\Big(4(1+3\alpha)^2B_1^2c^2 \\ &\qquad + 3(1+2\alpha)(1+4\alpha)B_1^2\left(4-c^2\right)\Big) + 8(1+2\alpha)(1+3\alpha)^2B_1^2c\left(4-c^2\right)\left(1-s^2\right)\Big) \\ &= T\Big(c^4\lambda + 2sB_1c^2\left(4-c^2\right)\left(u\,|B_2| + \alpha B_1^2\right) + 8(1+2\alpha)(1+3\alpha)^2B_1^2c\left(4-c^2\right) \\ &\qquad + s^2(1+2\alpha)(1+6\alpha+12\alpha^2)(4-c^2)B_1^2(c-2)(c-p)\Big) := F(c,s),\end{aligned}$$

where

$$p = \frac{6\left(1+6\alpha+8\alpha^2\right)}{1+6\alpha+12\alpha^2} > 2.$$

Proceeding similarly as in the previous proofs, it can be shown that $F(c,s)$ is an increasing function of $s$, whence

$$\max_{0\leqslant s\leqslant 1} F(c,s) = F(c,1) := G(c),$$

with

$$\begin{aligned}G(c) = T\Bigg(&c^4\Big(\lambda - B_1\left(u\,(B_1 + 2\,|B_2|) + 2\alpha B_1^2\right)\Big) + 8B_1c^2\left(u\,|B_2| - (1+2\alpha)\left(1+6\alpha+6\alpha^2\right)B_1 + \alpha B_1^2\right) \\ &+ 48(1+2\alpha)\left(1+6\alpha+8\alpha^2\right)B_1^2\Bigg).\end{aligned}$$

Writing $c^2 = t$ and

$$\begin{aligned}L &= \lambda - B_1\Big(u\,(B_1 + 2\,|B_2|) + 2\alpha B_1^2\Big), \\ M &= 8B_1\Big(u\,|B_2| - (1+2\alpha)\left(1+6\alpha+6\alpha^2\right)B_1 + \alpha B_1^2\Big), \\ N &= 48(1+2\alpha)\left(1+6\alpha+8\alpha^2\right)B_1^2.\end{aligned} \tag{20}$$

(6) yield

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leqslant T\begin{cases}\frac{4LN-M^2}{4L}, & M > 0,\ L \leqslant -\frac{M}{8}, \\ N, & M \leqslant 0,\ L \leqslant -\frac{M}{4}, \\ 16L + 4M + N, & M \geqslant 0,\ L \geqslant -\frac{M}{8} \text{ or } M \leqslant 0,\ L \geqslant -\frac{M}{4},\end{cases}$$

where $L, M, N$ are given by (20). This completes the proof. □

**Definition 2.10.** *Let $\varphi \in \mathcal{P}$ be given by Definition 2.1, and $\alpha \in [0,1]$. The class $L(\alpha,\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the subordination*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z).$$

This class is analogous to the $\alpha$-logarithmically convex functions introduced by Lewandowski *et al.* [26]. In [11], Darus *et al.* found sharp upper bounds for $|a_2|$, $|a_3|$ and $|a_3 - \mu a_2|$, $\mu$ real, for $f \in L(\alpha,\varphi)$. Evidently $L(1,\varphi)$ reduces to the class $\mathcal{S}^*(\varphi)$ treated in Definition 2.1.

**Theorem 2.11.** *Let $\varphi$ be given by* (7), $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in L(\alpha, \varphi)$, *and* $F(z) = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}$ *be its kth-root transform. Further let*

$$\lambda = \left|\delta B_1^4 + vB_1^2 B_2 + 4(2-\alpha)(3-2\alpha)^2 B_1 B_3 - 3(2-\alpha)^2(4-3\alpha)B_2^2\right|,$$
$$u = (2-\alpha)\left(7\alpha^2 - 18\alpha + 12\right),$$
$$v = (7\alpha^3 - 13\alpha^2 - 6\alpha + 12), \quad \text{and}$$
$$\delta = \Big(4(2\alpha-3)\Big(2k^2\alpha\left(15 - 17\alpha + 6\alpha^2 - 4\alpha^3\right) - 9k\left(8 - 10\alpha - \alpha^2 + 3\alpha^3\right) - 6\left(12 - 17\alpha + 6\alpha^2\right)\Big) + 9(3\alpha-4)\Big(2(2\alpha-3) + k\left(-2 + \alpha + \alpha^2\right)\Big)^2\Big)\Big/3\left(2k(2-\alpha)\right)^2.$$

1. *If $B_1, B_2$ and $B_3$ satisfy the conditions*

$$vB_1^2 + 2u|B_2| - 2(2-\alpha)\left(6 - 6\alpha + \alpha^2\right)B_1 \leqslant 0, \quad \text{and} \quad \lambda + 3(2-\alpha)^2(3\alpha-4)B_1^2 \leqslant 0,$$

*then the second Hankel determinant satisfies*

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leq \left(\frac{B_1}{2(3-2\alpha)k}\right)^2.$$

2. *If $B_1, B_2$ and $B_3$ satisfy the conditions*

$$vB_1^2 + 2u|B_2| - 2(2-\alpha)\left(6 - 6\alpha + \alpha^2\right)B_1 \geqslant 0, \quad \text{and} \quad 2\lambda - B_1\left(vB_1^2 + 2u|B_2| + 4(2-\alpha)(3-2\alpha)^2 B_1\right) \geqslant 0,$$

*or the conditions*

$$vB_1^2 + 2u|B_2| - 2(2-\alpha)\left(6 - 6\alpha + \alpha^2\right)B_1 \leqslant 0, \quad \text{and} \quad \lambda + 3(2-\alpha)^2(3\alpha-4)B_1^2 \geqslant 0,$$

*then the second Hankel determinant satisfies*

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leq \frac{\lambda}{3(4-3\alpha)\left(2(2-\alpha)(3-2\alpha)k\right)^2}.$$

3. *If $B_1, B_2$ and $B_3$ satisfy the conditions*

$$vB_1^2 + 2u|B_2| - 2(2-\alpha)\left(6 - 6\alpha + \alpha^2\right)B_1 > 0, \quad \text{and} \quad 2\lambda - B_1\left(vB_1^2 + 2u|B_2| + 4(2-\alpha)(3-2\alpha)^2 B_1\right) \leqslant 0,$$

*then the second Hankel determinant satisfies*

$$\begin{aligned}&\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \\ &\leqslant B_1^2\Big(-\left(vB_1^2 + 2(\alpha-2)\left(6 - 6\alpha + \alpha^2\right)B_1 + 2u|B_2|\right)^2 + 12(\alpha-2)^2(3\alpha-4)\left(uB_1^2 + vB_1^3 - \lambda + 2uB_1|B_2|\right)\Big)\Big/ \\ &\quad \Big(3(4-3\alpha)\left(4(2-\alpha)(3-2\alpha)k\right)^2\left(\lambda - uB_1^2 - vB_1^3 - 2uB_1|B_2|\right)\Big).\end{aligned}$$

*Proof.* With

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{(1-\alpha)} = \varphi\left(w(z)\right),$$

the above equation and (10) yield

$$\begin{aligned}
a_2 &= \frac{B_1 c_1}{2(2-\alpha)},\\
a_3 &= \frac{1}{2(3-2\alpha)}\left(\frac{-(\alpha^2+5\alpha-8)}{2\cdot 4(2-\alpha)^2}B_1^2c_1^2+\frac{B_1c_2}{2}-\frac{B_1c_1^2}{4}+\frac{B_2c_1^2}{4}\right),\\
a_4 &= \frac{1}{2^3\cdot 3(4-3\alpha)}\left(4B_2c_1c_2-2B_2c_1^3+B_3c_1^3+B_3c_1^3+4B_1c_3-4B_1c_1c_2+B_1c_1^3\right)\\
&\quad+\frac{\left(\alpha^3+21\alpha^2+20\alpha-48\right)}{2^4\cdot 3^2(2-\alpha)^3(4-3\alpha)}B_1^3c_1^3+\frac{\left(4\alpha^2+11\alpha-18\right)}{2^5\cdot 3(2-\alpha)^3(3-2\alpha)(4-3\alpha)}\\
&\quad\left(\left(\alpha^2+5\alpha-8\right)B_1^3c1^3-4(2-\alpha)^2B_1^2c_1c_2+2(2-\alpha)^2B_1^2c_1^3-2(2-\alpha)^2B_1B_2c_1^3\right).
\end{aligned}\tag{21}$$

From (21) and (3), and after some lengthy computations (validated by Mathematica), we find that

$$\begin{aligned}
\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| = T\Big|&\Big(\delta B_1^4+uB_1\left(B_1-2B_2\right)+vB_1^2\left(B_2-B_1\right)\\
&+4(2-\alpha)(3-2\alpha)^2B_1B_3-3(2-\alpha)^2(4-3\alpha)B_2^2\Big)c^4\\
&+2B_1\Big(2u\left(B_2-B_1\right)+vB_1^2\Big)c_1^2c_2+16B_1^2(2-\alpha)(3-2\alpha)^2c_1c_3\\
&-12B_1^2(2-\alpha)^2(4-3\alpha)c_2^2\Big|,
\end{aligned}$$

where

$$T=\frac{1}{3(4-3\alpha)\left(2^3(2-\alpha)(3-2\alpha)k\right)^2}.$$

Let

$$\begin{aligned}
d_1 &= 16B_1^2(2-\alpha)(3-2\alpha)^2,\\
d_2 &= 2B_1\Big(2u\left(B_2-B_1\right)+vB_1^2\Big),\\
d_3 &= -12B_1^2(2-\alpha)^2(4-3\alpha),\\
d_4 &= \delta B_1^4+uB_1\left(B_1-2B_2\right)+vB_1^2\left(B_2-B_1\right)+4(2-\alpha)(3-2\alpha)^2B_1B_3\\
&\quad-3(2-\alpha)^2(4-3\alpha)B_2^2.
\end{aligned}$$

Then

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| = T\left(d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right).$$

Thus

$$\begin{aligned}
\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| &= \frac{T}{4}\Big(c^4\left(d_1+2d_2+d_3+4d_4\right)+2xc^2\left(4-c^2\right)\left(d_1+d_2+d_3\right)\\
&\quad+\left(4-c^2\right)x^2\left(-d_1c^2+d_3\left(4-c^2\right)\right)+2d_1c\left(4-c^2\right)\left(1-|x|^2\right)z\Big)\\
&\leqslant T\Big(c^4\left(\lambda\right)+sc^2\left(4-c^2\right)B_1\left(2u\left|B_2\right|+vB_1^2\right)\\
&\quad+s^2\left(4-c^2\right)(2-\alpha)B_1^2\left(\left(7\alpha^2-18\alpha+12\right)c^2+12(2-\alpha)(4-3\alpha)\right)\\
&\quad+8c\left(4-c^2\right)(2-\alpha)(3-2\alpha)^2B_1^2\left(1-s^2\right)\Big)\\
&= T\Big(c^4\left(\lambda\right)+sc^2\left(4-c^2\right)B_1\left(2u\left|B_2\right|+vB_1^2\right)+8c\left(4-c^2\right)(2-\alpha)\\
&\quad(3-2\alpha)^2B_1^2+s^2(4-c^2)(2-\alpha)(12-18\alpha+7\alpha^2)B_1^2(c-2)(c-p)\Big)\\
&:= F(c,s),
\end{aligned}$$

where

$$p = \frac{6\left(8 - 10\alpha + 3\alpha^2\right)}{12 - 18\alpha + 7\alpha^2} > 2.$$

The function $F(c, s)$ is increasing relative to $s$, and so

$$\max_{0\leqslant s\leqslant 1} F(c, s) = F(c, 1) := G(c),$$

with

$$G(c) = T\Big(c^4\Big(\lambda - B_1\left(vB_1^2 + u(2|B_2| + B_1)\right)\Big) + c^2\Big(4B_1\left(2u\,|B_2| + vB_1^2 - 2(2-\alpha)\left(\alpha^2 - 6\alpha + 6\right)\right)B_1\Big) + 48(2-\alpha)^2(4-3\alpha)B_1^2\Big).$$

Letting $c^2 = t$ and

$$\begin{aligned} L &= \lambda - B_1\Big(vB_1^2 + u(2|B_2| + B_1)\Big),\\ M &= 4B_1\Big(2\left(u\,|B_2| - (2-\alpha)\left(\alpha^2 - 6\alpha + 6\right)B_1\right) + vB_1^2\Big),\\ N &= 48(2-\alpha)^2(4-3\alpha)B_1^2. \end{aligned} \tag{22}$$

(6) leads to

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leqslant T\begin{cases} \frac{4LN-M^2}{4L}, & M > 0,\ L \leqslant -\frac{M}{8},\\ N, & M \leqslant 0,\ L \leqslant -\frac{M}{4},\\ 16L + 4M + N, & M \geqslant 0,\ L \geqslant -\frac{M}{8} \text{ or } M \leqslant 0,\ L \geqslant -\frac{M}{4}, \end{cases}$$

where $L, M, N$ are given by (22). The proof is now evident. □

**Remark 2.12.**

1. *With $\alpha = 1$, Theorem 2.11 reduces to Theorem 2.2.*
2. *If $k = 1$ and $\alpha = 0$, then Theorem 2.11 reduces to [25, Theorem 2].*
3. *With the choice of $\varphi(z) = (1 + z)/(1 - z), k = 1$ and $\alpha = 0$, Theorem 2.11 reduces to [23, Theorem 3.2].*

We conclude with the following class of functions.

**Definition 2.13.** *Let $\varphi \in \mathcal{P}$ be given by Definition 2.1, and $\alpha \geqslant 0$. The class $M(\alpha, \varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the subordination*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z).$$

The class $M(\alpha, \varphi)$ is analogous to the $\alpha$-convex functions of Mocanu *et al.*[30], who investigated geometric properties of the class in the case $\varphi = (1 + z)/(1 - z)$.

**Theorem 2.14.** *Let $\varphi$ be given by (7), $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M(\alpha, \varphi)$, and $F(z) = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}$ be its kth-root transform. Further let*

$$\lambda = \left|\delta B_1^4 + 6\alpha B_1^2 B_2 + 4(1+2\alpha)^2 B_1 B_3 - 3(1+\alpha)(1+3\alpha)B_2^2\right|, \quad u = (7\alpha^2 + 4\alpha + 1),$$

*and*

$$\delta = -\frac{\left(1 + \left(7 + 4k^2\right)\alpha + \left(16 + 7k^2\right)\alpha^2 + \left(12 + k^2\right)\alpha^3\right)}{(1+\alpha)^3 k^2}.$$

1. *If $B_1, B_2$ and $B_3$ satisfy the conditions*

$$u\,|B_2| + 3\alpha B_1^2 - \left(\alpha^2 + 4\alpha + 1\right) B_1 \leqslant 0, \quad \text{and} \quad \lambda - 3(1+\alpha)(1+3\alpha)B_1^2 \leqslant 0,$$

*then the second Hankel determinant satisfies*

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leqslant \left(\frac{B_1}{2(1+2\alpha)k}\right)^2.$$

2. *If $B_1, B_2$ and $B_3$ satisfy the conditions*

$$u\,|B_2| + 3\alpha B_1^2 - \left(\alpha^2 + 4\alpha + 1\right) B_1 \geqslant 0, \quad \text{and} \quad \lambda - B_1\left(2(1+2\alpha)^2 B_1 + 3\alpha B_1^2 + u\,|B_2|\right) \geqslant 0,$$

*or the conditions*

$$u\,|B_2| + 3\alpha B_1^2 - \left(\alpha^2 + 4\alpha + 1\right) B_1 \leqslant 0, \quad \text{and} \quad \lambda - 3(1+\alpha)(1+3\alpha)B_1^2 \geqslant 0,$$

*then the second Hankel determinant satisfies*

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leqslant \frac{\lambda}{3(1+\alpha)(1+3\alpha)\left(2(1+2\alpha)k\right)^2}.$$

3. *If $B_1, B_2$ and $B_3$ satisfy the conditions*

$$u\,|B_2| + 3\alpha B_1^2 - \left(\alpha^2 + 4\alpha + 1\right) B_1 > 0, \quad \text{and} \quad \lambda - B_1\left(2(1+2\alpha)^2 B_1 + 3\alpha B_1^2 + u\,|B_2|\right) \leqslant 0,$$

*then the second Hankel determinant satisfies*

$$\begin{aligned}
&\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \\
&\leqslant B_1^2\Big( - \left(6\alpha B_1^2 - (1+4\alpha+\alpha^2)B_1 + u|B_2|\right)^2 - 3(1+\alpha)(1+3\alpha)\left(-\lambda + (u+12\alpha B_1)B_1^2 + 2uB_1|B_2|\right)\Big)\Big/ \\
&\Big(3(1+\alpha)(1+3\alpha)\left(4(1+2\alpha)k\right)^2\left(\lambda - (u+12\alpha B_1)B_1^2 - 2uB_1|B_2|\right)\Big).
\end{aligned}$$

*Proof.* For $f \in M(\alpha, \varphi)$, direct lengthy computations reveal that

$$\begin{aligned}
\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| = T\Big|&\Big(\delta B_1^4 + 6\alpha B_1^2\left(B_2 - B_1\right) + uB_1\left(B_1 - 2B_2\right) \\
&+ 4(1+2\alpha)^2 B_1 B_3 - 3(1+\alpha)(1+3\alpha)B_2^2\Big)c^4 + 4B_1\Big(3\alpha B_1^2 \\
&+ u\left(B_2 - B_1\right)\Big)c_1^2 c_2 + 16B_1^2(1+2\alpha)^2 c_1 c_3 - 12B_1^2(1+\alpha)(1+3\alpha)c_2^2\Big|,
\end{aligned}$$

where

$$T = \frac{1}{3(1+\alpha)(1+3\alpha)\left(2^3(1+2\alpha)k\right)^2}.$$

By writing

$$\begin{aligned}
d_1 &= 16(1+2\alpha)^2 B_1^2, \\
d_2 &= 4B_1\Big(3\alpha B_1^2 + u\left(B_2 - B_1\right)\Big), \\
d_3 &= -12(1+\alpha)(1+3\alpha)B_1^2, \\
d_4 &= \delta B_1^4 + 6\alpha B_1^2\left(B_2 - B_1\right) + uB_1\left(B_1 - 2B_2\right) + 4(1+2\alpha)^2 B_1 B_3 - 3(1+\alpha)(1+3\alpha)B_2^2,
\end{aligned}$$

then

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| = T\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|.$$

Consequently

$$\begin{aligned}\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| &= \frac{T}{4}\Big(c^4\left(d_1+2d_2+d_3+4d_4\right)+2xc^2\left(4-c^2\right)\left(d_1+d_2+d_3\right)\\ &\quad+\left(4-c^2\right)x^2\left(-d_1c^2+d_3\left(4-c^2\right)\right)+2d_1c\left(4-c^2\right)\left(1-|x|^2\right)y\Big)\\ &\leqslant T\Big(c^4\lambda+2sc^2\left(4-c^2\right)B_1\left(3\alpha B_1^2+u\,|B_2|\right)\\ &\quad+s^2\left(4-c^2\right)B_1^2\left(uc^2+12(1+\alpha)(1+3\alpha)\right)+8c\left(4-c^2\right)(1+2\alpha)^2B_1^2\left(1-s^2\right)\Big)\\ &= T\Big(c^4\lambda+2sc^2\left(4-c^2\right)B_1\left(3\alpha B_1^2+u\,|B_2|\right)\\ &\quad+8c(4-c^2)(1+2\alpha)^2B_1^2+s^2(4-c^2)uB_1^2(c-2)(c-p)\Big)\\ &:= F(c,s),\end{aligned}$$

where

$$p=\frac{6(1+\alpha)(1+3\alpha)}{u}>2.$$

The function $F(c,s)$ is an increasing function of $s$. Thus

$$\max_{0\leqslant s\leqslant 1} F(c,s)=F(c,1):=G(c),$$

with

$$\begin{aligned}G(c)&=T\Big(c^4\Big(\lambda-B_1\left(6\alpha B_1^2+u\left(B_1+2\,|B_2|\right)\right)\Big)\\ &\quad+c^2\Big(4B_1\left(6\alpha B_1^2+2u\,|B_2|-2\left(1+4\alpha+\alpha^2\right)B_1\right)\Big)+48(1+\alpha)(1+3\alpha)B_1^2\Big).\end{aligned}$$

Next let $c^2=t$ and

$$\begin{aligned}L&=\lambda-B_1\left(6\alpha B_1^2+u\left(B_1+2\,|B_2|\right)\right),\\ M&=8B_1\left(3\alpha B_1^2+u\,|B_2|-\left(1+4\alpha+\alpha^2\right)B_1\right),\\ N&=48(1+\alpha)(1+3\alpha)B_1^2.\end{aligned}\tag{23}$$

From (6), it follows that

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|\leqslant T\begin{cases}\frac{4LN-M^2}{4L}, & M>0,\ L\leqslant-\frac{M}{8};\\ N, & M\leqslant 0,\ L\leqslant-\frac{M}{4};\\ 16L+4M+N, & M\geqslant 0,\ L\geqslant-\frac{M}{8}\ \text{ or }\ M\leqslant 0,\ L\geqslant-\frac{M}{4},\end{cases}$$

where $L, M, N$ are given by (23). The rest of the proof is now evident. □

Theorem 2.14 yields the following special cases.

**Remark 2.15.**

1. *If $\alpha=0$, then Theorem 2.14 reduces to Theorem 2.2.*
2. *When $k=1$ and $\alpha=1$, Theorem 2.14 reduces to [25, Theorem 2].*
3. *Choosing $\varphi(z)=(1+z)/(1-z), k=1$ and $\alpha=1$, then Theorem 2.14 reduces to [23, Theorem 3.2].*

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