SUBORDINATION AND CONVOLUTION OF MULTIVALENT FUNCTIONS AND STARLIKENESS OF INTEGRAL TRANSFORMS

by

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SYMBOLS

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$\mathcal{A}_p$	Class of all p-valent analytic functions f of the form $f(z) = z^p + \sum_{k=1+p}^{\infty} a_k z^k \quad (z \in \mathcal{U})$	16
$\mathcal{A} := \mathcal{A}_1$	Class of analytic functions f of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathcal{U})$	2
$(a)_n$	Pochhammer symbol	79
$\mathbb{C}$	Complex plane	1
$\mathcal{CCV}$	Class of close-to-convex functions in $\mathcal{A}$	6
$\mathcal{CCV}_\alpha$	Class of close-to-convex functions of order α in $\mathcal{A}$	6
$\mathcal{CCV}(\varphi, \psi)$	$\left\{ f \in \mathcal{A} : \frac{f'(z)}{h'(z)} \prec \varphi(z), \quad h \in \mathcal{CV}(\psi) \right\}$	74
$\mathcal{CCV}(\alpha, \tau)$	$\left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{f'(z)}{h'(z)} \right) > \alpha, \quad h \in \mathcal{CV}(\tau) \right\}$	74
$\mathcal{CCV}_p^n(h)$	$\left\{ f \in \mathcal{A}_p : \frac{1}{p} \frac{zf'(z)}{\phi_n(z)} \prec h(z), \ \phi \in \mathcal{ST}_p^n(h) \right\}$	56
$\mathcal{CCV}_{p,g}^n(h)$	$\left\{ f \in \mathcal{A}_p : \frac{1}{p} \frac{z(g*f)'(z)}{(g*\phi)_n(z)} \prec h(z), \ \phi \in \mathcal{ST}_{p,g}^n(h) \right\}$	56
$\mathcal{CP}_p(\alpha, \lambda)$	$\left\{ f \in \mathcal{A}_p : \mathrm{Re}\left(\frac{1}{p} \frac{(zf'(z))'}{p(1-\lambda)z^{p-1}+\lambda f'(z)} \right) + \alpha > \left| \frac{(zf'(z))'}{p(1-\lambda)z^{p-1}+\lambda f'(z)} - \alpha \right|, \quad z \in \mathcal{U} \right\}$	111
$\mathcal{CV}$	Class of convex functions in $\mathcal{A}$	5
$\mathcal{CV}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$	5
$\mathcal{CV}_g(h)$	$\left\{ f \in \mathcal{A} : 1 + \frac{z(f*g)''(z)}{(f*g)'(z)} \prec h(z) \right\}$	23
$\mathcal{CV}[A, B]$	$\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1) \}$	12
$\mathcal{CV}(\varphi)$	$\left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}$	12

$\mathcal{CV}_p$	Class of convex functions in $\mathcal{A}_p$	16
$\mathcal{CV}_p(\beta)$	Class of convex functions of order β in $\mathcal{A}_p$	18
$\mathcal{CV}_p(\varphi)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \phi(z)\right\}$	17
$\mathcal{CV}_p^n(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{(zf')'(z)}{f_n'(z)} \prec h(z)\right\}$	51
$\mathcal{CV}_{p,g}^n(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{CV}_p^n(h)\right\}$	52
$\mathcal{CVS}_p^n(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{2(zf')'(z)}{f_n'(z)+f_n'(-z)} \prec h(z),\ \frac{f_n'(z)+f_n'(-z)}{z^{p-1}} \neq 0 \text{ in } \mathcal{U}\right\}$	59
$\mathcal{CVS}_{p,g}^n(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{CVS}_p^n(h)\right\}$	59
$\mathcal{CVC}_p^n(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{2(zf')'(z)}{f_n'(z)+\overline{f_n'(\overline{z})}} \prec h(z),\ \frac{f_n'(z)+\overline{f_n'(\overline{z})}}{z^{p-1}} \neq 0 \text{ in } \mathcal{U}\right\}$	62
$\mathcal{CVC}_{p,g}^n(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{CVC}_p^n(h)\right\}$	62
$\mathcal{CVSC}_p^n(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{2(zf')'(z)}{f_n'(z)+\overline{f_n'(-\overline{z})}} \prec h(z),\ \frac{f_n'(z)+\overline{f_n'(-\overline{z})}}{z^{p-1}} \neq 0 \text{ in } \mathcal{U}\right\}$	66
$\mathcal{CVSC}_{p,g}^n(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{CVSC}_p^n(h)\right\}$	66
$\overline{co}(D)$	The closed convex hull of a set D	28
$f * g$	Convolution or Hadamard product of functions f and g	14
$\mathcal{H}(\mathcal{U})$	Class of analytic functions in $\mathcal{U}$	1
$\mathcal{H}[b,n]$	Class of analytic functions f in $\mathcal{U}$ of the form	1
	$f(z) = b + b_n z^n + b_{n+1}z^{n+1} + \cdots$	
$\mathcal{H}_0 := \mathcal{H}[0,1]$	Class of analytic functions f in $\mathcal{U}$ of the form	1
	$f(z) = b_1 z + b_2 z^2 + \cdots$	
$\mathcal{H} := \mathcal{H}[1,1]$	Class of analytic functions f in $\mathcal{U}$ of the form	1
	$f(z) = 1 + b_1 z + b_2 z^2 + \cdots$	
$\prec$	Subordinate to	11
k	Koebe function $k(z) = z/(1-z)^2$	2

$\mathbb{N}$	$\mathbb{N} := \{1, 2, \cdots\}$	15
$N_\delta(f)$	$\left\{z + \sum_{k=2}^{\infty} b_k z^k \ : \ \sum_{k=2}^{\infty} k\lvert a_k - b_k\rvert \le \delta\right\}$	32
$N_\delta^p(f)$	$\left\{z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \ : \ \sum_{k=1}^{\infty} \frac{(p+k)}{p} \left\lvert a_{p+k} - b_{p+k}\right\rvert \le \delta\right\}$	111
$\mathcal{P}(\beta)$	$\left\{f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\left(f'(z) - \beta\right) > 0,\ z \in \mathcal{U}\right\}$	30
$\mathcal{P}_\alpha(\beta)$	$\Big\{f \in \mathcal{A} : \exists \phi \in \mathbb{R}$ with $\operatorname{Re} e^{i\phi}\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - \beta\right) > 0,\ z \in \mathcal{U}\Big\}$	31
$\mathcal{PST}$	Class of parabolic starlike functions in $\mathcal{A}$	8
$\mathcal{PST}(\alpha)$	Class of parabolic starlike functions of order α in $\mathcal{A}$	9
$\mathcal{PST}(\alpha, \beta)$	Class of parabolic β-starlike functions of order α in $\mathcal{A}$	9
$\mathcal{QCV}$	Class of quasi-convex functions in $\mathcal{A}$	6
$\mathbb{R}$	Set of all real numbers	2
$\mathcal{QCCV}_p^n(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{\left(zf'\right)'(z)}{\phi_n'(z)} \prec h(z),\ \phi \in \mathcal{CV}_p^n(h)\right\}$	56
$\mathcal{QCCV}_{p,g}^n(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{\left(z(g*f)'\right)'(z)}{(g*\phi)_n'(z)} \prec h(z),\ \phi \in \mathcal{CV}_{p,g}^n(h)\right\}$	56
Re	Real part of a complex number	5
$\mathcal{R}_\alpha$	Class of prestarlike functions of order α in $\mathcal{A}$	15
$\mathcal{R}_p(\alpha)$	Class of prestarlike functions of order α in $\mathcal{A}_p$	18
$\mathcal{R}(\beta)$	$\left\{f \in \mathcal{A} : \operatorname{Re}\left(f'(z) + zf''(z)\right) > \beta,\ z \in \mathcal{U}\right\}$	29
$\mathcal{R}_\gamma(\beta)$	$\{f \in \mathcal{A} : \exists \phi \in \mathbb{R}$ with $\operatorname{Re} e^{i\phi}\left(f'(z) + \gamma z f''(z) - \beta\right) > 0, \quad z \in \mathcal{U}\Big\}$	31
$\mathcal{S}$	Class of all normalized univalent functions f of the form $f(z) = z + a_2 z^2 + \cdots,\ z \in \mathcal{U}$	2
$\mathcal{SCCV}_\alpha$	Class of strongly close-to-convex functions of order α in $\mathcal{A}$	6

$\mathcal{SCV}_\alpha$	Class of strongly convex functions of order α in $\mathcal{A}$	6
$\mathcal{SP}_p(\alpha,\lambda)$	$\left\{f\in\mathcal{A}_p:\operatorname{Re}\left(\frac{1}{p}\frac{zf'(z)}{(1-\lambda)z^p+\lambda f(z)}\right)+\alpha > \left\|\frac{1}{p}\frac{zf'(z)}{(1-\lambda)z^p+\lambda f(z)}-\alpha\right\|,\quad z\in\mathcal{U}\right\}$	111
$\mathcal{SST}_\alpha$	Class of strongly starlike functions of order α in $\mathcal{A}$	6
$\mathcal{ST}$	Class of starlike functions in $\mathcal{A}$	5
$\mathcal{ST}[A,B]$	$\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz}\quad(-1\le B<A\le 1)\}$	12
$\mathcal{ST}(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$	5
$\mathcal{ST}(\varphi)$	$\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\varphi(z)\right\}$	12
$\mathcal{ST}_p$	Class of starlike functions in $\mathcal{A}_p$	16
$\mathcal{ST}_p(\beta)$	Class of starlike functions of order β in $\mathcal{A}_p$	18
$\mathcal{ST}_p(\varphi)$	$\left\{f\in\mathcal{A}_p:\frac{1}{p}\frac{zf'(z)}{f(z)}\prec\phi(z)\right\}$	16
$\mathcal{ST}_s$	Class of starlike functions with respect to symmetric points in $\mathcal{A}$	7
$\mathcal{ST}_c$	Class of starlike functions with respect to conjugate points in $\mathcal{A}$	7
$\mathcal{ST}_{sc}$	Class of starlike functions with respect to symmetric conjugate points in $\mathcal{A}$	7
$\mathcal{ST}_g(h)$	$\left\{f\in\mathcal{A}:\frac{z(f*g)'(z)}{(f*g)(z)}\prec h(z)\right\}$	23
$\mathcal{ST}_s^n$	Class of starlike functions with respect to n-ply symmetric points in $\mathcal{A}$	24
$\mathcal{ST}_c^n$	Class of starlike functions with respect to n-ply conjugate points in $\mathcal{A}$	25

$\mathcal{ST}^n_{sc}$	Class of starlike functions with respect to n-ply symmetric conjugate points in $\mathcal{A}$	25
$\mathcal{ST}^n_p(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{zf'(z)}{f_n(z)} \prec h(z),\ \frac{f_n(z)}{z^p} \neq 0 \text{ in } \mathcal{U}\right\}$	51
$\mathcal{ST}^n_{p,g}(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{ST}^n_p(h)\right\}$	51
$\mathcal{STS}^n_p(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{2zf'(z)}{f_n(z)-f_n(-z)} \prec h(z),\ \frac{f_n(z)-f_n(-z)}{z^p} \neq 0 \text{ in } \mathcal{U}\right\}$	58
$\mathcal{STS}^n_{p,g}(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{STS}^n_p(h)\right\}$	58
$\mathcal{STC}^n_p(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{2zf'(z)}{f_n(z)+\overline{f_n(\overline{z})}} \prec h(z),\ \frac{f_n(z)+\overline{f_n(\overline{z})}}{z^p} \neq 0 \text{ in } \mathcal{U}\right\}$	62
$\mathcal{STC}^n_{p,g}(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{STC}^n_p(h)\right\}$	62
$\mathcal{STSC}^n_p(h)$	$\left\{f \in \mathcal{A}_p : \frac{1}{p}\frac{2zf'(z)}{f_n(z)-\overline{f_n(-\overline{z})}} \prec h(z),\ \frac{f_n(z)-\overline{f_n(-\overline{z})}}{z^p} \neq 0 \text{ in } \mathcal{U}\right\}$	65
$\mathcal{STSC}^n_{p,g}(h)$	$\left\{f \in \mathcal{A}_p : f * g \in \mathcal{STSC}^n_p(h)\right\}$	66
$\mathcal{U}$	Open unit disk $\{z \in \mathcal{C} : \lvert z\rvert < 1\}$	1
$\mathcal{U}_r$	Open disk $\{z \in \mathcal{C} : \lvert z\rvert < r\}$ of radius r	7
$\mathcal{UST}$	Class of uniformly starlike functions in $\mathcal{A}$	8
$\mathcal{UCV}$	Class of uniformly convex functions in $\mathcal{A}$	8
$\mathcal{UCV}(\alpha)$	Class of uniformly convex functions of order α in $\mathcal{A}$	9
$\mathcal{UCV}(\alpha,\beta)$	Class of uniformly β-convex functions of order α in $\mathcal{A}$	10
$\mathcal{V}^*$	The dual set of $\mathcal{V}$	27
$\mathcal{V}^{**}$	The second dual of $\mathcal{V}$	27
$\mathcal{W}_\beta(\alpha,\gamma)$	$\Big\{f \in \mathcal{A} : \exists\phi \in \mathbb{R}$ with $\operatorname{Re} e^{i\phi}\Big((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma zf''(z) - \beta\Big) > 0, z \in \mathcal{U}\Big\}$	31
$\Psi_n[\Omega,q]$	Class of admissible functions	19

SUBORDINASI DAN KONVOLUSI FUNGSI MULTIVALEN DAN PENJELMAAN KAMIRAN BAK–BINTANG

ABSTRAK

Tesis ini membincangkan fungsi analisis dan fungsi multivalen yang tertakrif pada cakera unit terbuka $\mathcal{U}$. Umumnya, fungsi-fungsi tersebut diandaikan ternormal, sama ada dalam bentuk

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

atau

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p},$$

dengan p integer positif tetap. Andaikan $\mathcal{A}$ sebagai kelas yang terdiri daripada fungsi-fungsi f dengan penormalan pertama, manakala $\mathcal{A}_p$ terdiri daripada fungsi-fungsi f dengan penormalan kedua. Tesis ini mengkaji lima masalah penyelidikan.

Pertama, andaikan $f^{(q)}$ sebagai terbitan peringkat ke-q bagi fungsi $f \in \mathcal{A}_p$. Dengan menggunakan teori subordinasi pembeza, syarat cukup diperoleh agar rantai pembeza berikut dipenuhi:

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z), \text{ atau } \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z).$$

Di sini, Q ialah fungsi superordinasi yang bersesuaian, $\lambda(p,q) = p!/(p-q)!$, dan $\prec$ menandai subordinasi. Sebagai hasil susulan penting, beberapa kriteria sifat univalen dan cembung diperoleh bagi kes $p = q = 1$.

Sifat bak-bintang terhadap titik n-lipat juga diitlakkan kepada kes fungsi mul-

tivalen. Hal ini melibatkan fungsi-fungsi $f \in \mathcal{A}_p$ yang memenuhi subordinasi

$$\frac{1}{p}\frac{zf'(z)}{\frac{1}{n}\sum_{k=0}^{n-1}\epsilon^{n-k}f(\epsilon^k z)} \prec h(z),$$

dengan h sebagai fungsi cembung ternormalkan yang mempunyai bahagian nyata positif serta $h(0) = 1$, n integer positif tetap, dan ϵ memenuhi $\epsilon^n = 1$, $\epsilon \neq 1$. Dengan cara yang serupa, kelas fungsi p-valen cembung, hampir-cembung dan kuasicembung terhadap titik n-lipat diperkenalkan, serta juga fungsi p-valen bak-bintang dan fungsi cembung terhadap titik simetri n-lipat, titik konjugat dan titik konjugat simetri. Sifat rangkuman kelas dan konvolusi bagi kelas-kelas tersebut dikaji.

Sifat mengawetkan rangkuman bagi pengoperasian kamiran juga diperluaskan. Dua pengoperasian kamiran $F : \mathcal{A}^n \times \overline{\mathcal{U}}^2 \to \mathcal{A}$ dan $G : \mathcal{A}^n \times \overline{\mathcal{U}}^2 \to \mathcal{A}$ dibincangkan, dengan

$$F(z) = F_{f_1,\cdots,f_n;z_1,z_2}(z) = \int_0^z \prod_{j=1}^{n}\left(\frac{f_j(z_2\zeta) - f_j(z_1\zeta)}{(z_2 - z_1)\zeta}\right)^{\alpha_j} d\zeta \quad (z_1, z_2 \in \overline{\mathcal{U}}),$$

$$G(z) = G_{f_1,\cdots,f_n;z_1,z_2}(z) = z\prod_{j=1}^{n}\left(\frac{f_j(z_2 z) - f_j(z_1 z)}{(z_2 - z_1)z}\right)^{\alpha_j} \quad (z_1, z_2 \in \overline{\mathcal{U}}).$$

Pengoperasian tersebut merupakan pengitlakan hasil kajian-kajian terdahulu. Sifat mengawetkan bak-bintang, cembung, dan hampir-cembung dikaji, bukan sahaja bagi fungsi f_j yang terletak di dalam kelas-kelas tertentu, tetapi juga bagi fungsi f_j yang terletak di dalam kelas fungsi bak-bintang ala Ma-Minda dan cembung ala Ma-Minda.

Satu penjelmaan kamiran menarik yang mendapat perhatian pelbagai kajian dewasa ini ialah $V_\lambda : \mathcal{A} \to \mathcal{A}$ dengan

$$V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt.$$

Di sini λ merupakan fungsi nyata tak negatif terkamirkan pada $[0,1]$ yang memenuhi syarat $\int_0^1 \lambda(t)dt = 1$. Penjelmaan tersebut mempunyai penggunaan signifikan dalam teori fungsi geometri. Tesis ini mengkaji sifat bak-bintang penjelmaan V_λ pada kelas

$$\mathcal{W}_\beta(\alpha,\gamma) := \Bigg\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\Big((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) - \beta\Big) > 0, z \in \mathcal{U} \Bigg\}$$

dengan menggunakan Prinsip Dual. Sebagai hasil susulan penting, nilai terbaik $\beta < 1$ diperoleh yang mempastikan fungsi-fungsi $f \in \mathcal{A}$ yang memenuhi syarat

$$\operatorname{Re}\Big(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)\Big) > \beta$$

in $\mathcal{U}$ adalah semestinya bak-bintang pada $\mathcal{U}$. Contoh-contoh penting turut dibangunkan sepadan dengan pilihan tertentu fungsi teraku λ.

Tesis ini diakhiri dengan memperkenalkan dua subkelas multivalen pada $\mathcal{A}_p$. Kelas-kelas tersebut terdiri daripada fungsi bak-bintang parabola teritlak peringkat α jenis λ, ditandai $\mathcal{SP}_p(\alpha,\lambda)$, dan kelas fungsi cembung parabola peringkat α jenis λ, ditandai $\mathcal{CP}_p(\alpha,\lambda)$. Kedua-dua kelas tersebut ditunjukkan tertutup terhadap konvolusi dengan fungsi prabak-bintang. Turut diperoleh adalah kriteria baru bagi fungsi-fungsi untuk terletak di dalam kelas $\mathcal{SP}_p(\alpha,\lambda)$. Jiranan- δ bagi fungsi-fungsi di dalam kelas-kelas tersebut juga dicirikan.

SUBORDINATION AND CONVOLUTION OF MULTIVALENT FUNCTIONS AND STARLIKENESS OF INTEGRAL TRANSFORMS

ABSTRACT

This thesis deals with analytic functions as well as multivalent functions defined on the unit disk $\mathcal{U}$. In most cases, these functions are assumed to be normalized, either of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

or

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p},$$

p a fixed positive integer. Let $\mathcal{A}$ be the class of functions f with the first normalization, while $\mathcal{A}_p$ consists of functions f with the latter normalization. Five research problems are discussed in this work.

First, let $f^{(q)}$ denote the q-th derivative of a function $f \in \mathcal{A}_p$. Using the theory of differential subordination, sufficient conditions are obtained for the following differential chain to hold:

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z), \text{ or } \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z).$$

Here Q is an appropriate superordinate function, $\lambda(p;q) = p!/(p-q)!$, and $\prec$ denotes subordination. As important consequences, several criteria for univalence and convexity are obtained for the case $p = q = 1$.

The notion of starlikeness with respect to $n-$ply points is also generalized to

the case of multivalent functions. These are functions $f \in \mathcal{A}_p$ satisfying

$$\frac{1}{p}\frac{zf'(z)}{\frac{1}{n}\sum_{k=0}^{n-1}\epsilon^{n-k}f(\epsilon^k z)} \prec h(z),$$

where h is a normalized convex function with positive real part satisfying $h(0) = 1$, n a fixed positive integer, and ϵ satisfies $\epsilon^n = 1,\ \epsilon \neq 1$. Similar classes of p-valent functions to be convex, close-to-convex and quasi-convex functions with respect to n-ply points, as well as p-valent starlike and convex functions with respect to n-ply symmetric points, conjugate points and symmetric conjugate points respectively are introduced. Inclusion and convolution properties of these classes are investigated.

Membership preservation properties of integral operators are also extended. Two integral operators $F : \mathcal{A}^n \times \overline{\mathcal{U}}^2 \to \mathcal{A}$ and $G : \mathcal{A}^n \times \overline{\mathcal{U}}^2 \to \mathcal{A}$ are considered, where

$$F(z) = F_{f_1,\cdots,f_n;z_1,z_2}(z) = \int_0^z \prod_{j=1}^n \left(\frac{f_j(z_2\zeta) - f_j(z_1\zeta)}{(z_2 - z_1)\zeta}\right)^{\alpha_j} d\zeta \quad (z_1, z_2 \in \overline{\mathcal{U}}),$$

$$G(z) = G_{f_1,\cdots,f_n;z_1,z_2}(z) = z\prod_{j=1}^n \left(\frac{f_j(z_2 z) - f_j(z_1 z)}{(z_2 - z_1)z}\right)^{\alpha_j} \quad (z_1, z_2 \in \overline{\mathcal{U}}).$$

These operators are generalization of earlier works. Preservation properties of starlikeness, convexity, and close-to-convexity are investigated, not only for functions f_j belonging to those respective classes, but also for functions f_j in the classes of Ma-Minda type starlike and convex functions.

An interesting integral transform that has attracted many recent works is the transform $V_\lambda : \mathcal{A} \to \mathcal{A}$ given by

$$V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}dt,$$

where λ is an integrable non-negative real-valued function on $[0,1]$ satisfying $\int_0^1 \lambda(t)dt = 1$. This transform has significant applications in geometric function theory. This thesis investigates starlikeness of the transform V_λ over the class

$$\mathcal{W}_\beta(\alpha,\gamma) := \Bigg\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\Big((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta\Big) > 0,\ z \in \mathcal{U} \Bigg\}$$

using the Duality Principle. As a significant consequence, the best value of $\beta < 1$ is obtained that ensures functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\Big(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)\Big) > \beta$$

in $\mathcal{U}$ are necessarily starlike. Important examples are also determined for specific choices of the admissible function λ.

Finally, two multivalent subclasses of $\mathcal{A}_p$ are introduced. These classes consist of generalized parabolic starlike functions of order α and type λ, denoted by $\mathcal{SP}_p(\alpha,\lambda)$, and parabolic convex functions of order α and type λ, denoted by $\mathcal{CP}_p(\alpha,\lambda)$. It is shown that these two classes are closed under convolution with prestarlike functions. Additionally, a new criterion for functions to belong to the class $\mathcal{SP}_p(\alpha,\lambda)$ is derived. We also describe the δ-neighborhood of functions belonging to these classes.

CHAPTER 1

INTRODUCTION

1.1 Univalent Functions

Let $\mathbb{C}$ be the complex plane. A function f is *analytic* at z_0 in a domain D if it is differentiable in some neighborhood of z_0, and it is *analytic on a domain* D if it is analytic at all points in D. A function f which is analytic on a domain D is said to be *univalent* there if it is a one-to-one mapping on D, and f is *locally univalent* at $z_0 \in D$ if it is univalent in some neighborhood of z_0. It is evident that f is locally univalent at z_0 provided $f'(z_0) \neq 0$. The Riemann Mapping theorem is an important theorem in geometric function theory. It states that every simply connected domain which is not the whole complex plane can be mapped conformally onto the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem 1.1 (Riemann Mapping Theorem) [40, p. 11] *Let D be a simply connected domain which is a proper subset of the complex plane. Let ζ be a given point in D. Then there is a unique univalent analytic function f which maps D onto the unit disk $\mathcal{U}$ satisfying $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

Let $\mathcal{H}(\mathcal{U})$ be the class of analytic functions in $\mathcal{U}$ and $\mathcal{H}[b, n]$ be the subclass of $\mathcal{H}(\mathcal{U})$ consisting of functions of the form

$$g(z) = b + b_n z^n + b_{n+1} z^{n+1} + \cdots . \tag{1.1}$$

Denote by $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$. If $g \in \mathcal{H}[b_0, 1]$ is univalent in $\mathcal{U}$, then $g(z) - b_0$ is again univalent in $\mathcal{U}$ as the addition of a constant only translates the image. Since g is univalent in $\mathcal{U}$, then $b_1 = g'(0) \neq 0$, and hence $f(z) = (g(z) - b_0)/b_1$ is also univalent in $\mathcal{U}$. Conversely, if f is univalent in $\mathcal{U}$, then so is

g. Putting $b_n/b_1 = a_n,\ n = 1, 2, 3 \cdots$ in (1.1) gives the normalized form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

In the treatment of univalent analytic functions in $\mathcal{U}$, it is sufficient to consider the class $\mathcal{A}$ consisting of all functions f analytic in $\mathcal{U}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. A function f in $\mathcal{A}$ has a Taylor series of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathcal{U}).$$

The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. The function k in the class $\mathcal{S}$ given by

$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = \sum_{n=1}^{\infty} n z^n \quad (z \in \mathcal{U}) \tag{1.2}$$

is called the Koebe function. It maps $\mathcal{U}$ onto the complex plane except for a slit along the half-line $(-\infty, -1/4]$. The Koebe function and its rotations $e^{-i\beta}k(e^{i\beta}z),\ \beta \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers), play a very important role in the study of $\mathcal{S}$. They often are the extremal functions for various problems in $\mathcal{S}$. In 1916, Bieberbach [20] proved the following theorem for functions in $\mathcal{S}$.

Theorem 1.2 (Bieberbach's Theorem) [40, p. 30] *If $f \in \mathcal{S}$, then $|a_2| \le 2$, with equality if and only if f is a rotation of the Koebe function k.*

In the same paper, Bieberbach conjectured that, for $f \in \mathcal{S}$, $|a_n| \le n$ is generally valid. For the cases $n = 3$, and $n = 4$, the conjecture was proved respectively by Löwner [69], and Garabedian and Schiffer [50]. Much later in 1985, de Branges [22] proved the Bieberbach's conjecture for all coefficients with the help of the hypergeometric functions. Bieberbach's theorem has important implications in the theory of univalent functions. These include the famous covering theorem which states that if $f \in \mathcal{S}$, then the image of $\mathcal{U}$ under f must cover the open disk centered at the origin of radius 1/4.

Theorem 1.3 (Koebe One-Quarter Theorem) [40, p. 31] *The range of every function $f \in \mathcal{S}$ contains the disk $\{w : |w| < 1/4\}$.*

The Koebe function shows that the radius one-quarter is sharp. Another important consequence of the Bieberbach's theorem is the Distortion Theorem which provides sharp upper and lower bounds for $|f'(z)|$.

Theorem 1.4 (Distortion Theorem) [40, p. 32] *For each $f \in \mathcal{S}$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} \quad (|z| = r < 1).$$

The Distortion Theorem can be applied to obtain sharp upper and lower bounds for $|f(z)|$, known as the Growth Theorem.

Theorem 1.5 (Growth Theorem) [40, p. 33] *For each $f \in \mathcal{S}$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2} \quad (|z| = r < 1).$$

Again the Koebe function demonstrates sharpness of both theorems above.

Another implication of the Bieberbach's theorem is the Rotation Theorem which provides sharp upper bound for $|\arg f'(z)|$.

Theorem 1.6 (Rotation Theorem) [40, p. 99] *For each $f \in \mathcal{S}$,*

$$|\arg f'(z)| \leq \begin{cases} 4sin^{-1}r & (r \leq \dfrac{1}{\sqrt{2}}), \\ \\ \pi + \log \frac{r^2}{1-r^2} & (r \geq \frac{1}{\sqrt{2}}), \end{cases}$$

where $r = |z| < 1$. The bound is sharp for each $z \in \mathcal{U}$.

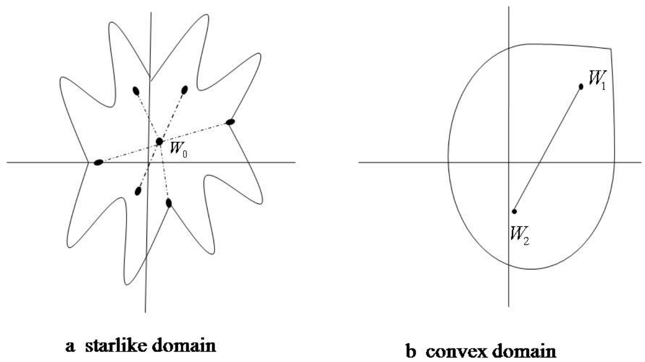

Figure 1.1: Starlike and convex domains

The very long gap between the Bieberbach's conjecture [20] of 1916 and its proof in 1985 by de Branges [22] motivated researchers to work in several directions. One of these directions was to prove the Bieberbach's conjecture $|a_n| \leq n$ for subclasses of $\mathcal{S}$ defined by geometric conditions. Among these classes are the classes of starlike functions, convex functions, close-to-convex functions, and quasi-convex functions. A set $D \subset \mathbb{C}$ is called *starlike* with respect to $w_0 \in D$ if the line segment joining w_0 to every other point $w \in D$ lies in the interior of D (see Figure 1.1a). The set D is called *convex* if for every pair of points w_1 and w_2 in D, the line segment joining w_1 and w_2 lies in the interior of D (see Figure 1.1b). A function $f \in \mathcal{A}$ is said to be a starlike function if $f(\mathcal{U})$ is a starlike domain with respect to 0, and $f \in \mathcal{A}$ is a convex function if $f(\mathcal{U})$ is a convex domain. Analytically, these geometric properties are respectively equivalent to the conditions [40, 51, 52, 55, 93]

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \text{and} \quad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,$$

where $\mathrm{Re}(w)$ is the real part of the complex number w. The Koebe function k in (1.2) is an example of a starlike function. The function

$$f(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n$$

which maps $\mathcal{U}$ onto the half-plane $\{w : \mathrm{Re}\, w > -1/2\}$ is convex. The subclasses of $\mathcal{A}$ consisting of starlike and convex functions are denoted respectively by $\mathcal{ST}$ and $\mathcal{CV}$. An important relationship between convex and starlike functions was first observed by Alexander [5] in 1915.

Theorem 1.7 (Alexander's Theorem) [40, p. 43] *Let $f \in \mathcal{A}$. Then $f \in \mathcal{CV}$ if and only if $zf' \in \mathcal{ST}$.*

Robertson [105] in 1936 introduced the concepts of starlike and convex functions of order α, $0 \le \alpha < 1$. A function $f \in \mathcal{A}$ is said to be *starlike or convex of order* α if it satisfies the condition

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \alpha \quad \text{or} \quad \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \ge \alpha \quad (0 \le \alpha < 1).$$

These classes will be denoted respectively by $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$. Evidently $\mathcal{ST}(0) = \mathcal{ST}$ and $\mathcal{CV}(0) = \mathcal{CV}$.

For $0 < \alpha \le 1$, Brannan and Kirwan [23] introduced the classes of strongly starlike and strongly convex functions of order α. A function $f \in \mathcal{A}$ is said to be *strongly starlike of order* α if it satisfies

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \le \frac{\alpha\pi}{2} \quad (z \in \mathcal{U},\ 0 < \alpha \le 1),$$

and is *strongly convex of order* α if

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| \le \frac{\alpha\pi}{2} \quad (z \in \mathcal{U},\ 0 < \alpha \le 1).$$

The subclasses of $\mathcal{A}$ consisting of strongly starlike and strongly convex functions of order α are denoted respectively by $\mathcal{SST}_\alpha$ and $\mathcal{SCV}_\alpha$. Since the condition $\operatorname{Re} w(z) > 0$ is equivalent to $|\arg w(z)| < \pi/2$, it follows that $\mathcal{SST}_1 \equiv \mathcal{ST}$ and $\mathcal{SCV}_1 \equiv \mathcal{CV}$.

In 1952, Kaplan [61] introduced the class of close-to-convex functions. A function $f \in \mathcal{A}$ is said to be *close-to-convex* if there is a function $g \in \mathcal{CV}$ such that $\operatorname{Re}\big(f'(z)/g'(z)\big) > 0$ for all $z \in \mathcal{U}$, or equivalently, if there is a function $g \in \mathcal{ST}$ such that $\operatorname{Re}\big(zf'(z)/g(z)\big) > 0$ for all $z \in \mathcal{U}$. The class of all close-to-convex functions in $\mathcal{A}$ is denoted by $\mathcal{CCV}$. A function $f \in \mathcal{A}$ is said to be *close-to-convex of order* α, $0 \leq \alpha < 1$, if there is a function $g \in \mathcal{CV}$ such that $\operatorname{Re}\big(f'(z)/g'(z)\big) > \alpha$ for all $z \in \mathcal{U}$. This class is denoted by $\mathcal{CCV}_\alpha$.

Reade [104] introduced the class of strongly close-to-convex functions of order α, $0 < \alpha \leq 1$. A function $f \in \mathcal{A}$ is said to be *strongly close-to-convex of order* α if there is function $\phi \in \mathcal{CV}$ satisfying

$$\left|\arg \frac{f'(z)}{\phi'(z)}\right| \leq \frac{\alpha\pi}{2} \quad (z \in \mathcal{U},\ 0 < \alpha \leq 1).$$

The subclass of $\mathcal{A}$ consisting of strongly close-to-convex functions of order α is denoted by $\mathcal{SCCV}_\alpha$. When $\alpha = 1$, $\mathcal{SCCV}_1 \equiv \mathcal{CCV}$.

In 1980, Noor and Thomas [80] introduced the class of quasi-convex functions. A function $f \in \mathcal{A}$ is said to be *quasi-convex* if there is a function $g \in \mathcal{CV}$ such that $\operatorname{Re}\big((zf'(z))'/g'(z)\big) > 0$ for all $z \in \mathcal{U}$. The class of all quasi-convex functions in $\mathcal{A}$ is denoted by $\mathcal{QCV}$.

A function $f \in \mathcal{A}$ is said to be *starlike with respect to symmetric points in* $\mathcal{U}$ if for every r less than and sufficiently close to one and every ζ on $|z| = r$, the angular velocity of $f(z)$ about the point $f(-\zeta)$ is positive at $z = \zeta$ as z traverses the circle $|z| = r$ in the positive direction, that is,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z) - f(-\zeta)}\right) > 0 \quad (z = \zeta,\ |\zeta| = r).$$

This class was introduced and studied in 1959 by Sakaguchi [115]. Let the class of these functions be denoted by $\mathcal{ST}_s$. An equivalent description of this class is given by the following theorem.

Theorem 1.8 [115] *Let $f \in \mathcal{A}$. Then $f \in \mathcal{ST}_s$ if and only if*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0 \quad (z \in \mathcal{U}).$$

Further investigations into the class of starlike functions with respect to symmetric points can be found in [35, 79, 85, 117, 128, 130–132, 135]. El-Ashwah and Thomas [41] introduced and studied the classes consisting of starlike functions with respect to conjugate points, and starlike functions with respect to symmetric conjugate points defined respectively by the conditions

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z) + \overline{f(\overline{z})}}\right) > 0, \quad \mathrm{Re}\left(\frac{zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right) > 0.$$

Let the classes of these functions be denoted respectively by $\mathcal{ST}_c$ and $\mathcal{ST}_{sc}$.

Ford [44] observed that convex or starlike functions inherit hereditary properties. In other words, if $f \in \mathcal{S}$ is starlike or convex, then $f(\mathcal{U}_r)$ is also a starlike or a convex domain, where $\mathcal{U}_r = \{z : |z| < r\}$.

Theorem 1.9 (Ford's Theorem) [52, p. 114] *Let f be in $\mathcal{S}$. If $f(\mathcal{U})$ is a convex domain, then for each positive $r < 1$, $f(\mathcal{U}_r)$ is also a convex domain. If $f(\mathcal{U})$ is starlike with respect to the origin, then for each positive $r < 1$, $f(\mathcal{U}_r)$ is also starlike with respect to the origin.*

It follows from the above theorem that convex (starlike) functions map circles centered at the origin in the unit disk onto convex (starlike) area. However this geometric property does not hold in general for circles whose centers are not at the origin. This motivated Goodman [53, 54] to introduce the classes $\mathcal{UCV}$ and $\mathcal{UST}$ of uniformly convex and uniformly starlike functions. An analytic function $f \in \mathcal{S}$ is uniformly convex (uniformly starlike) if f maps every circular arc γ contained in $\mathcal{U}$ with center ζ also in $\mathcal{U}$ onto a convex (starlike with respect to $f(\zeta)$) arc. Analytic descriptions of these classes are given by the following theorem.

Theorem 1.10 [53, 54] *Let $f \in \mathcal{A}$. Then*

(a) $f \in \mathcal{UCV}$ if and only if

$$\mathrm{Re}\left(1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right) \geq 0 \quad (\,(z,\zeta) \in \mathcal{U} \times \mathcal{U})\,.$$

(b) $f \in \mathcal{UST}$ if and only if

$$\mathrm{Re}\,\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)} \geq 0 \quad (\,(z,\zeta) \in \mathcal{U} \times \mathcal{U})\,.$$

Rønning [106] (also, see [70]) gave a more applicable one variable analytic characterization for $\mathcal{UCV}$. A normalized analytic function $f \in \mathcal{A}$ belongs to $\mathcal{UCV}$ if and only if it satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathcal{U}).$$

Goodman [54] gave examples that demonstrated the Alexander's relation (Theorem 1.7) does not hold between the classes $\mathcal{UCV}$ and $\mathcal{UST}$. Later Rønning [107] introduced the class of parabolic starlike functions $\mathcal{PST}$ consisting of functions

$F = zf'$ such that $f \in \mathcal{UCV}$. It is evident that $f \in \mathcal{PST}$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \mathcal{U}).$$

Let

$$\Omega = \{w : \operatorname{Re} w > |w - 1|\} = \left\{w : (\operatorname{Im} w)^2 < 2\operatorname{Re} w - 1\right\}.$$

Clearly, Ω is a parabolic region bounded by $y^2 = 2x - 1$. The function $f \in \mathcal{UCV}$ if and only if $\left(1 + zf''/f'\right) \in \Omega$. Similarly, $f \in \mathcal{PST}$ if and only if $zf'/f \in \Omega$. For this reason, a function $f \in \mathcal{PST}$ is called a parabolic starlike function. A survey of these functions can be found in [108]. In [106, 109], Rønning generalized the classes $\mathcal{UCV}$ and $\mathcal{PST}$ by introducing a parameter α in the following way: a function $f \in \mathcal{A}$ is in $\mathcal{PST}(\alpha)$ if it satisfies the analytic characterization

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (\alpha \in \mathbb{R},\ z \in \mathcal{U}),$$

and $f \in \mathcal{UCV}(\alpha)$, the class of uniformly convex functions of order α, if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \left|\frac{zf''(z)}{f'(z)}\right| \quad (\alpha \in \mathbb{R},\ z \in \mathcal{U}).$$

He also introduced the more general classes $\mathcal{PST}(\alpha, \beta)$ consisting of parabolic β-starlike functions of order α that satisfies the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (-1 < \alpha \leq 1,\ \beta \geq 0,\ z \in \mathcal{U}). \tag{1.3}$$

Analogously, the class $\mathcal{UCV}(\alpha,\beta)$ consists of uniformly β-convex functions of order α satisfying the condition

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right)>\beta\left|\frac{zf''(z)}{f'(z)}\right| \quad (-1<\alpha\leq 1,\ \beta\geq 0,\ z\in\mathcal{U}). \tag{1.4}$$

Indeed, it follows from (1.3) and (1.4) that $f\in\mathcal{UCV}(\alpha,\beta)$ if and only if $zf'\in\mathcal{PST}(\alpha,\beta)$. The geometric representation of the relation (1.3) is that the class $\mathcal{PST}(\alpha,\beta)$ consists of functions f for which the function $\left(zf'/f\right)$ takes values in the parabolic region Ω, where

$$\Omega=\{w:\operatorname{Re}w-\alpha>\beta|w-1|\}=\left\{w:\operatorname{Im}w<\frac{1}{\beta}\sqrt{\left(\operatorname{Re}w-\alpha\right)^2-\beta^2\left(\operatorname{Re}w-1\right)^2}\right\}.$$

Clearly, $\mathcal{PST}(\alpha,1)=\mathcal{PST}(\alpha)$ and $\mathcal{UCV}(\alpha,1)=\mathcal{UCV}(\alpha)$.

The transform

$$\int_0^z\frac{f(t)}{t}dt\equiv\int_0^1\frac{f(tz)}{t}dt$$

introduced by Alexander [5] is known as Alexander transform of f. Using Alexander's Theorem (Theorem 1.7), it is clear that $f\in\mathcal{ST}$ if and only if the Alexander transform of f is in $\mathcal{CV}$. Libera [67] and Livingston [68] investigated the transform

$$2\int_0^1 f(tz)dt,$$

and Bernardi [17] later considered the transform

$$(c+1)\int_0^1 t^{c-1}f(tz)dt,\quad (c>-1) \tag{1.5}$$

which generalizes the Libera and Livingston transform. For that reason, the transform (1.5) is called the generalized Bernardi-Libera-Livingston integral transform. It is well-known [17] that the classes of starlike, convex and close-to-convex functions are closed under the Bernardi-Libera-Livingston transform for all $c > -1$.

An analytic function f is *subordinate* to g in $\mathcal{U}$, written $f \prec g$, or $f(z) \prec g(z)$ $(z \in \mathcal{U})$, if there exists a function w analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathcal{U}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Recall that a function $f \in \mathcal{A}$ belongs to the class of starlike functions $\mathcal{ST}$, convex functions $\mathcal{CV}$, or close-to-convex functions $\mathcal{CCV}$ if it satisfies respectively the analytic condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad \text{and} \quad \operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad g(z) \in \mathcal{CV}.$$

A function in any one of these classes is characterized by either of the quantities $zf'(z)/f(z)$, $1 + zf''(z)/f'(z)$ or $f'(z)/g'(z)$ lying in a given region in the right half plane; the region is convex and symmetric with respect to the real axis [71]. Since $p(z) = (1 + z)/(1 - z)$ is a normalized analytic function mapping $\mathcal{U}$ onto $\{w : \operatorname{Re} w > 0\}$, in terms of subordination, the above conditions are respectively equivalent to

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \qquad 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} \quad \text{and} \quad \frac{f'(z)}{g'(z)} \prec \frac{1+z}{1-z}.$$

Ma and Minda [71] gave a unified presentation of various subclasses of starlike and convex functions by replacing the superordinate function $p(z) = (1+z)/(1-z)$ by a more general analytic function φ with positive real part and normalized by the conditions $\varphi(0) = 1$ and $\varphi'(0) > 0$. Further it is assumed that φ maps the unit disk $\mathcal{U}$ onto a region starlike with respect to 1 that is symmetric with respect to the real axis. They introduced the following general classes that enveloped several well-known classes as special cases:

$$\mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\},$$

and

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}.$$

When

$$\varphi(z) = \varphi_\alpha(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1),$$

the classes $\mathcal{CV}(\varphi_\alpha)$ and $\mathcal{ST}(\varphi_\alpha)$ reduce to the familiar classes $\mathcal{CV}(\alpha)$ and $\mathcal{ST}(\alpha)$ of univalent convex and starlike functions of order α.

When

$$\varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B \leq A \leq 1),$$

the classes $\mathcal{CV}(\varphi)$ and $\mathcal{ST}(\varphi)$ reduce respectively to the class $\mathcal{CV}[A, B]$ of *Janowski convex functions* and the class $\mathcal{ST}[A, B]$ of *Janowski starlike functions* [60, 90]. Thus

$$\mathcal{CV}[A, B] =: \mathcal{CV}\left(\frac{1 + Az}{1 + Bz}\right) \text{ and } \quad \mathcal{ST}[A, B] =: \mathcal{ST}\left(\frac{1 + Az}{1 + Bz}\right).$$

When

$$\varphi(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)^2,$$

the classes $\mathcal{CV}(\varphi)$ and $\mathcal{ST}(\varphi)$ reduce to the familiar classes of uniformly convex functions $\mathcal{UCV}$ and its associated class $\mathcal{PST}$.

Define the functions $h_\varphi \in \mathcal{ST}(\varphi)$ and $k_\varphi \in \mathcal{CV}(\varphi)$ respectively by

$$\frac{zh'_\varphi(z)}{h_\varphi(z)} = \varphi(z) \quad (z \in \mathcal{U},\ h_\varphi \in \mathcal{A})$$

$$1 + \frac{zk''_\varphi(z)}{k'_\varphi(z)} = \varphi(z) \quad (z \in \mathcal{U},\ k_\varphi \in \mathcal{A}).$$

In [71], Ma and Minda showed that the functions h_φ and k_φ turned out to be extremal for certain functionals in $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$. In addition, they derived distortion, growth, covering and rotation theorems for the classes $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$ and obtained sharp order of growth for coefficients of these classes.

Theorem 1.11 (Distortion Theorem for $\mathcal{CV}(\varphi)$) [71, Corollary 1] *For each $f \in \mathcal{CV}(\varphi)$,*

$$k'_\varphi(-r) \leq |f'(z)| \leq k'_\varphi(r) \quad (|z| = r < 1).$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of k_φ.

Theorem 1.12 (Growth Theorem for $\mathcal{CV}(\varphi)$) [71, Corollary 2] *For each $f \in \mathcal{CV}(\varphi)$,*

$$-k_\varphi(-r) \leq |f(z)| \leq k_\varphi(r) \quad (|z| = r < 1).$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of k_φ.

Theorem 1.13 (Covering Theorem for $\mathcal{CV}(\varphi)$) [71, Corollary 3] *Suppose $f \in \mathcal{CV}(\varphi)$. Then either f is a rotation of k_φ or $f(\mathcal{U}) \supseteq \{w : |w| \leq -k_\varphi(-1)\}$. Here $-k_\varphi(-1)$ is understood to be the limit of $-k_\varphi(-r)$ as r tends to 1.*

Theorem 1.14 (Rotation Theorem for $\mathcal{CV}(\varphi)$) [71, Corollary 4] *For each $f \in \mathcal{CV}(\varphi)$,*

$$|\arg f'(z)| \leq \max_{|z|=r} \arg\left(k'_\varphi(z)\right) \quad (|z| = r < 1).$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of k_φ.

Next, we state the corresponding results for the class $\mathcal{ST}(\varphi)$. These results follows from the correspondence between $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$.

Theorem 1.15 (Distortion Theorem for $\mathcal{ST}(\varphi)$) [71, Theorem 2] *If $f \in \mathcal{ST}(\varphi)$ with $\min_{|z|=r} |\varphi(z)| = |\varphi(-r)|$ and $\max_{|z|=r} |\varphi(z)| = |\varphi(r)|$, then*

$$h'_\varphi(-r) \leq |f'(z)| \leq h'_\varphi(r) \quad (|z| = r < 1).$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of h_φ.

Theorem 1.16 (Growth Theorem for $\mathcal{ST}(\varphi)$) [71, Corollary 1'] *If $f \in \mathcal{ST}(\varphi)$, then*

$$-h_\varphi(-r) \leq |f(z)| \leq h_\varphi(r) \quad (|z| = r < 1).$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of h_φ.

Theorem 1.17 (Covering Theorem for $\mathcal{ST}(\varphi)$) [71, Corollary 2'] *Suppose $f \in \mathcal{ST}(\varphi)$. Then either f is a rotation of h_φ or $f(\mathcal{U}) \supseteq \{w : |w| \leq -h_\varphi(-1)\}$. Here $-h_\varphi(-1)$ is the limit of $-h_\varphi(-r)$ as r tends to 1.*

Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in $|z| < R_1$, and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be analytic in $|z| < R_2$. The *convolution*, or *Hadamard product*, of f and g is the function $h = f * g$ given by the power series

$$h(z) = (f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n. \tag{1.6}$$

This power series is convergent in $|z| < R_1 R_2$. The term "convolution" arose from the following equivalent representation

$$(f * g)(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} f\left(\frac{z}{\zeta}\right) g(\zeta) \frac{d\zeta}{\zeta} \quad \left(\frac{|z|}{R_1} < \rho < R_2\right).$$

The geometric series

$$l(z) = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z},$$

acts as the identity element under convolution [40, pp. 246-247] for the class $\mathcal{A}$.

The functions f and zf' can be represented in terms of convolution as

$$f(z) = f * \frac{z}{1-z} \quad \text{and} \quad zf'(z) = f * \frac{z}{(1-z)^2}.$$

Using Alexander's theorem (Theorem 1.7), a function $f \in \mathcal{A}$ is convex if and only if $f * \big(z/(1-z)^2\big)$ is starlike. So the classes $\mathcal{ST}$ and $\mathcal{CV}$ can be unified by considering $\mathcal{S}_g = \{f \in \mathcal{A} : f * g \in \mathcal{ST}\}$ for an appropriate g. For $g(z) = z/(1-z)$, $\mathcal{S}_g = \mathcal{ST}$, while for $g(z) = z/(1-z)^2$, $\mathcal{S}_g = \mathcal{CV}$.

An important subclass of $\mathcal{A}$ defined by using convolution is the class of prestarlike functions introduced by Ruscheweyh [111]. For $\alpha < 1$, the class $\mathcal{R}_\alpha$ of prestarlike functions of order α is defined by

$$\mathcal{R}_\alpha := \left\{ f \in \mathcal{A} : f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{ST}(\alpha) \right\},$$

while $\mathcal{R}_1$ consists of $f \in \mathcal{A}$ satisfying $\operatorname{Re} f(z)/z > 1/2$. Prestarlike functions have a number of interesting geometric properties. For instance, $\mathcal{R}_0$ is the class of univalent convex functions $\mathcal{CV}$, and $\mathcal{R}_{1/2}$ is the class of univalent starlike functions $\mathcal{ST}(1/2)$ of order $1/2$.

1.2 Multivalent Functions

A function f is *p-valent* (or *multivalent* of order p) if for each w_0 (w_0 may be infinity), the equation $f(z) = w_0$ has at most p roots in $\mathcal{U}$, where the roots are counted with their multiplicities, and for some w_1 the equation $f(z) = w_1$ has exactly p roots in $\mathcal{U}$ [52]. For a fixed $p \in \mathbb{N} := \{1, 2, \cdots\}$, let $\mathcal{A}_p$ denote the class of all analytic functions f of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \tag{1.7}$$

that are p-valent in the open unit disk $\mathcal{U}$, and for $p = 1$, let $\mathcal{A}_1 := \mathcal{A}$.

The *convolution*, or *Hadamard product*, of two p-valent functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad \text{and} \quad g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$$

is defined as

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p}.$$

A p-valent function $f \in \mathcal{A}_p$ is *starlike* if it satisfies the condition

$$\frac{1}{p} \mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0 \quad (f(z)/z \neq 0,\ z \in \mathcal{U}),$$

and is *convex* if it satisfies the condition

$$\frac{1}{p} \mathrm{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (f'(z) \neq 0,\ z \in \mathcal{U}).$$

The subclasses of $\mathcal{A}_p$ consisting of starlike and convex functions are denoted respectively by $\mathcal{ST}_p$ and $\mathcal{CV}_p$. More generally, let φ be an analytic function with positive real part in $\mathcal{U}$, $\varphi(0) = 1$, $\varphi'(0) > 0$, and φ maps the unit disk $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The classes $\mathcal{ST}_p(\varphi)$ and $\mathcal{CV}_p(\varphi)$ consist respectively of p-valent functions f *starlike* with respect to φ and p-valent functions f *convex* with respect to φ in $\mathcal{U}$ given by

$$\mathcal{ST}_p(\varphi) := \left\{ f \in \mathcal{A}_p : \frac{1}{p} \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$

and

$$\mathcal{CV}_p(\varphi) := \left\{ f \in \mathcal{A}_p : \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\}.$$

These classes were introduced and investigated by Ali et al. in [8]. The functions $h_{\varphi,p}$ and $k_{\varphi,p}$ defined respectively by

$$\frac{1}{p}\frac{zh'_{\varphi,p}(z)}{h_{\varphi,p}(z)} = \varphi(z) \quad (z \in \mathcal{U}, h_{\varphi,p} \in \mathcal{A}_p),$$
$$\frac{1}{p}\left(1 + \frac{zk''_{\varphi,p}(z)}{k'_{\varphi,p}(z)}\right) = \varphi(z) \quad (z \in \mathcal{U}, k_{\varphi,p} \in \mathcal{A}_p),$$

are important examples of functions in $\mathcal{ST}_p(\varphi)$ and $\mathcal{CV}_p(\varphi)$. A result analogues to Alexander' theorem (Theorem 1.7) was obtained by Ali et al. in [8].

Theorem 1.18 [8, Theorem 2.1] *The function f belongs to $\mathcal{CV}_p(\varphi)$ if and only if $(1/p)zf' \in \mathcal{ST}_p(\varphi)$.*

When $p = 1$ these classes reduced to the $\mathcal{ST}(\varphi)$ and $\mathcal{CV}(\varphi)$ classes introduced by Ma and Minda [71].

When

$$\varphi(z) = \frac{1+z}{1-z},$$

the classes $\mathcal{ST}_p(\varphi)$ and $\mathcal{CV}_p(\varphi)$ reduce to the familiar classes of p-valent starlike and convex functions $\mathcal{ST}_p$ and $\mathcal{CV}_p$. In addition if $p = 1$ these classes are respectively the classes of univalent starlike and convex functions $\mathcal{ST}$ and $\mathcal{CV}$.

When

$$\varphi(z) = \varphi_\beta(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1)$$

the classes $\mathcal{ST}_p(\varphi_\beta)$ and $\mathcal{CV}_p(\varphi_\beta)$ reduce to the familiar classes of p-valent starlike and convex functions of order β introduced by Patil and Thakare in [88]:

$$\mathcal{ST}_p(\beta) := \left\{ f \in \mathcal{A}_p : \frac{1}{p}\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \right\},$$
$$\mathcal{CV}_p(\beta) := \left\{ f \in \mathcal{A}_p : \frac{1}{p}\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta \right\}.$$

For $p \in \mathbb{N}$ and $\alpha < 1$, Kumar and Reddy [12] defined the class $\mathcal{R}_p(\alpha)$ of p-valent prestarlike functions of order α by

$$\mathcal{R}_p(\alpha) = \left\{ f \in \mathcal{A}_p : f(z) * \frac{z^p}{(1-z)^{2p(1-\alpha)}} \in \mathcal{ST}_p(\alpha) \right\}.$$

They obtained necessary and sufficient coefficient conditions for a function f to be in the class $\mathcal{R}_p(\alpha)$. Evidently, this class reduces to the class of prestarlike functions $\mathcal{R}(\alpha)$ introduced by Ruscheweyh [111] for $p = 1$.

1.3 Differential Subordination

In the theory of complex-valued functions there are many differential conditions which shape the characteristics of a function. A simple example is the Noshiro-Warschawski Theorem [40, Theorem 2.16, p.47]: If f is analytic in a convex domain D, then

$$\mathrm{Re}\,\big(f'(z)\big) > 0 \Rightarrow f \text{ is univalent in } D.$$

This theorem and many known similar differential implications dealt with real-valued inequalities that involved the real part, imaginary part or modulus of a complex expression. In 1981 Miller and Mocanu [74] replaced the differential inequality, a real valued concept, with its complex analogue of differential subordination.

Let $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ be an analytic function and let h be univalent in the unit disk $\mathcal{U}$. If p is analytic in $\mathcal{U}$ and satisfies the second-order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.8}$$

then p is called a *solution* of the differential subordination. A univalent function q is called a *dominant* if $p(z) \prec q(z)$ for all p satisfying (1.8). A dominant q_1 satisfying $q_1(z) \prec q(z)$ for all dominants q of (1.8) is said to be the *best dominant* of (1.8). The best dominant is unique up to a rotation of $\mathcal{U}$. If $p \in \mathcal{H}[a, n]$, then p is called an (a, n)-*solution*, q an (a, n)-*dominant*, and q_1 the *best* (a, n)-*dominant*. Let $\Omega \subset \mathbb{C}$ and let (1.8) be replaced by

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \quad (z \in \mathcal{U}), \tag{1.9}$$

where Ω is a simply connected domain containing $h(\mathcal{U})$. Even though this is a "differential inclusion" and $\psi(p(z), zp'(z), z^2p''(z); z)$ may not be analytic in $\mathcal{U}$, the condition in (1.9) will also be referred to as a *second-order differential subordination*, and the same definition for solution, dominant and best dominant as given above can be extended to this generalization. The monograph [75] by Miller and Mocanu provides a detailed account on the theory of differential subordination.

Denote by $\mathcal{Q}$ the set of all functions q that are analytic and injective on $\overline{\mathcal{U}} \backslash E(q)$, where

$$E(q) = \{\zeta \in \partial\mathcal{U} : \lim_{z \to \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} \setminus E(q)$.

Definition 1.1 [75, Definition 2.3a, p. 27] *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfying the admissibility condition*

$$\psi(r, s, t; z) \notin \Omega \tag{1.10}$$

whenever $r = q(\zeta),\ s = k\zeta q'(\zeta),$ *and*

$$\mathrm{Re}\left(\frac{t}{s}+1\right) \geq k\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right),$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ *and* $k \geq n$. *Additionally,* $\Psi_1[\Omega, q]$ *will be written as* $\Psi[\Omega, q]$.

If $\psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$, *then the admissibility condition (1.10) reduces to*

$$\psi(q(\zeta), k\zeta q'(\zeta); z) \notin \Omega,$$

$z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U} \setminus E(q)$ *and* $k \geq n$.

If $\psi : \mathbb{C} \times \mathcal{U} \to \mathbb{C}$, *then the admissibility condition (1.10) becomes*

$$\psi(q(\zeta); z) \notin \Omega,$$

$z \in \mathcal{U}$, *and* $\zeta \in \partial\mathcal{U} \setminus E(q)$.

A foundation result in the theory of first and second order differential subordination is the following theorem:

Theorem 1.19 [75, Theorem 2.3b, p.28] *Let* $\psi \in \Psi_n[\Omega, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{H}[a, n]$ *satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega, \tag{1.11}$$

then $p \prec q$.

It is evident that the dominant of a differential subordination of the form (1.11) can be obtained by checking that the function ψ is an admissible function. This requires that the function ψ satisfies (1.10). Considering the special case when $\Omega = h(\mathcal{U})$ is a simply connected domain, and h is a conformal mapping of $\mathcal{U}$ onto

Ω, the following second-order differential subordination result is an immediate consequence of Theorem 1.19. The set $\Psi_n[h(\mathcal{U}), q]$ is written as $\Psi_n[h, q]$.

Theorem 1.20 [75, Theorem 2.3c, p.30] *Let* $\psi \in \Psi_n[h, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{H}[a, n]$, $\psi(p(z), zp'(z), z^2p''(z); z)$ *is analytic in* $\mathcal{U}$, *and*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.12}$$

then $p \prec q$.

The next theorem yields best dominant of the differential subordination (1.12)

Theorem 1.21 [75, Theorem 2.3f, p.32] *Let* h *be univalent in* $\mathcal{U}$ *and* $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. *Suppose that the differential equation*

$$\psi(q(z), nzq'(z), n(n-1)zq'(z) + n^2z^2q''(z); z) = h(z)$$

has a solution q, *with* $q(0) = a$, *and one of the following conditions is satisfied:*

(*i*) $q \in \mathcal{Q}$ *and* $\psi \in \Psi_n[h, q]$,

(*ii*) q *is univalent in* $\mathcal{U}$ *and* $\psi \in \Psi_n[h, q_\rho]$ *for some* $\rho \in (0, 1)$, *or*

(*iii*) q *is univalent in* $\mathcal{U}$ *and there exists* $\rho_0 \in (0, 1)$ *such that* $\psi \in \Psi_n[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $p \in \mathcal{H}[a, n]$, $\psi(p(z), zp'(z), z^2p''(z); z)$ *is analytic in* $\mathcal{U}$, *and* p *satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z),$$

then $p \prec q$, *and* q *is the best* (a, n)*-dominant.*

When dealing with first-order differential subordination, the following theorem is useful.

Theorem 1.22 [75, Theorem 3.4h, p.132] *Let q be univalent in $\mathcal{U}$ and let θ and ϕ be analytic in a domain D containing $q(\mathcal{U})$, with $\phi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that either*

(i) *h is convex, or*

(ii) *Q is starlike.*

In addition, assume that

(iii) $\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right) > 0.$

If p is analytic in $\mathcal{U}$, with $p(0) = q(0)$, $p(\mathcal{U}) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p \prec q$, and q is the best dominant.

Let $f \in \mathcal{A}_p$ be given by (1.7). Upon differentiating both sides of f q-times with respect to z, the following differential operator is obtained:

$$f^{(q)}(z) = \lambda(p;q)z^{p-q} + \sum_{k=1}^{\infty} \lambda(k+p;q)a_{k+p}z^{k+p-q},$$

where

$$\lambda(p;q) := \frac{p!}{(p-q)!} \quad (p \geq q; p \in \mathbb{N}; q \in \mathbb{N} \cup \{0\}).$$

Several researchers have investigated higher-order derivatives of multivalent functions, see for example [10, 11, 37, 56–58, 81, 89, 120, 141]. Recently, by use of the well-known Jack's Lemma [59, 75], Irmak and Cho [57] obtained interesting results for certain classes of functions defined by higher-order derivatives. We shall continue this investigation in Chapter 2.

1.4 Functions with Respect to n-ply Points

As defined on p. 14, the convolution of two functions f and g with power series

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n \text{ and } g(z) = z + \sum_{n=1}^{\infty} b_n z^n$$

convergent in $\mathcal{U}$ is defined by

$$(f * g)(z) := z + \sum_{n=1}^{\infty} a_n b_n z^n \quad (z \in \mathcal{U}).$$

Pólya and Schoenberg in 1958 [91] posed two important conjectures:

1. If f and $g \in \mathcal{CV}$, then $f * g \in \mathcal{CV}$.
2. If $f \in \mathcal{CV}$ and $g \in \mathcal{ST}$, then $f * g \in \mathcal{ST}$.

Using Alexander's theorem, (Theorem 1.7), it is clear that any one of these conjectures implies the other. These conjunctures were later proved by Ruscheweyh and Sheil-Small [114].

Let $h : \mathcal{U} \to \mathbb{C}$ be a convex function with positive real part in $\mathcal{U}$, $h(0) = 1$, and g be a given fixed function in $\mathcal{A}$. Shanmugam [116] introduced the classes $\mathcal{ST}_g(h)$ and $\mathcal{CV}_g(h)$ consisting of functions f satisfying

$$\frac{z(f * g)'(z)}{(f * g)(z)} \prec h(z) \text{ and } 1 + \frac{z(f * g)''(z)}{(f * g)'(z)} \prec h(z).$$

Note that for $g(z) = z/(1 - z)$, the class $\mathcal{ST}_g(h) \equiv \mathcal{ST}(h)$ and the class $\mathcal{CV}_g(h) \equiv \mathcal{CV}(h)$. He introduced these classes [116] and other related classes, and investigated inclusion and convolution properties by using the convex hull method [113, 114] and the method of differential subordination [75]. Ali *et al.* [8] investigated the subclasses of $p-$valent starlike and convex functions, and obtained several subordination and convolution properties, as well as sharp distortion, growth and rotation estimates. These works were recently extended by Supramaniam *et al.* [127]. Similar problems but for the class of meromorphic functions were also recently investigated by Mohd *et al.* [78].

For a fixed positive integer n, $\epsilon^n = 1$, $\epsilon \neq 1$ and $f \in \mathcal{A}$, define the function with $n-$ply points $f_n \in \mathcal{A}$ by

$$f_n(z) := \frac{1}{n} \sum_{k=0}^{n-1} \epsilon^{n-k} f(\epsilon^k z). \tag{1.13}$$

It is clear that $f_1(z) = f(z)$ and $f_2(z) = (f(z) - f(-z))/2$. A function $f \in \mathcal{A}$ is *starlike with respect to n-symmetric points* if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f_n(z)} > 0. \tag{1.14}$$

Denote the class of these functions by $\mathcal{ST}_s^n$. For $n = 2$, the class $\mathcal{ST}_s^n$ reduces to the class $\mathcal{ST}_s$ consisting of the starlike functions with respect to symmetric points in $\mathcal{U}$ introduced by Sakaguchi [115]. If k is an integer, then the following identities follow directly from (1.13) :

$$\begin{aligned} f_n(\epsilon^k z) &= \epsilon^k f_n(z), \\ f_n'(\epsilon^k z) &= f_n'(z) = \frac{1}{n} \sum_{m=0}^{n-1} f'(\epsilon^m z), \\ \epsilon^k f_n''(\epsilon^k z) &= f_n''(z) = \frac{1}{n} \sum_{m=0}^{n-1} \epsilon^m f''(\epsilon^m z). \end{aligned}$$

More generally, the condition (1.14) can be generalized to the subordination

$$\frac{zf'(z)}{f_n(z)} \prec h(z),$$

where h is a given convex function, with $h(0) = 1$ and $\operatorname{Re}(h) > 0$. El-Ashwah and Thomas [41] introduced the classes $\mathcal{ST}_c$ and $\mathcal{ST}_{sc}$ consisting of the starlike functions with respect to conjugate points in $\mathcal{U}$ and the starlike functions with respect to symmetric conjugate points in $\mathcal{U}$ respectively. In 2004, Ravichandran [101] introduced the classes of starlike, convex and close-to-convex functions with respect to $n-$ply symmetric points, conjugate points and symmetric conjugate points, and obtained several convolution properties. Other investigations into the classes defined by using conjugate and symmetric conjugate points can be found in [4,38,62,133,134,136,137,139]. These classes of functions will be treated further in Chapter 3.

1.5 Integral Operators

The study of integral operators is an important problem in the field of Geometric Function Theory. In [21], Biernacki falsely claimed that $\int_0^z (f(\zeta)/\zeta)\, d\zeta$ is univalent whenever f is univalent. Moved by this, Causey [34] considered a related problem of finding conditions on $\delta \in \mathbb{C}$ such that the integral operator $F_\delta : \mathcal{A} \to \mathcal{A}$ given by

$$(F_\delta f)(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^\delta d\zeta$$

is univalent whenever f is univalent. It is known [64] that $F_\delta \in \mathcal{S}$ when $|\delta| < 1/4$. The case $\delta = 1$ was earlier considered by Alexander [5] and he proved that F_1 is in $\mathcal{CV}$ whenever f is in $\mathcal{ST}$. In [73], Merkes obtained various extension of inclusion results for certain subclasses of $\mathcal{S}$. He showed that

$$F_\delta(\mathcal{ST}) \subset \mathcal{S} \text{ whenever } |\delta| \leq 1/2. \tag{1.15}$$

There is no larger disk $|\delta| \leq R,\ R > 1/2$, such that the inclusion (1.15) holds.

In the case δ is real, Merkes [73, Theorem 2] obtained

$$F_\delta(\mathcal{ST}) \subset \mathcal{ST} \text{ whenever } -1/2 \le \delta \le 1$$

and

$$F_\delta(\mathcal{ST}) \subset \mathcal{CCV} \text{ whenever } 1 < \delta \le 3/2.$$

He also obtained necessary and sufficient conditions on α such that $F_\delta(\mathcal{ST}) \subset \mathcal{SCCV}_\alpha$, the class of strongly close-to-convex functions of order α defined on p. 6. In recent years, considerable attention has been given to the problem for various classes of univalent functions, see for example, the works of [1, 8, 24–32, 34, 42, 43, 64, 77, 87, 118, 126].

Suppose $|z_1| \le 1$, $|z_2| \le 1$, and $f \in \mathcal{SCCV}_\alpha$. Pommerenke [92] proved that

$$\int_0^z \frac{f(z_2\zeta) - f(z_1\zeta)}{(z_2 - z_1)\zeta}\, d\zeta \in \mathcal{SCCV}_\alpha.$$

Singh [121] showed that $1/2 \int_0^z (f(t) - f(-t))t^{-1}\, dt$ is starlike if f is starlike. Analogous results were also proven for convexity and close-to-convexity. Extending the results of Singh [121], Chandra and Singh [36] proved that the integral

$$\int_0^z \frac{f(e^{i\mu}\zeta) - f(e^{i\psi}\zeta)}{(e^{i\mu} - e^{i\psi})\zeta}\, d\zeta \quad (\mu \ne \psi, 0 \le \mu, \psi < 2\pi)$$

preserves membership in the classes of starlike, convex and close-to-convex functions.

For $\alpha_j \ge 0$ and $f_j \in \mathcal{A}$, define the operators $F : \mathcal{A}^n \times \overline{\mathcal{U}}^2 \to \mathcal{A}$ and G :

$\mathcal{A}^n \times \overline{\mathcal{U}}^2 \to \mathcal{A}$ respectively by

$$F(z) = F_{f_1,\cdots,f_n;z_1,z_2}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{f_j(z_2\zeta) - f_j(z_1\zeta)}{(z_2 - z_1)\zeta} \right)^{\alpha_j} d\zeta \quad (z_1, z_2 \in \overline{\mathcal{U}}) \tag{1.16}$$

$$G(z) = G_{f_1,\cdots,f_n;z_1,z_2}(z) = z \prod_{j=1}^{n} \left(\frac{f_j(z_2 z) - f_j(z_1 z)}{(z_2 - z_1)z} \right)^{\alpha_j} \quad (z_1, z_2 \in \overline{\mathcal{U}}) \tag{1.17}$$

where $\overline{\mathcal{U}}$ is the closed unit disk. Ponnusamy and Singh [97] introduced the operator F, given in (1.16), and investigated its univalence. In particular, they proved that for certain α_j the integral operator F is close-to-convex of order $1/2$ for all $f_j \in \mathcal{A}$. In Chapter 4, we shall explore further inclusion properties for the operators F and G.

1.6 Dual Set and the Duality Principle

Let $\mathcal{H}$ denote the class of all analytic functions in $\mathcal{U}$ of the form $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$. For a set $\mathcal{V} \subset \mathcal{H}$, the dual of $\mathcal{V}$, denoted by $\mathcal{V}^*$, is defined as

$$\mathcal{V}^* = \{ g \in \mathcal{H} : (f * g)(z) \neq 0 \text{ in } \mathcal{U} \text{ for all } f \in \mathcal{V} \} .$$

The set $\mathcal{V}^{**} = (\mathcal{V}^*)^*$ is called the second dual of $\mathcal{V}$. The standard reference to duality theory for convolutions is the monograph by Ruscheweyh [113] and his paper [110]. A subset $\mathcal{V} \subset \mathcal{H}$ is said to be *complete*, if it has the following property:

$$f \in \mathcal{V} \Rightarrow f_x \in \mathcal{V} \quad (|x| \leq 1)$$

where $f_x(z) = f(xz), \ z \in \mathcal{U}$. Let Λ be the space of continuous linear functionals on $\mathcal{H}$. The Duality Principle states that, subject to certain conditions on $\mathcal{V}$, the range $\lambda(\mathcal{V}) = \{\lambda(f) : f \in \mathcal{V}\}$ of a continuous complex-valued linear functional on

$\mathcal{V}$ equals its range on $\mathcal{V}^{**}$.

Theorem 1.23 (The Duality Principle) [113, p. 15] *Let $\mathcal{V} \subset \mathcal{H}$ be compact and complete. Then $\lambda(\mathcal{V}) = \lambda(\mathcal{V}^{**})$ for all $\lambda \in \Lambda$. Moreover, $\overline{co}(\mathcal{V}) = \overline{co}(\mathcal{V}^{**})$, where $\overline{co}$ stands for the closed convex hull of a set.*

Ruscheweyh proved the following important result as a corollary of the Duality Principle.

Theorem 1.24 [113, Corollary1.1, p. 17] *Let $\mathcal{V} \subset \mathcal{H}$ be compact and complete. Let $\lambda_1, \lambda_2 \in \Lambda$ with $0 \notin \lambda_2(\mathcal{V})$. Then for any $f \in \mathcal{V}^{**}$ there exists a function $g \in \mathcal{V}$ such that*

$$\frac{\lambda_1(f)}{\lambda_2(f)} = \frac{\lambda_1(g)}{\lambda_2(g)}.$$

Ruscheweyh obtained the second dual of some widely used subsets of $\mathcal{H}$. One of these sets is $\mathcal{V}_\beta$ which is described in the following theorem.

Theorem 1.25 [110, Theorem 1] *Let $\beta \neq 1$ be real, and*

$$\mathcal{V}_\beta = \left\{ \beta + (1-\beta)\left(\frac{1+xz}{1+yz}\right) \ : \ |x| = |y| = 1 \right\}.$$

Then

$$\mathcal{V}_\beta^* = \left\{ g \in \mathcal{H} \ : \ \operatorname{Re} g(z) > \frac{1-2\beta}{2-2\beta}, \ z \in \mathcal{U} \right\},$$

and

$$\mathcal{V}_\beta{}^{**} = \left\{ g \in \mathcal{H} \ : \ \exists \phi \in \mathbb{R} \textit{ such that } \operatorname{Re}\left(e^{i\phi}(g(z) - \beta)\right) > 0, \ z \in \mathcal{U} \right\}.$$

For $\beta < 1$, let

$$\mathcal{R}(\beta) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(f'(z) + zf''(z)\right) > \beta, \ z \in \mathcal{U} \right\}.$$

Several authors discussed the problem of finding the smallest value of β for which a function $f \in \mathcal{R}(\beta)$ would be starlike. For $\beta = 0$, Chichra [39] proved that if $f \in \mathcal{R}(0)$, then f is univalent, i.e $\mathcal{R}(0) \subset \mathcal{S}$. Singh and Singh [122] showed that $f \in \mathcal{R}(0)$ would imply that f is starlike, while Krzyz [65] gave an example to show that $f \in \mathcal{R}(0)$ is not necessarily convex. Later Singh and Singh [123] proved that for $\beta \leq -1/4$ the function $f \in \mathcal{R}(\beta)$ is starlike. This estimate was improved by Nunokawa and Thomas [82]. They showed that $\mathcal{R}(\beta) \subset \mathcal{ST}$ if β satisfies the equation

$$3\beta + (1-\beta)(2 - \log(4/e))\log(4/e) = 0,$$

and thus $\beta \cong -0.262$. In 1994, Ali [6] improved the lower bound for β. He proved that $\mathcal{R}(\beta_0) \subset \mathcal{S}$ for $\beta_0 = -(2\log 2 - 1)/2(1-\log 2) \cong -0.629$, and $\mathcal{R}(\beta) \subset \mathcal{ST}$ if $\beta = (6-\pi^2)/(24-\pi^2) \cong -0.2739$. Ali conjectured that $\mathcal{R}(\beta_0) \subset \mathcal{ST}$, and that β_0 is the best estimate. This conjecture was proved to be affirmative by Fournier and Ruscheweyh [48].

The implication

$$\operatorname{Re}\left(f'(z) + zf''(z)\right) > \beta \Rightarrow f \in \mathcal{ST}$$

is equivalent to $V(f) \in \mathcal{ST}$ whenever $\operatorname{Re}\left(f'(z)\right) > \beta$, where $V : \mathcal{A} \to \mathcal{A}$ is given by

$$V(f)(z) = \int_0^1 \frac{f(tz)}{t} dt.$$

Fournier and Ruscheweyh [48] introduced a more general operator $V_\lambda : \mathcal{A} \to \mathcal{A}$ given by

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \tag{1.18}$$

where λ is a non-negative real-valued integrable function satisfying the condition

$$\int_0^1 \lambda(t)dt = 1.$$

They used the Duality Principle [110,113] to prove starlikeness of the linear integral transform V_λ over functions f in the class

$$\mathcal{P}(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi} \left(f'(z) - \beta \right) > 0, \quad z \in \mathcal{U} \right\}.$$

Such problems were previously handled by using the theory of subordination (see for example [94]). The duality methodology seems to work best in this case since it gives sharp estimates on the parameter β. The following result was proved by Fournier and Ruscheweyh [48].

Theorem 1.26 [48] *Let Λ be integrable function over $[0,1]$ and positive on $(0,1)$. If*

$$\frac{\Lambda(t)}{1-t^2} \textit{ is decreasing on } (0,1),$$

then $\mathcal{L}_\Lambda(\mathcal{CCV}) = 0$, where

$$\mathcal{L}_\Lambda(f) := \inf_{z \in \mathcal{U}} \int_0^1 \Lambda(t) \left(\operatorname{Re} \frac{f(tz)}{tz} - \frac{1}{(1+t)^2} \right) dt \quad (\textit{for } f \in \mathcal{S}),$$

and

$$\mathcal{L}_\Lambda(\mathcal{S}) := \inf_{z \in \mathcal{U}} \mathcal{L}_\Lambda(f).$$

This duality technique is now popularly used by several authors to discuss similar problems. In 2001, Kim and Rønning [63] investigated starlikeness property of the integral transform (1.18) for functions f in the class

$$\mathcal{P}_\alpha(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \right.$$
$$\left. \operatorname{Re} e^{i\phi}\Big((1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - \beta\Big) > 0, \quad z \in \mathcal{U} \right\}.$$

In a recent paper Ponnusamy and Rønning [96] discussed this problem for functions f in the class

$$\mathcal{R}_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\left(f'(z) + \gamma z f''(z) - \beta\right) > 0, \quad z \in \mathcal{U} \right\}.$$

For $\alpha \geq 0$, $\gamma \geq 0$ and $\beta < 1$, define the class

$$\mathcal{W}_\beta(\alpha,\gamma) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\Big((1-\alpha+2\gamma)\frac{f(z)}{z} \right.$$
$$\left. + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) - \beta\Big) > 0, \quad z \in \mathcal{U} \right\}. \tag{1.19}$$

It is evident that $\mathcal{P}(\beta) \equiv \mathcal{W}_\beta(1,0)$, $\mathcal{P}_\alpha(\beta) \equiv \mathcal{W}_\beta(\alpha,0)$, and $\mathcal{R}_\gamma(\beta) \equiv \mathcal{W}_\beta(1+2\gamma,\gamma)$. In Chapter 5, starlikeness of the integral transform (1.18) over the class $\mathcal{W}_\beta(\alpha,\gamma)$ will be investigated.

1.7 Neighborhood Sets

For $\delta \geq 0$, Rusheweyh [112] defined the *δ-neighborhood* of a function

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathcal{U})$$

to be the set

$$N_\delta(f) := \left\{ g \in \mathcal{A} : g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \text{ and } \sum_{k=2}^{\infty} k|a_k - b_k| \le \delta \right\}.$$

Among other results, Ruscheweyh [112] proved that

$$N_{1/4}(f) \subset \mathcal{ST}$$

for $f \in \mathcal{CV}$. Sheil-Small and Silvia [119] introduced a more general notion of T-δ-neighborhood of an analytic function. The neighborhood problems for analytic functions were considered by many others, for example, see [2, 3, 18, 19, 45–47, 86, 98, 129].

Ruscheweyh in [112] developed new inclusion criteria for some known subclasses of analytic functions. These criteria are very useful in solving extremal problems associated with several subclasses of univalent functions. In particular, he proved that

$$f \in \mathcal{ST} \iff (f * h)(z) \neq 0 \text{ in } \mathcal{U},$$

where

$$h(z) = \frac{1}{1+it}\left(\frac{z}{(1-z)^2} - it\frac{z}{1-z}\right) \qquad (t \in \mathbb{R},\ z \in \mathcal{U}).$$

Several results of this type for different classes of functions in $\mathcal{U}$ were obtained by Ruscheweyh [112], Rahman and Stankiewicz [98] and Silverman et al. [124]. Padmanabhan [84] investigated the δ-neighborhood problem for the class $\mathcal{UCV}$ of uniformly convex functions. In Chapter 6, we shall introduce two generalized p-valent parabolic starlike and p-valent parabolic convex subclasses of $\mathcal{A}_p$. The δ-neighborhood problems for functions belonging to these classes are investigated and a new inclusion criterion for the subclass of p-valent parabolic starlike functions is obtained.

1.8 Scope of the Thesis

This thesis investigates five research problems. In Chapter 2, corresponding to an appropriate superordinate function Q defined on the unit disk $\mathcal{U}$, sufficient conditions are obtained for a p-valent function f to satisfy the subordination

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z), \text{ or } \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z),$$

where

$$f^{(q)}(z) = \lambda(p;q)z^{p-q} + \sum_{k=1}^{\infty} \lambda(k+p;q)a_{k+p}z^{k+p-q},$$

and

$$\lambda(p;q) := \frac{p!}{(p-q)!} \quad (p \geq q; p \in \mathbb{N}; q \in \mathbb{N} \cup \{0\}).$$

For the case $p = q = 1$, criteria for univalece and convexity of analytic functions are obtained. Additionally, the second subordination gives conditions for starlikeness of functions for the case $q = 0$ and $p = 1$.

The aim of Chapter 3 is to give a unified treatment for classes of starlike, convex and close-to-convex functions with respect to $n-$ply symmetric points, conjugate points and symmetric conjugate points. For this purpose, general classes of p-valent starlike, convex, close-to-convex and quasi-convex functions with respect to n-ply points, as well as p-valent starlike and convex functions with respect to symmetric points, conjugate points and symmetric conjugate points respectively are introduced. Inclusion and convolution properties of these classes will be investigated, and it would be evident that previous earlier works are special instances of our work.

In Chapter 4, membership preservation properties of the operators F and G given by (1.16) and (1.17) on the subclasses of starlike, convex and close-to-convex functions will be investigated. We shall also make connections with various earlier works.

In Chapter 5, the Duality Principle is used to determine the best value of $\beta < 1$ that ensures the integral transform $V_\lambda(f)$ in (1.18) maps the class $\mathcal{W}_\beta(\alpha,\gamma)$ defined in (1.19) respectively into the class $\mathcal{S}$ of normalized univalent functions and the class $\mathcal{ST}$ of starlike univalent functions. Simple sufficient conditions for $V_\lambda(f)$ to be starlike are obtained. This will lead to several applications for specific choices of the admissible function λ. In addition, the smallest value $\beta < 1$ is obtained that ensures a function f satisfying

$$\operatorname{Re}\left(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)\right) > \beta$$

is starlike. This result generalizes the earlier work of Fournier and Ruscheweyh [48], Kim and Rønning [63], and Ponnusamy and Rønning [96].

In Chapter 6, a subclass $\mathcal{SP}_p(\alpha,\lambda)$ of p-valent parabolic starlike functions of order α and type λ, and a subclass $\mathcal{CP}_p(\alpha,\lambda)$ of p-valent parabolic convex functions of order α and type λ will be introduced and studied. It is shown that these two classes are closed under convolution with prestarlike functions. In addition, new inclusion criterion for functions to belong to the class $\mathcal{SP}_p(\alpha,\lambda)$ will be derived, and the δ-neighborhood for functions belonging to the classes $\mathcal{SP}_p(\alpha,\lambda)$ and $\mathcal{CP}_p(\alpha,\lambda)$ will be investigated.

CHAPTER 2
SUBORDINATION PROPERTIES OF HIGHER-ORDER DERIVATIVES OF MULTIVALENT FUNCTIONS

2.1 Higher-Order Derivatives

Let $f \in \mathcal{A}_p$ be given by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}.$$

Upon differentiating both sides of f q-times with respect to z, the following differential operator is obtained:

$$f^{(q)}(z) = \lambda(p;q) z^{p-q} + \sum_{k=1}^{\infty} \lambda(k+p;q) a_{k+p} z^{k+p-q},$$

where

$$\lambda(p;q) := \frac{p!}{(p-q)!} \quad (p \geq q; p \in \mathbb{N}; q \in \mathbb{N} \cup \{0\}).$$

As defined in Section 1.2, p. 16, a p-valent function $f \in \mathcal{A}_p$ is *starlike* if it satisfies the condition $\frac{1}{p}\mathrm{Re}\,(zf'(z)/f(z)) > 0 \quad (z \in \mathcal{U})$. More generally, let ϕ be an analytic function with positive real part in $\mathcal{U}$, $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps the unit disk $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The classes $\mathcal{ST}_p(\phi)$ and $\mathcal{CV}_p(\phi)$ consist respectively of p-valent functions f *starlike* with respect to ϕ and p-valent functions f *convex* with respect to ϕ in $\mathcal{U}$ given by

$$f \in \mathcal{ST}_p(\phi) \Leftrightarrow \frac{1}{p}\frac{zf'(z)}{f(z)} \prec \phi(z), \text{ and } \quad f \in \mathcal{CV}_p(\phi) \Leftrightarrow \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z).$$

These classes were introduced and investigated in [8]. The functions $h_{\phi,p}$ and $k_{\phi,p}$

defined respectively by

$$\frac{1}{p}\frac{zh'_{\phi,p}}{h_{\phi,p}} = \phi(z) \quad (z \in \mathcal{U}, h_{\phi,p} \in \mathcal{A}_p), \tag{2.1}$$

$$\frac{1}{p}\left(1 + \frac{zk''_{\phi,p}}{k'_{\phi,p}}\right) = \phi(z) \quad (z \in \mathcal{U}, k_{\phi,p} \in \mathcal{A}_p), \tag{2.2}$$

are important examples of functions in $\mathcal{ST}_p(\phi)$ and $\mathcal{CV}_p(\phi)$.

In this chapter, corresponding to an appropriate superordinate function Q defined on the unit disk $\mathcal{U}$, sufficient conditions are obtained for a p-valent function f to satisfy the subordination

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z), \text{ or } \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z).$$

In the particular case when $q = 1$ and $p = 1$, and Q a function with positive real part, the first subordination gives sufficient conditions for univalence of analytic functions, while the second subordination implication gives conditions for convexity of functions. If $q = 0$ and $p = 1$, the second subordination gives conditions for starlikeness of functions. Thus results obtained in this work give important information on the geometric properties of functions satisfying differential subordination conditions involving higher-order derivatives.

The following lemmas are needed to prove our main results. These results are special cases of Theorem 1.22.

Lemma 2.1 [75, Corollary 3.4h.1, p. 135] *Let Q be univalent in $\mathcal{U}$, and φ be analytic in a domain D containing $Q(\mathcal{U})$. If $zQ'(z)\varphi(Q(z))$ is starlike, and P is analytic in $\mathcal{U}$ with $P(0) = Q(0)$ and $P(\mathcal{U}) \subset D$, then*

$$zP'(z)\varphi(P(z)) \prec zQ'(z)\varphi(Q(z)) \Rightarrow P(z) \prec Q(z),$$

and Q is the best dominant.

Lemma 2.2 [75, Corollary 3.4h.2, p. 135] *Let Q be convex univalent in $\mathcal{U}$, and θ be analytic in a domain D containing $Q(\mathcal{U})$. Assume that*

$$\operatorname{Re}\left(\theta'(Q(z)) + 1 + \frac{zQ''(z)}{Q'(z)}\right) > 0.$$

If P is analytic in $\mathcal{U}$ with $P(0) = Q(0)$ and $P(\mathcal{U}) \subset D$, then

$$zP'(z) + \theta(P(z)) \prec zQ'(z) + \theta(Q(z)) \ \Rightarrow P(z) \prec Q(z),$$

and Q is the best dominant.

2.2 Subordination Conditions for Univalence

The theorems below give sufficient conditions for the differential subordination

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z)$$

to hold. In addition, as corollary to these theorems, three sufficient conditions for functions to be univalent are obtained.

Theorem 2.1 *Let Q be univalent and nonzero in $\mathcal{U}$ with $Q(0) = 1$ and $zQ'(z)/Q(z)$ be starlike in $\mathcal{U}$. If a function $f \in \mathcal{A}_p$ satisfies the subordination*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \prec \frac{zQ'(z)}{Q(z)} + p - q, \tag{2.3}$$

then

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z),$$

and Q is the best dominant.

Proof. Define the analytic function P by

$$P(z) := \frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}}.$$

Then a computation shows that

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = \frac{zP'(z)}{P(z)} + p - q. \tag{2.4}$$

The subordination (2.3) yields

$$\frac{zP'(z)}{P(z)} + p - q \prec \frac{zQ'(z)}{Q(z)} + p - q,$$

or equivalently

$$\frac{zP'(z)}{P(z)} \prec \frac{zQ'(z)}{Q(z)}. \tag{2.5}$$

Define the function $\varphi : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $\varphi(w) := 1/w$. Then (2.5) can be written as

$$zP'(z) \cdot \varphi(P(z)) \prec zQ'(z) \cdot \varphi(Q(z)).$$

Since $Q(z) \neq 0$, φ is analytic in a domain containing $Q(\mathcal{U})$. Also $zQ'(z) \cdot \varphi(Q(z)) = zQ'(z)/Q(z)$ is starlike. The result now follows from Lemma 2.1. ∎

Remark 2.1 *For $f \in \mathcal{A}_p$, Irmak and Cho* [57, Theorem 2.1, p.2] *showed that*

$$\operatorname{Re} \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} < p - q \Rightarrow |f^{(q)}(z)| < \lambda(p;q)|z|^{p-q-1}.$$

However it is evident that the hypothesis of this implication cannot be satisfied by any function in $\mathcal{A}_p$ as the quantity

$$\left.\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right|_{z=0} = p - q.$$

Theorem 2.1 is the correct formulation of their result and under a more general setting.

Corollary 2.1 *Let* $-1 \le B < A \le 1$*. If* $f \in \mathcal{A}_p$ *satisfies*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \prec \frac{z(A-B)}{(1+Az)(1+Bz)} + p - q,$$

then

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec \frac{1+Az}{1+Bz}.$$

Proof. For $-1 \le B < A \le 1$, define the function Q by

$$Q(z) = \frac{1+Az}{1+Bz}.$$

Then a computation shows that

$$F(z) := \frac{zQ'(z)}{Q(z)} = \frac{(A-B)z}{(1+Az)(1+Bz)}$$

and

$$h(z) := \frac{zF'(z)}{F(z)} = \frac{1-ABz^2}{(1+Az)(1+Bz)}.$$

With $z = re^{i\theta}$, note that

$$\begin{aligned} \operatorname{Re}\left(h(re^{i\theta})\right) &= \operatorname{Re}\frac{1-ABr^2e^{2i\theta}}{(1+Are^{i\theta})(1+Bre^{i\theta})} \\ &= \frac{(1-ABr^2)(1+ABr^2+(A+B)r\cos\theta)}{\left|(1+Are^{i\theta})(1+Bre^{i\theta})\right|^2}. \end{aligned}$$

Now $(A+B) \ge 0$ yields

$$1+ABr^2+(A+B)r\cos\theta \ge (1-Ar)(1-Br) > 0,$$

while $(A+B) \leq 0$ gives

$$1 + ABr^2 + (A+B)r\cos\theta \geq (1+Ar)(1+Br) > 0.$$

Thus $\operatorname{Re} h(z) > 0$, and hence $zQ'(z)/Q(z)$ is starlike. The desired result now follows from Theorem 2.1. ∎

Example 2.1

1. *For* $0 < \beta \leq 1$, *choose* $A = \beta$, $B = 0$ *in Corollary 2.1. Since* $w \prec \beta z/(1+\beta z)$ *is equivalent to* $|w| \leq \beta\,|1-w|$, *it follows that if* $f \in \mathcal{A}_p$ *satisfies*

$$\left| \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + \frac{\beta^2}{1-\beta^2} \right| < \frac{\beta}{1-\beta^2},$$

then

$$\left| \frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} - 1 \right| < \beta.$$

2. *With* $A = 1$ *and* $B = 0$, *it follows from Corollary 2.1 that whenever* $f \in \mathcal{A}_p$ *satisfies*

$$\operatorname{Re}\left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right) < \frac{1}{2},$$

then

$$\left| \frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} - 1 \right| < 1.$$

Taking $q = 0$ and $Q(z) = h_{\phi,p}/z^p$ given by (2.1), Theorem 2.1 yields the following corollary:

Corollary 2.2 [8] *If* $f \in \mathcal{ST}_p(\phi)$, *then*

$$\frac{f(z)}{z^p} \prec \frac{h_{\phi,p}}{z^p}.$$

Similarly choosing $q = 1$ and $Q(z) = k'_{\phi,p}/pz^{p-1}$ given by (2.2), Theorem 2.1 yields the following corollary:

Corollary 2.3 [8] *If $f \in \mathcal{CV}_p(\phi)$, then*

$$\frac{f'(z)}{z^{p-1}} \prec \frac{k'_{\phi,p}}{z^{p-1}}.$$

Theorem 2.2 *Let Q be convex univalent in $\mathcal{U}$ with $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}}\left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q\right) \prec zQ'(z),$$

then

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z),$$

and Q is the best dominant.

Proof. Define the analytic function P by $P(z) := f^{(q)}(z)/\lambda(p;q)z^{p-q}$. Then it follows from (2.4) that

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}}\left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q\right) = zP'(z).$$

By assumption,

$$zP'(z)\varphi(P(z)) \prec zQ'(z)\varphi(Q(z)),$$

where $\varphi : \mathbb{C} \to \mathbb{C}$ defined by $\varphi(w) := 1$. Since Q is convex, and $zQ'(z)\varphi(Q(z)) = zQ'(z)$ is starlike, Lemma 2.1 gives the desired result. ∎

Example 2.2 *When*

$$Q(z) := 1 + \frac{z}{\lambda(p;q)},$$

Theorem 2.2 reduces to Theorem 2.4 in [57]*: If $f \in \mathcal{A}_p$ satisfies*

$$\left| f^{(q)}(z) \left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right) \right| \leq |z|^{p-q},$$

then

$$\left| f^{(q)}(z) - \lambda(p;q) z^{p-q} \right| \leq |z|^{p-q}.$$

In the special case $q = 1$, this result provides a sufficient condition for multivalent functions f to be close-to-convex.

Theorem 2.3 *Let Q be convex univalent in $\mathcal{U}$ with $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf^{(q+1)}(z)}{\lambda(p;q)z^{p-q}} \prec zQ'(z) + (p-q)Q(z),$$

then

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z),$$

and Q is the best dominant.

Proof. Define the function P by

$$P(z) = \frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}}.$$

It follows from (2.4) that

$$zP'(z) + (p-q)P(z) \prec zQ'(z) + (p-q)Q(z),$$

that is,

$$zP'(z) + \theta(P(z)) \prec zQ'(z) + \theta(Q(z)),$$

where $\theta : \mathbb{C} \to \mathbb{C}$ defined by $\theta(w) := (p-q)w$. The conditions in Lemma 2.2 are clearly satisfied. Thus $f^{(q)}(z)/\lambda(p;q)z^{p-q} \prec Q(z)$, and Q is the best dominant. ∎

Taking $q = 0$, Theorem 2.3 yields the following corollary:

Corollary 2.4 [144, Corollary 2.11] *Let Q be convex univalent in $\mathcal{U}$, and $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{f'(z)}{z^{p-1}} \prec zQ'(z) + pQ(z),$$

then

$$\frac{f(z)}{z^p} \prec Q(z).$$

With $p = 1$, Corollary 2.4 yields the following corollary:

Corollary 2.5 [144, Corollary 2.9] *Let Q be convex univalent in $\mathcal{U}$, and $Q(0) = 1$. If $f \in \mathcal{A}$ satisfies*

$$f'(z) \prec zQ'(z) + Q(z),$$

then

$$\frac{f(z)}{z} \prec Q(z).$$

Theorem 2.4 *Let Q be univalent and nonzero in $\mathcal{U}$ with $Q(0) = 1$ and $zQ'(z)/Q^2(z)$ be starlike. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{\lambda(p;q)z^{p-q}}{f^{(q)}(z)}\left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q\right) \prec \frac{zQ'(z)}{Q^2(z)},$$

then

$$\frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}} \prec Q(z),$$

and Q is the best dominant.

Proof. Define the function P by

$$P(z) = \frac{f^{(q)}(z)}{\lambda(p;q)z^{p-q}}.$$

It follows from (2.4) that

$$\frac{\lambda(p;q)z^{p-q}}{f^{(q)}(z)}\left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}-(p-q)\right)=\frac{1}{P(z)}\frac{zP'(z)}{P(z)}=\frac{zP'(z)}{P^2(z)}.$$

By assumption,

$$\frac{zP'(z)}{P^2(z)} \prec \frac{zQ'(z)}{Q^2(z)}. \tag{2.6}$$

With $\varphi : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ defined by $\varphi(w) := 1/w^2$, equation (2.6) can be written as

$$zP'(z)\varphi(P(z)) \prec zQ'(z)\varphi(Q(z)).$$

The function φ is analytic in $\mathbb{C}-\{0\}$. Since $zQ'(z)\varphi(Q(z))$ is starlike, it follows from Lemma 2.1 that $P(z) \prec Q(z)$, and Q is the best dominant. ∎

Taking $q = 1,\ p = 1$, and $Q(z) = (1+z)/(1-z)$ in Theorem 2.1, Theorem 2.2 and Theorem 2.3 yields three sufficient conditions for f to be univalent:

Corollary 2.6 *A function $f \in \mathcal{A}$ is univalent if it satisfies one of the following subordinations:*

(i) $\frac{zf''(z)}{f'(z)} \prec \frac{2z}{1-z^2}$,

(ii) $zf''(z) \prec \frac{2z}{(1-z)^2}$, *or*

(iii) $\frac{zf''(z)}{(f'(z))^2} \prec \frac{2z}{(1+z)^2}$.

2.3 Subordination Related to Convexity

In this section we look for sufficient conditions to ensure the following differential subordination

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z)$$

holds. As corollaries, sufficient conditions are obtained for functions f to be convex.

Theorem 2.5 *Let Q be univalent and nonzero in $\mathcal{U}$ with $Q(0) = 1$, $Q(z) \neq q - p + 1$, and $zQ'(z)/(Q(z)(Q(z) + p - q - 1))$ be starlike in $\mathcal{U}$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q + 1}{\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1} \prec 1 + \frac{zQ'(z)}{Q(z)(Q(z) + p - q - 1)}, \tag{2.7}$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z),$$

and Q is the best dominant.

Proof. Let the function P be defined by

$$P(z) = \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1. \tag{2.8}$$

Upon differentiating logarithmically both sides of (2.8), it follows that

$$\frac{zP'(z)}{P(z) + p - q - 1} = 1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)}. \tag{2.9}$$

Thus

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q + 1 = \frac{zP'(z)}{P(z) + p - q - 1} + P(z). \tag{2.10}$$

The equations (2.8) and (2.10) yield

$$\frac{1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q + 1}{\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q - 1} = \frac{zP'(z)}{P(z)(P(z) + p - q - 1)} + 1. \tag{2.11}$$

If $f \in \mathcal{A}_p$ satisfies the subordination (2.7), equation (2.11) gives

$$\frac{zP'(z)}{P(z)(P(z) + p - q - 1)} \prec \frac{zQ'(z)}{Q(z)(Q(z) + p - q - 1)},$$

that is,

$$zP'(z)\varphi(P(z)) \prec zQ'(z)\varphi(Q(z))$$

with $\varphi : \mathbb{C}\setminus\{0, 1-p+q\} \to \mathbb{C}$ defined by $\varphi(w) := 1/w(w+p-q-1)$. The desired result is now established by an application of Lemma 2.1. ∎

Theorem 2.5 contains Corollary 4 in [102] as a special case. In particular, we note that Theorem 2.5 with $p = 1$, $q = 0$, and $Q(z) = (1 + Az)/(1 + Bz)$ for $-1 \leq B < A \leq 1$ yields the following corollary:

Corollary 2.7 [102, Corollary 6, p. 123] *Let $-1 \leq B < A \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1 + \frac{(A-B)z}{(1+Az)^2},$$

then $f \in \mathcal{ST}[A, B]$.

For $A = 0$, $B = b$ and $A = 1$, $B = -1$, Corollary 2.7 gives the results of Obradovič and Tuneski [83].

Theorem 2.6 *Let Q be univalent and nonzero in $\mathcal{U}$ with $Q(0) = 1$, $Q(z) \neq q-p+1$ and $zQ'(z)/(Q(z) + p - q - 1)$, be starlike in $\mathcal{U}$. If $f \in \mathcal{A}_p$ satisfies*

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \prec \frac{zQ'(z)}{Q(z) + p - q - 1},$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z),$$

and Q is the best dominant.

Proof. Let the function P be defined by (2.8). It follows from (2.9) and the hypothesis that

$$\frac{zP'(z)}{P(z) + p - q - 1} \prec \frac{zQ'(z)}{Q(z) + p - q - 1}. \tag{2.12}$$

Define the function $\varphi : \mathbb{C} \setminus \{1 - p + q\} \to \mathbb{C}$ by

$$\varphi(w) := \frac{1}{w + p - q - 1}.$$

Then (2.12) can be written as

$$zP'(z)\varphi(P(z)) \prec zQ'(z)\varphi(Q(z)).$$

Since φ is analytic in a domain containing $Q(\mathcal{U})$, and $zQ'(z)\varphi(Q(z))$ is starlike, the result follows from Lemma 2.1. ∎

Theorem 2.7 *Let Q be a convex function in $\mathcal{U}$ with $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\left(2 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right) \prec zQ'(z) + Q(z) + p - q - 1,$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z),$$

and Q is the best dominant.

Proof. Let the function P be defined by (2.8). Using (2.9), it follows that

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\left(1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right) = zP'(z),$$

and therefore

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\left(2 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right) = zP'(z) + P(z) + p - q - 1.$$

By assumption,

$$zP'(z) + P(z) + p - q - 1 \prec zQ'(z) + Q(z) + p - q - 1,$$

or

$$zP'(z) + \theta(P(z)) \prec zQ'(z) + \theta(Q(z)),$$

where the function $\theta : \mathbb{C} \setminus \{1-p+q\} \to \mathbb{C}$ defined by $\theta := 1$. The proof now follows from Lemma 2.2. ∎

Theorem 2.8 *Let Q be a convex function in $\mathcal{U}$ with $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\left(1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right) \prec zQ'(z),$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 \prec Q(z),$$

and Q is the best dominant.

Proof. Let the function P be defined by (2.8). It follows from (2.9) that

$$zP'(z)\varphi(P(z)) \prec zQ'(z)\varphi(Q(z)),$$

where $\varphi : \mathbb{C} \to \mathbb{C}$ defined by $\varphi(w) := 1$. The result follows easily from Lemma 2.1. ∎

Taking $q = 1,\ p = 1$, and $Q(z) = (1+z)(1-z)$ in Theorem 2.5, Theorem 2.6, Theorem 2.7 and Theorem 2.8 yield four convexity conditions:

Corollary 2.8 *A function $f \in \mathcal{A}$ is convex if it satisfies one of the following subordinations:*

(i) $\frac{2+zf'''(z)/f''(z)}{1+zf''(z)/f'(z)} \prec 1 + \frac{z(1-z)^2}{1-z^2}$,

(ii) $1 + \frac{zf'''(z)}{f''(z)} - \frac{zf''(z)}{f'(z)} \prec \frac{1}{1-z}$,

(iii) $\frac{zf''(z)}{f'(z)}\left(2 + \frac{zf'''(z)}{f''(z)} - \frac{zf''(z)}{f'(z)}\right) \prec \frac{2z(2-z)}{(1-z)^2}$, *or*

(iv) $\frac{zf''(z)}{f'(z)}\left(1 + \frac{zf'''(z)}{f''(z)} - \frac{zf''(z)}{f'(z)}\right) \prec \frac{2z}{(1-z)^2}$.

CHAPTER 3
CONVOLUTION PROPERTIES OF MULTIVALENT FUNCTIONS WITH RESPECT TO N-PLY POINTS AND SYMMETRIC CONJUGATE POINTS

3.1 Motivation and Preliminaries

As defined in Section 1.1, p. 7, a function $f \in \mathcal{A}$ is starlike with respect to symmetric points in $\mathcal{U}$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0$$

for all $z \in \mathcal{U}$. The class of all such functions, denoted by $\mathcal{ST}_s$, was introduced and investigated by Sakaguchi [115]. El-Ashwah and Thomas [41] introduced the class $\mathcal{ST}_c$ consisting of starlike functions with respect to conjugate points, and the class $\mathcal{ST}_{sc}$ of starlike functions with respect to symmetric conjugate points defined respectively by the conditions

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}}\right) > 0, \quad \text{and} \quad \operatorname{Re}\left(\frac{zf'(z)}{f(z)-\overline{f(-\overline{z})}}\right) > 0.$$

In 2004, Ravichandran [101] introduced the classes of starlike, convex and close-to-convex functions with respect to $n-$ply symmetric points, conjugate points and symmetric conjugate points, and obtained several convolution properties. Other investigations into the classes defined by using conjugate and symmetric conjugate points can be found in [4, 38, 62, 133, 134, 136, 137, 139].

All these many investigations can be unified, and it is the aim of this chapter to show such a unified method. For this purpose, several general subclasses of p-valent functions are introduced such as starlike, convex, close-to-convex and quasi-convex functions with respect to n-ply points, as well as p-valent starlike and convex functions with respect to symmetric points, conjugate points and symmetric conjugate points respectively. Inclusion and convolution properties of these classes will be investigated, and it would be evident that previous earlier works are special instances of our present work.

The following theorems would be required.

Theorem 3.1 [75, Corollary 4.1h.1, p. 200] *Let h be convex in $\mathcal{U}$, and S and T be analytic functions in $\mathcal{U}$ with $S(0) = T(0)$. If $\mathrm{Re}\left(zS'(z)/S(z)\right) > 0$, then*

$$\frac{T'(z)}{S'(z)} \prec h(z) \Longrightarrow \frac{T(z)}{S(z)} \prec h(z).$$

The following theorem provides a convolution result between a prestalike function of order α, $f \in \mathcal{R}_\alpha$ (cf. p. 15), and a starlike function of the same order α, $g \in \mathcal{ST}(\alpha)$ (cf. p. 5).

Theorem 3.2 [113, Theorem 2.4, p. 54] *Let $\alpha \leq 1$, $f \in \mathcal{R}_\alpha$ and $g \in \mathcal{ST}(\alpha)$. Then*

$$\frac{f * (Hg)}{f * g}(\mathcal{U}) \subset \overline{co}(H(\mathcal{U})),$$

for any analytic function H in $\mathcal{U}$, where $\overline{co}(H(\mathcal{U}))$ denote the closed convex hull of $H(\mathcal{U})$.

Theorem 3.2 due to Ruscheweyh [113] can easily be adapted to yield the following result for multivalent functions.

Theorem 3.3 *If $f(z)/z^{p-1} \in \mathcal{R}_\alpha$ and $g(z)/z^{p-1} \in \mathcal{ST}(\alpha)$, then*

$$\frac{f * (Hg)}{f * g}(\mathcal{U}) \subset \overline{co}\left(H\left(\mathcal{U}\right)\right)$$

for any analytic function H defined in $\mathcal{U}$.

Proof. It is evident that

$$\frac{f(z) * (Hg)(z)}{f(z) * g(z)} = \frac{\frac{f(z)}{z^{p-1}} * H(z) \cdot \frac{g(z)}{z^{p-1}}}{\frac{f(z)}{z^{p-1}} * \frac{g(z)}{z^{p-1}}}.$$

Since $f/z^{p-1} \in \mathcal{R}_\alpha$ and $g/z^{p-1} \in \mathcal{ST}(\alpha)$, Theorem 3.2 yields

$$\frac{f(z) * (Hg)(z)}{f(z) * g(z)} \subset \overline{co} H(\mathcal{U}).$$ ∎

3.2 Multivalent Functions with Respect to n-ply Points

In the following sequel, the function $g \in \mathcal{A}_p$ is a fixed function and the function h is a convex univalent function with positive real part satisfying $h(0) = 1$. On certain occasions, for example in Theorem 3.4, we would additionally require that $\operatorname{Re} h(z) > 1 - (1-\alpha)/p$, where $0 \le \alpha < 1$. Multivalent functions starlike and convex with respect to $n-$ply points are given below:

Definition 3.1 *Let $n \ge 1$ be an integer, $\epsilon^n = 1$, and $\epsilon \ne 1$. For $f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \in \mathcal{A}_p$, define the function $f_n \in \mathcal{A}_p$ with respect to $n-$ply points by*

$$f_n(z) := \frac{1}{n} \sum_{k=0}^{n-1} \epsilon^{n-pk} f(\epsilon^k z) = z^p + a_{p+n} z^{p+n} + a_{p+2n} z^{p+2n} + \cdots.$$

Definition 3.2 *The class $\mathcal{ST}_p^n(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying $f_n(z)/z^p \ne 0$ in $\mathcal{U}$ and the subordination*

$$\frac{1}{p} \frac{z f'(z)}{f_n(z)} \prec h(z).$$

Denote by $\mathcal{ST}_{p,g}^n(h)$ the class

$$\mathcal{ST}_{p,g}^n(h) := \left\{ f \in \mathcal{A}_p : f * g \in \mathcal{ST}_p^n(h) \right\}.$$

Similarly, $\mathcal{CV}_p^n(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying $f_n'(z)/z^{p-1} \neq 0$ in $\mathcal{U}$ and the subordination

$$\frac{1}{p}\frac{\left(zf'\right)'(z)}{f_n'(z)} \prec h(z),$$

and

$$\mathcal{CV}_{p,g}^n(h) := \left\{f \in \mathcal{A}_p : f * g \in \mathcal{CV}_p^n(h)\right\}.$$

Remark 3.1 *For $n = 1$, the classes $\mathcal{ST}_{p,g}^1(h)$ and $\mathcal{CV}_{p,g}^1(h)$ were studied by Supramaniam et al.* [127].

Evidently when $g(z) = z^p/(1-z)$, the classes $\mathcal{ST}_{p,g}^n(h)$ and $\mathcal{CV}_{p,g}^n(h)$ reduced respectively to the classes $\mathcal{ST}_p^n(h)$ and $\mathcal{CV}_p^n(h)$. Thus these new classes of p-valent starlike and convex functions with respect to n-ply points unify the classes $\mathcal{ST}_p^n(h)$ and $\mathcal{CV}_p^n(h)$. Note that for $n = 1$, $\mathcal{ST}_p^1(h) := \mathcal{ST}_p(h)$ and $\mathcal{CV}_p^1(h) := \mathcal{CV}_p(h)$.

It is clear that $\mathcal{ST}_{p,zg'}^n(h) = \mathcal{CV}_{p,g}^n(h)$. Interestingly the property that every convex function is necessarily starlike remains valid even for multivalent functions with respect to n-ply points. Indeed the following result holds:

Lemma 3.1 *Let g be a fixed function in $\mathcal{A}_p$, and h a convex univalent function having positive real part with $h(0) = 1$.*

(i) If $f \in \mathcal{ST}_{p,g}^n(h)$, then $f_n \in \mathcal{ST}_{p,g}(h)$.

(ii) The function $f \in \mathcal{CV}_{p,g}^n(h)$ if and only if $zf'/p \in \mathcal{ST}_{p,g}^n(h)$.

(iii) The inclusion $\mathcal{CV}_{p,g}^n(h) \subset \mathcal{ST}_{p,g}^n(h)$ holds.

Proof. It is sufficient to prove the result for $g(z) = z^p/(1-z)$.

(i) Let $f \in \mathcal{ST}_p^n(h)$. For any fixed $z \in \mathcal{U}$,

$$\frac{1}{p}\frac{zf'(z)}{f_n(z)} \in h(\mathcal{U}). \tag{3.1}$$

Replacing z by $\epsilon^k z$ in (3.1), it follows that

$$\frac{1}{p}\frac{\epsilon^k z f'(\epsilon^k z)}{f_n(\epsilon^k z)} \in h(\mathcal{U}). \tag{3.2}$$

In light of the fact that

$$f_n(\epsilon^k z) = \epsilon^{pk} f_n(z), \tag{3.3}$$

the containment (3.2) becomes

$$\frac{1}{p}\frac{\epsilon^{k(1-p)} z f'(\epsilon^k z)}{f_n(z)} \in h(\mathcal{U}).$$

Since $h(\mathcal{U})$ is convex, it follows that

$$\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{p}\frac{\epsilon^{k(1-p)} z f'(\epsilon^k z)}{f_n(z)} \in h(\mathcal{U}). \tag{3.4}$$

Using the identity

$$f_n'(z) = \frac{1}{n}\sum_{k=0}^{n-1}\epsilon^{k(1-p)} f'(\epsilon^k z), \tag{3.5}$$

it is seen that (3.4) becomes

$$\frac{1}{p}\frac{z f_n'(z)}{f_n(z)} \in h(\mathcal{U}).$$

Thus

$$\frac{1}{p}\frac{z f_n'(z)}{f_n(z)} \prec h(z),$$

that is, $f_n \in \mathcal{ST}_p(h)$.

(ii) Since $(zf'/p)_n(z) = zf_n'(z)/p$, it is evident that

$$\frac{1}{p}\frac{\left(zf'\right)'(z)}{f_n'(z)} = \frac{1}{p}\frac{z\left(\frac{1}{p}zf'\right)'(z)}{(\frac{1}{p}zf')_n(z)}.$$

Thus $f \in \mathcal{CV}_p^n(h)$ if and only if $zf^{'}/p \in \mathcal{ST}_p^n(h)$.

(iii) Let $f \in \mathcal{CV}_p^n(h)$. Then part (ii) shows that $zf^{'}/p \in \mathcal{ST}_p^n(h)$. We deduce from part (i) that $(zf^{'}/p)_n \in \mathcal{ST}_p(h)$. From $(zf^{'}/p)_n = zf_n^{'}/p$, part (ii) now shows that $f_n \in \mathcal{CV}_p(h)$. Since $\mathcal{CV}_p(h)$ is subset of $\mathcal{ST}_p(h)$ [127, Theorem 2.1], it follows that $f_n \in \mathcal{ST}_p(h)$, and because h is a function with positive real part, the function f_n is starlike.

Define the functions T and S by

$$T(z) := \frac{1}{p} z f'(z) \quad \text{and} \quad S(z) := f_n(z).$$

Since the function S is starlike and

$$\frac{T'(z)}{S'(z)} = \frac{1}{p} \frac{\left(zf'\right)'(z)}{f_n'(z)} \prec h(z),$$

Theorem 3.1 implies that

$$\frac{1}{p} \frac{zf'(z)}{f_n(z)} = \frac{T(z)}{S(z)} \prec h(z),$$

whence $f \in \mathcal{ST}_p^n(h)$. ∎

Ruscheweyh and Sheil-Small [114] proved the Polya-Schoenberg conjecture that the classes of convex functions, starlike functions and close-to-convex functions are closed under convolution with convex functions. In the following theorem, this result is extended for the convolution between prestarlike functions and multivalent functions with respect to n-ply points.

Theorem 3.4 *Let h be a convex univalent function satisfying the condition*

$$\operatorname{Re} h(z) > 1 - \frac{1-\alpha}{p} \quad (0 \leq \alpha < 1),$$

and $\phi \in \mathcal{A}_p$ with $\phi/z^{p-1} \in \mathcal{R}_\alpha$.

(i) If $f \in \mathcal{ST}^n_{p,g}(h)$, then $\phi * f \in \mathcal{ST}^n_{p,g}(h)$. Equivalently, $\mathcal{ST}^n_{p,g}(h) \subset \mathcal{ST}^n_{p,g*\phi}(h)$.

(ii) If $f \in \mathcal{CV}^n_{p,g}(h)$, then $\phi * f \in \mathcal{CV}^n_{p,g}(h)$. Equivalently, $\mathcal{CV}^n_{p,g}(h) \subset \mathcal{CV}^n_{p,g*\phi}(h)$.

Proof. (i) Let $f \in \mathcal{ST}^n_p(h)$. From Lemma 3.1 (i), it follows that $f_n \in \mathcal{ST}_p(h)$. The function ψ_n defined by

$$\psi_n(z) := \frac{f_n(z)}{z^{p-1}}$$

is analytic and satisfies

$$\frac{z\psi_n'(z)}{\psi_n(z)} = \frac{zf_n'(z)}{f_n(z)} - (p-1) \prec p\,h(z) - (p-1).$$

Since $\operatorname{Re} h(z) > 1 - (1-\alpha)/p$, it follows that

$$\operatorname{Re} \frac{z\psi_n'(z)}{\psi_n(z)} > \alpha,$$

and hence $\psi_n \in \mathcal{ST}(\alpha)$. Define the function H by

$$H(z) := \frac{1}{p}\frac{zf'(z)}{f_n(z)}.$$

Since $H \prec h$ and h is convex, an application of Theorem 3.3 shows that

$$\frac{1}{p}\frac{z(\phi * f)'(z)}{(\phi * f)_n(z)} = \frac{\phi(z) * \frac{1}{p}zf'(z)}{\phi(z) * f_n(z)} = \frac{(\phi * Hf_n)(z)}{(\phi * f_n)(z)} \prec h(z),$$

and thus $\phi * f \in \mathcal{ST}^n_p(h)$.

The general result for $f \in \mathcal{ST}^n_{p,g}(h)$ follows from the fact that

$$f \in \mathcal{ST}^n_{p,g}(h) \Leftrightarrow f * g \in \mathcal{ST}^n_p(h).$$

(ii) Now let $f \in \mathcal{CV}^n_{p,g}(h)$ so that $zf'/p \in \mathcal{ST}^n_{p,g}(h)$. The result of part (i) yields

$(zf'/p) * \phi = z(f * \phi)'/p \in \mathcal{ST}^n_{p,g}(h)$, and thus $\phi * f \in \mathcal{CV}^n_{p,g}(h)$. ∎

Close-to-convex and quasi-convex multivalent functions with respect to $n-$ply points are defined as follows:

Definition 3.3 *The class $\mathcal{CCV}^n_p(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying the subordination*

$$\frac{1}{p}\frac{zf'(z)}{\phi_n(z)} \prec h(z)$$

for some $\phi \in \mathcal{ST}^n_p(h)$. The general class $\mathcal{CCV}^n_{p,g}(h)$ then consists of functions $f \in \mathcal{A}_p$ satisfying the subordination

$$\frac{1}{p}\frac{z(g * f)'(z)}{(g * \phi)_n(z)} \prec h(z)$$

for some $\phi \in \mathcal{ST}^n_{p,g}(h)$. The class $\mathcal{QCV}^n_p(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying the subordination

$$\frac{1}{p}\frac{\left(zf'\right)'(z)}{\phi'_n(z)} \prec h(z)$$

for some $\phi \in \mathcal{CV}^n_p(h)$, while the class $\mathcal{QCV}^n_{p,g}(h)$ consists of $f \in \mathcal{A}_p$ such that

$$\frac{1}{p}\frac{\left(z(g * f)'\right)'(z)}{(g * \phi)'_n(z)} \prec h(z)$$

for some $\phi \in \mathcal{CV}^n_{p,g}(h)$.

Lemma 3.2 *Let g be a fixed function in $\mathcal{A}_p$, and h a convex univalent function with positive real part satisfying $h(0) = 1$. Then*

(i) $\mathcal{CV}^n_{p,g}(h) \subset \mathcal{QCV}^n_{p,g}(h) \subset \mathcal{CCV}^n_{p,g}(h)$,

(ii) $f \in \mathcal{QCV}^n_{p,g}(h)$ if and only if $zf'/p \in \mathcal{CCV}^n_{p,g}(h)$.

Proof. (i) By taking $\phi = f$, it is evident from the definition that $\mathcal{CV}^n_{p,g}(h) \subset \mathcal{QCV}^n_{p,g}(h)$. To prove the second inclusion, suppose that $f \in \mathcal{QCV}^n_{p,g}(h)$. Then there exists $\phi \in \mathcal{CV}^n_{p,g}(h)$ such that

$$\frac{1}{p}\frac{\left(z(g*f)'\right)'(z)}{(g*\phi)'_n(z)} \prec h(z).$$

Since $\phi \in \mathcal{CV}^n_{p,g}(h)$, it follows that $(g*\phi)_n \in \mathcal{CV}_p(h)$ which is subset of $\mathcal{ST}_p(h)$ [127, Theorem 2.1]. Thus $(g*\phi)_n \in \mathcal{ST}_p(h)$. The result now follows from Theorem 3.1 with

$$T(z) = \frac{1}{p}z(g*f)'(z) \quad \text{and} \quad S(z) = (g*\phi)_n(z).$$

(ii) Here the proof follows from the identity

$$\frac{1}{p}\frac{\left(z(g*f)'\right)'(z)}{(g*\phi)'_n(z)} = \frac{1}{p}\frac{z\left((g*\frac{1}{p}zf')\right)'(z)}{(g*\frac{1}{p}z\phi')_n(z)},$$

and Lemma 3.1 (ii). ∎

Theorem 3.5 *Let h and ϕ satisfy the conditions of Theorem 3.4.*

(i) If $f \in \mathcal{CCV}^n_{p,g}(h)$ with respect to a function $f_1 \in \mathcal{ST}^n_{p,g}(h)$, then $\phi * f \in \mathcal{CCV}^n_{p,g}(h)$ with respect to the function $\phi * f_1 \in \mathcal{ST}^n_{p,g}(h)$. Also $\mathcal{CCV}^n_{p,g}(h) \subset \mathcal{CCV}^n_{p,g*\phi}(h)$.

(ii) If $f \in \mathcal{QCV}^n_{p,g}(h)$ with respect to $f_1 \in \mathcal{CV}^n_{p,g}(h)$, then $\phi * f \in \mathcal{QCV}^n_{p,g}(h)$ with respect to $\phi * f_1 \in \mathcal{CV}^n_{p,g}(h)$. Also $\mathcal{QCV}^n_{p,g}(h) \subset \mathcal{QCV}^n_{p,g*\phi}(h)$.

Proof. (i) It is sufficient to prove the result for the case $g(z) = z^p/(1-z)$. Let $f \in \mathcal{CCV}^n_p(h)$ with respect to a function $f_1 \in \mathcal{ST}^n_p(h)$. Theorem 3.4 yields $\phi * f_1 \in \mathcal{ST}^n_p(h)$, and Lemma 3.1 (i) gives $(f_1)_n$ is in $\mathcal{ST}_p(h)$. Also it is easy to see that the function $(f_1)_n/z^{p-1} \in \mathcal{ST}(\alpha)$. Now define the analytic function H by

$$H(z) := \frac{1}{p}\frac{zf'(z)}{(f_1)_n(z)}.$$

Since $H(z) \prec h(z)$, an application of Theorem 3.3 shows that

$$\frac{1}{p}\frac{z(\phi * f)'(z)}{(\phi * f_1)_n(z)} = \frac{\phi(z) * \frac{1}{p}zf'(z)}{\phi(z) * (f_1)_n(z)} = \frac{(\phi * H(f_1)_n)(z)}{(\phi * (f_1)_n)(z)} \prec h(z).$$

This completes the proof of part (i).

(ii) If $f \in \mathcal{QCV}^n_{p,g}(h)$, then Lemma 3.2 (ii) gives $zf'/p \in \mathcal{CCV}^n_{p,g}(h)$. Since

$$\frac{1}{p}z(\phi * f)'(z) = \phi(z) * \frac{1}{p}zf'(z),$$

the result of part (i) shows that $z(\phi * f)'/p \in \mathcal{CCV}^n_{p,g}(h)$. From Lemma 3.2 (ii), $\phi * f \in \mathcal{QCV}^n_{p,g}(h)$. ∎

3.3 Multivalent Functions with Respect to n-ply Symmetric Points

In this and the following two sections, it is assumed that p **is an odd number.** Also, the function $g \in \mathcal{A}_p$ is a fixed function and the function h is convex univalent with positive real part satisfying $h(0) = 1$. The classes of multivalent functions that are $p-$valent starlike and $p-$valent convex with respect to n-ply symmetric, conjugate, and symmetric conjugate points are introduced, and their convolution properties discussed.

Definition 3.4 *For odd positive integer p , the class $\mathcal{STS}^n_p(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying $(f_n(z) - f_n(-z))/z^p \neq 0$ in $\mathcal{U}$ and the subordination*

$$\frac{1}{p}\frac{2zf'(z)}{f_n(z) - f_n(-z)} \prec h(z).$$

Denote by $\mathcal{STS}^n_{p,g}(h)$ the class

$$\mathcal{STS}^n_{p,g}(h) := \left\{f \in \mathcal{A}_p : f * g \in \mathcal{STS}^n_p(h)\right\}.$$

Similarly, $\mathcal{CVS}_p^n(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying $\left(f_n'(z) + f_n'(-z)\right)/z^{p-1}$ $\neq 0$ in $\mathcal{U}$ and the subordination

$$\frac{1}{p}\frac{2\left(zf'\right)'(z)}{f_n'(z) + f_n'(-z)} \prec h(z),$$

and

$$\mathcal{CVS}_{p,g}^n(h) := \left\{f \in \mathcal{A}_p : f * g \in \mathcal{CVS}_p^n(h)\right\}.$$

In the special case $n = 1$, we shall adopt the following usual notations: $\mathcal{STS}_{p,g}^1(h) =: \mathcal{STS}_{p,g}(h)$ and $\mathcal{CVS}_{p,g}^1(h) =: \mathcal{CVS}_{p,g}(h)$.

Remark 3.2 *When $p = 1$, these two classes were investigated by Ravichandran* [101]*. We also took note that these classes reduced to the classes studied in* [100] *when $n = 1$ and $g(z) = z/(1-z)$.*

Lemma 3.3 *Let g be a fixed function in $\mathcal{A}_p$, and h a convex univalent function with positive real part satisfying $h(0) = 1$.*

(i) If $f \in \mathcal{STS}_{p,g}^n(h)$ and $F(z) := (f(z) - f(-z))/2$, then $F_n \in \mathcal{ST}_{p,g}(h)$.

(ii) If $f \in \mathcal{STS}_{p,g}^n(h)$, then $f_n \in \mathcal{STS}_{p,g}(h)$.

(iii) The function $f \in \mathcal{CVS}_{p,g}^n(h)$ if and only if $zf'/p \in \mathcal{STS}_{p,g}^n(h)$,

(iv) The inclusion $\mathcal{CVS}_{p,g}^n(h) \subset \mathcal{STS}_{p,g}^n(h)$ holds.

Proof. Again it is enough to prove the results for $g(z) = z^p/(1-z)$.
(i) Let $f \in \mathcal{STS}_p^n(h)$. For any fixed $z \in \mathcal{U}$,

$$\frac{1}{p}\frac{zf'(z)}{F_n(z)} \in h(\mathcal{U}).$$

Replacing z by $-z$ and taking the convex combination of these two expressions, it readily follows that

$$\frac{1}{2p}\left(\frac{zf'(z)}{F_n(z)} + \frac{(-z)f'(-z)}{F_n(-z)}\right) = \frac{1}{p}\frac{zF'(z)}{F_n(z)} \in h(\mathcal{U}).$$

This shows that the function $F \in \mathcal{ST}_p^n(h)$ and Lemma 3.1 (i) now yields $F_n \in \mathcal{ST}_p(h)$.

(ii) Replacing z by $\epsilon^k z$ in

$$\frac{1}{p}\frac{2zf'(z)}{f_n(z) - f_n(-z)} \in h(\mathcal{U}),$$

and using (3.3) and (3.5), it follows from the convexity of $h(\mathcal{U})$ that

$$\frac{1}{p}\frac{2zf_n'(z)}{f_n(z) - f_n(-z)} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{p}\frac{2\epsilon^k zf'(\epsilon^k z)}{f_n(\epsilon^k z) - f_n(-\epsilon^k z)} \in h(\mathcal{U}).$$

Thus $f_n \in \mathcal{STS}_p(h)$.

(iii) Since $(zf'/p)_n(-z) = -zf_n'(-z)/p$, it is clear that

$$\frac{1}{p}\frac{2\left(zf'\right)'(z)}{f_n'(z) + f_n'(-z)} = \frac{1}{p}\frac{2z(\frac{1}{p}zf')'(z)}{(\frac{1}{p}zf')_n(z) - (\frac{1}{p}zf')_n(-z)}.$$

Thus $f \in \mathcal{CVS}_p^n(h)$ if and only if $zf'/p \in \mathcal{STS}_p^n(h)$.

(iv) Let $f \in \mathcal{CVS}_p^n(h)$ and $F(z) := (f(z) - f(-z))/2$. The result in part (iii) shows that $zf^{'}/p \in \mathcal{STS}_p^n(h)$. Hence, by part (i), $(zF^{'}/p)_n \in \mathcal{ST}_p(h)$. Since $(zF^{'}/p)_n = zF_n^{'}/p$, Lemma 3.1(ii) shows that $F_n \in \mathcal{CV}_p(h)$. So it follows from Lemma 3.1(iii) that $F_n \in \mathcal{ST}_p(h)$. Since h is a function with positive real part, we deduce that the function F_n is starlike.

Now let $T(z) := zf'(z)/p$ and $S(z) := (f_n(z) - f_n(-z))/2 = F_n(z)$. Since

$f \in \mathcal{CVS}^n_p(h)$,

$$\frac{T'(z)}{S'(z)} = \frac{1}{p}\frac{2\left(zf'\right)'(z)}{f_n'(z)+f_n'(-z)} \prec h(z).$$

Since S is starlike, the above subordination together with Theorem 3.1 implies that

$$\frac{1}{p}\frac{2zf'(z)}{f_n(z)-f_n(-z)} = \frac{T(z)}{S(z)} \prec h(z),$$

and hence $f \in \mathcal{STS}^n_p(h)$. ∎

Theorem 3.6 *Let h and ϕ satisfy the conditions of Theorem 3.4.*

(i) If $f \in \mathcal{STS}^n_{p,g}(h)$, then $\phi * f \in \mathcal{STS}^n_{p,g}(h)$. Equivalently, $\mathcal{STS}^n_{p,g}(h) \subset \mathcal{STS}^n_{p,g*\phi}(h)$.

(ii) If $f \in \mathcal{CVS}^n_{p,g}(h)$, then $\phi * f \in \mathcal{CVS}^n_{p,g}(h)$. Equivalently, $\mathcal{CVS}^n_{p,g}(h) \subset \mathcal{CVS}^n_{p,g*\phi}(h)$.

Proof. It is enough to prove the results when $g(z) = z^p/(1-z)$.

(i) Define the functions F and H by

$$F(z) := \frac{1}{2}(f(z)-f(-z)) \quad \text{and} \quad H(z) := \frac{1}{p}\frac{zf'(z)}{F_n(z)}.$$

Lemma 3.3 (i) shows that $F_n \in \mathcal{ST}_p(h)$. Since h is a convex function with $\operatorname{Re} h(z) > 1-(1-\alpha)/p$, it follows that

$$\operatorname{Re}\frac{zF_n'(z)}{F_n(z)} > p-1+\alpha,$$

and whence the function $F_n(z)/z^{p-1}$ is starlike of order α. Since $H(z) \prec h(z)$, Theorem 3.3 yields

$$\frac{1}{p}\frac{2z(\phi*f)'(z)}{(\phi*f)_n(z)-(\phi*f)_n(-z)} = \frac{\phi(z)*\frac{1}{p}zf'(z)}{\phi(z)*(f_n(z)-f_n(-z))/2} = \frac{(\phi*HF_n)(z)}{(\phi*F_n)(z)} \prec h(z),$$

and thus $\phi * f \in \mathcal{STS}_p^n(h)$.

(ii) If $f \in \mathcal{CVS}_p^n(h)$, Lemma 3.3 (iii) and the result of part (i) above yield

$$\phi * \frac{1}{p} z f' = \frac{1}{p} z(\phi * f)' \in \mathcal{STS}_p^n(h).$$

Hence $\phi * f \in \mathcal{CVS}_p^n(h)$. ∎

3.4 Multivalent Functions with Respect to n-ply Conjugate Points

Definition 3.5 *For odd positive integer p , the class $\mathcal{STC}_p^n(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying $\left(f_n(z) + \overline{f_n(\overline{z})}\right)/z^p \neq 0$ in $\mathcal{U}$ and the subordination*

$$\frac{1}{p}\frac{2zf'(z)}{f_n(z) + \overline{f_n(\overline{z})}} \prec h(z).$$

Denote by $\mathcal{STC}_{p,g}^n(h)$ the class

$$\mathcal{STC}_{p,g}^n(h) := \left\{ f \in \mathcal{A}_p : f * g \in \mathcal{STC}_{p,g}^n(h) \right\}.$$

Similarly, $\mathcal{CVC}_p^n(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying $\left(f_n'(z) + \overline{f_n'(\overline{z})}\right)/z^{p-1}$ $\neq 0$ in $\mathcal{U}$ and the subordination

$$\frac{1}{p}\frac{2\left(zf'\right)'(z)}{f_n'(z) + \overline{f_n'(\overline{z})}} \prec h(z),$$

and

$$\mathcal{CVC}_{p,g}^n(h) := \left\{ f \in \mathcal{A}_p : f * g \in \mathcal{CVC}_p^n(h) \right\}.$$

For $n = 1$, we adopt the following notations: $\mathcal{STC}_{p,g}^1(h) =: \mathcal{STC}_{p,g}$ and $\mathcal{CVC}_{p,g}^1(h) =:$ $\mathcal{CVC}_{p,g}(h)$.

Remark 3.3 *Ravichandran* [101] *investigated these two classes for $p = 1$. These classes reduced to the classes studied in* [100] *when $n = 1$ and $g(z) = z/(1 - z)$.*

Lemma 3.4 *Let g be a fixed function in $\mathcal{A}_p$, and h a convex univalent function with positive real part satisfying $h(0) = 1$.*

(i) If $f \in \mathcal{STC}^n_{p,g}(h)$ and $F(z) := \left(f(z) + \overline{f(\overline{z})}\right)/2$, then $F_n \in \mathcal{ST}_{p,g}(h)$.

(ii) If $f \in \mathcal{STC}^n_{p,g}(h)$, then $f_n \in \mathcal{STC}_{p,g}(h)$.

(iii) The function $f \in \mathcal{CVC}^n_{p,g}(h)$ if and only if $zf'/p \in \mathcal{STC}^n_{p,g}(h)$.

(iv) The inclusion $\mathcal{CVC}^n_{p,g}(h) \subset \mathcal{STC}^n_{p,g}(h)$ holds.

Proof. Again it is enough to prove the results when $g(z) = z^p/(1-z)$.

(i) Since $F_n(z) = (f_n(z) + \overline{f_n(\overline{z})})/2$, if $f \in \mathcal{STC}^n_p(h)$, then

$$\frac{1}{p}\frac{zf'(z)}{F_n(z)} \in h(\mathcal{U})$$

for any fixed $z \in \mathcal{U}$. Thus

$$\frac{1}{2p}\left(\frac{zf'(z)}{F_n(z)} + \overline{\left(\frac{\overline{z}f'(\overline{z})}{F_n(\overline{z})}\right)}\right) = \frac{1}{p}\frac{zF'(z)}{F_n(z)} \in h(\mathcal{U}).$$

This shows that the function $F \in \mathcal{ST}^n_p(h)$ and by Lemma 3.1 (i) it follows that $F_n \in \mathcal{ST}_p(h)$.

(ii) Replacing z by $\epsilon^k z$ in

$$\frac{1}{p}\frac{2zf'(z)}{f_n(z) + \overline{f_n(\overline{z})}} \in h(\mathcal{U}),$$

and using (3.3) and (3.5), it follows from the convexity of $h(\mathcal{U})$ that

$$\frac{1}{p}\frac{2zf_n'(z)}{f_n(z) + \overline{f_n(\overline{z})}} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{p}\frac{2\epsilon^k zf'(\epsilon^k z)}{f_n(\epsilon^k z) + \overline{f_n(\epsilon^{-k}\overline{z})}} \in h(\mathcal{U}).$$

Thus $f_n \in \mathcal{STC}_p(h)$.

(iii) Since

$$\overline{\left(\frac{1}{p}zf'\right)_n}(\overline{z}) = \frac{1}{p}z\overline{f_n'(\overline{z})},$$

it follows that

$$\frac{1}{p}\frac{2\left(zf'\right)'(z)}{f_n'(z)+\overline{f_n'(\overline{z})}} = \frac{1}{p}\frac{2z(\frac{1}{p}zf')'(z)}{(\frac{1}{p}zf')_n(z)+\overline{(\frac{1}{p}zf')_n(\overline{z})}}.$$

Thus $f \in \mathcal{CVC}_{p,g}^n(h)$ if and only if $zf'/p \in \mathcal{STC}_{p,g}^n(h)$.

(iv) If $f \in \mathcal{CVC}_p^n(h)$, then part (iii) gives $zf^{'}/p \in \mathcal{STC}_p^n(h)$. With $F(z) = (f(z) + \overline{f(\overline{z})})/2$, it follows from part (i) that

$$\begin{aligned}\left(\frac{1}{p}zF'\right)_n &= \frac{\left(\frac{1}{p}zf'\right)_n + \left(\overline{\left(\frac{1}{p}zf'\right)(\overline{z})}\right)_n}{2} \\ &= \frac{\frac{1}{p}zf_n^{'} + \frac{1}{p}z\overline{f_n^{'}(\overline{z})}}{2} \\ &= \frac{1}{p}zF_n^{'} \in \mathcal{ST}_p(h).\end{aligned}$$

Lemma 3.1(ii) now gives $F_n \in \mathcal{CV}_p(h)$, and so $F_n \in \mathcal{ST}_p(h)$. Thus F_n is starlike.

Next let $T(z) := zf'(z)/p$ and $S(z) := (f_n(z) + \overline{f_n(\overline{z})})/2 = F_n(z)$. Since

$$\frac{T'(z)}{S'(z)} = \frac{1}{p}\frac{2\left(zf'\right)'(z)}{f_n'(z)+\overline{f_n'(\overline{z})}} \prec h(z),$$

and S is starlike, Theorem 3.1 shows that

$$\frac{1}{p}\frac{2zf'(z)}{f_n(z)+\overline{f_n'(\overline{z})}} = \frac{T(z)}{S(z)} \prec h(z),$$

whence $f \in \mathcal{STC}_{p,g}^n(h)$. ∎

Theorem 3.7 *Let h and ϕ satisfy the conditions of Theorem 3.4 and ϕ has real coefficients.*

(i) If $f \in \mathcal{STC}^n_{p,g}$, then $\phi * f \in \mathcal{STC}^n_{p,g}(h)$. Equivalently, $\mathcal{STC}^n_{p,g}(h) \subset \mathcal{STC}^n_{p,g*\phi}(h)$.

(ii) If $f \in \mathcal{CVC}^n_{p,g}(h)$, then $\phi * f \in \mathcal{CVC}^n_{p,g}(h)$, and $\mathcal{CVC}^n_{p,g}(h) \subset \mathcal{CVC}^n_{p,g*\phi}(h)$.

Proof. (i) Let $f \in \mathcal{STC}^n_p(h)$.Define the functions $F(z)$ and $H(z)$ by

$$F(z) = \frac{f(z) + \overline{f(\bar{z})}}{2} \quad \text{and} \quad H(z) = \frac{1}{p}\frac{zf'(z)}{F_n(z)}.$$

Using Lemma 3.4, and proceeding similarly as in the proof of Theorem 3.6, it can be shown that the function $F_n(z)/z^{p-1}$ is starlike of order α, where $F_n(z) = (f_n(z) + \overline{f_n(\bar{z})})/2$.

Since $H(z) \prec h(z)$ and because ϕ has real coefficients, Theorem 3.3 yields

$$\frac{1}{p}\frac{2z(\phi * f)'(z)}{(\phi * f)_n(z) + \overline{(\phi * f)_n(\bar{z})}} = \frac{\phi(z) * \frac{1}{p}zf'(z)}{\phi(z) * \left(f_n(z) + \overline{f_n(\bar{z})}\right)/2} = \frac{(\phi * HF_n)(z)}{(\phi * F_n)(z)} \prec h(z),$$

or $\phi * f \in \mathcal{STC}^n_p(h)$.

(ii) If $f \in \mathcal{CVC}^n_{p,g}(h)$, it follows from Lemma 3.4 (iii) that $zf'/p \in \mathcal{STC}^n_{p,g}(h)$. By part (i), it is now evident that

$$\phi * \frac{1}{p}zf' = \frac{1}{p}z(\phi * f)' \in \mathcal{STC}^n_{p,g}(h),$$

and thus we deduce that $\phi * f \in \mathcal{CVC}^n_{p,g}(h)$ from Lemma 3.4 (iii). ∎

3.5 Multivalent Functions with Respect to n-ply Symmetric Conjugate Points

Definition 3.6 *For odd positive integer p , the class $\mathcal{STSC}^n_p(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying $\left(f_n(z) - \overline{f_n(-\bar{z})}\right)/z^p \neq 0$ in $\mathcal{U}$ and the subordination*

$$\frac{1}{p}\frac{2zf'(z)}{f_n(z) - \overline{f_n(-\bar{z})}} \prec h(z).$$

Denote by $\mathcal{STSC}^n_{p,g}(h)$ *the class*

$$\mathcal{STSC}^n_{p,g}(h) := \left\{ f \in \mathcal{A}_p : f * g \in \mathcal{STSC}^n_p(h) \right\}.$$

Similarly, $\mathcal{CVSC}^n_p(h)$ *consists of functions* $f \in \mathcal{A}_p$ *satisfying* $\left(f'_n(z) + \overline{f'_n(-\overline{z})} \right) / z^{p-1}$ $\neq 0$ *in* $\mathcal{U}$ *and the subordination*

$$\frac{1}{p} \frac{2\left(zf'\right)'(z)}{f'_n(z) + \overline{f'_n(-\overline{z})}} \prec h(z),$$

and

$$\mathcal{CVSC}^n_{p,g}(h) := \left\{ f \in \mathcal{A}_p : f * g \in \mathcal{CVSC}^n_p(h) \right\}.$$

The following usual notations $\mathcal{STSC}^1_{p,g}(h) := \mathcal{STSC}_{p,g}(h)$ and $\mathcal{CVSC}^1_{p,g}(h) :=$ $\mathcal{CVSC}_{p,g}(h)$ are used for the case $n = 1$.

Remark 3.4 *The above two classes were investigated by Ravichandran* [101] *for* $p = 1$. *Evidently, these classes reduced to the classes studied in* [100] *when* $n = 1$ *and* $g(z) = z/(1-z)$.

The following two results can readily be established by proceeding analogously as in the proofs of Lemmas 3.3 and 3.4, and Theorems 3.6 and 3.7. We omit these proofs.

Lemma 3.5 *Let* g *be a fixed function in* $\mathcal{A}_p$, *and* h *a convex univalent function with positive real part satisfying* $h(0) = 1$.

(i) If $f \in \mathcal{STSC}^n_{p,g}(h)$ *and* $F(z) := (f(z) - \overline{f(-\overline{z})})/2$, *then* $F_n \in \mathcal{ST}_{p,g}(h)$.

(ii) If $f \in \mathcal{STSC}^n_{p,g}(h)$, *then* $f_n \in \mathcal{STSC}_{p,g}(h)$.

(iii) The function $f \in \mathcal{CVSC}^n_{p,g}(h)$ *if and only if* $zf'/p \in \mathcal{STSC}^n_{p,g}(h)$.

(iv) The inclusion $\mathcal{CVSC}^n_{p,g}(h) \subset \mathcal{STSC}^n_{p,g}(h)$ *holds.*

Theorem 3.8 *Let h and ϕ satisfy the conditions of Theorem 3.4 and ϕ has real coefficients.*

(i) If $f \in \mathcal{STSC}^n_{p,g}$, then $\phi * f \in \mathcal{STSC}^n_{p,g}(h)$.

(ii) If $f \in \mathcal{CVSC}^n_{p,g}(h)$, then $\phi * f \in \mathcal{CVSC}^n_{p,g}(h)$.

CHAPTER 4

CLOSURE PROPERTIES OF OPERATORS ON MA-MINDA TYPE STARLIKE AND CONVEX FUNCTIONS

4.1 Two Operators

As mentioned in Section 1.5, p. 26, Chandra and Singh [36] proved that the integral

$$\int_0^z \frac{f(e^{i\mu}\zeta) - f(e^{i\psi}\zeta)}{(e^{i\mu} - e^{i\psi})\zeta}\, d\zeta \quad (\mu \neq \psi, 0 \leq \mu, \psi < 2\pi)$$

preserves membership in the classes of starlike, convex and close-to-convex functions. This integral operator has been generalized in the following manner:

Definition 4.1 *For $\alpha_j \geq 0$ and $f_j \in \mathcal{A}$, define the operators $F : \mathcal{A}^n \times \overline{\mathcal{U}}^2 \to \mathcal{A}$ and $G : \mathcal{A}^n \times \overline{\mathcal{U}}^2 \to \mathcal{A}$ by*

$$F(z) = F_{f_1,\cdots,f_n;z_1,z_2}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{f_j(z_2\zeta) - f_j(z_1\zeta)}{(z_2 - z_1)\zeta} \right)^{\alpha_j} d\zeta \quad (z_1, z_2 \in \overline{\mathcal{U}}), \tag{4.1}$$

$$G(z) = G_{f_1,\cdots,f_n;z_1,z_2}(z) = z \prod_{j=1}^{n} \left(\frac{f_j(z_2 z) - f_j(z_1 z)}{(z_2 - z_1)z} \right)^{\alpha_j} \quad (z_1, z_2 \in \overline{\mathcal{U}}). \tag{4.2}$$

Here the powers are chosen to be principal. It is clear that $G(z) = zF'(z)$. With F and G as above, define the classes $\mathcal{F}_n$ and $\mathcal{G}_n$ respectively by

$$\mathcal{F}_n(f_1, ..., f_n) := \left\{ F_{f_1,\cdots,f_n;z_1,z_2} : f_j \in \mathcal{A}, z_1, z_2 \in \overline{\mathcal{U}} \right\}, \tag{4.3}$$

and

$$\mathcal{G}_n(f_1, ..., f_n) := \left\{ G_{f_1,\cdots,f_n;z_1,z_2} : f_j \in \mathcal{A}, z_1, z_2 \in \overline{\mathcal{U}} \right\}. \tag{4.4}$$

In the case $n = 1$, it is assumed that $\alpha_1 = 1$ in (4.1) and (4.2), and we write $\mathcal{F}(f) := \mathcal{F}_1(f)$ and $\mathcal{G}(f) := \mathcal{G}_1(f)$ respectively.

Ponnusamy and Singh [97] introduced the operator F in (4.1) and investigated its univalence. In this chapter, membership preservation properties of the operators F and G on the subclasses of starlike, convex and close-to-convex functions will be investigated. We shall also make connections with various earlier works. The following lemma will be required.

Lemma 4.1 [72] *Let G be analytic, and H convex univalent in $\mathcal{U}$. If the range of the function G'/H' is contained in a convex set Δ, then so do the values $(G(z_2) - G(z_1)) / (H(z_2) - H(z_1))$ for $z_1, z_2 \in \mathcal{U}$.*

This lemma yields the following immediate result:

Lemma 4.2 *Let φ be a convex function with $\varphi(0) = 1$ and $z_1, z_2 \in \overline{\mathcal{U}}$. If $f \in \mathcal{A}$ satisfies the subordination $f'(z)/g'(z) \prec \varphi(z)$ for some $g \in \mathcal{CV}$, then*

$$\frac{f(z_2 z) - f(z_1 z)}{g(z_2 z) - g(z_1 z)} \prec \varphi(z).$$

4.2 Operators on Subclasses of Convex Functions

Theorem 4.1 *For $j = 1, 2, \ldots, n$, let $\alpha_j \geq 0$, $0 \leq \beta_j < 1$ and $\gamma := 1 - \sum_{j=1}^{n} \alpha_j (1 - \beta_j)$. For $f_j \in \mathcal{A}$, let F and G be given by (4.1) and (4.2) respectively. If $f_j \in \mathcal{CV}(\beta_j)$, then $F \in \mathcal{CV}(\gamma)$ and $G \in \mathcal{ST}(\gamma)$. In particular, if $\sum_{j=1}^{n} \alpha_j (1 - \beta_j) \leq 1$, then $F \in \mathcal{CV}$ and $G \in \mathcal{ST}$.*

Proof. Let $f_j \in \mathcal{CV}(\beta_j)$ so that

$$\frac{(z f_j'(z))'}{f_j'(z)} = 1 + \frac{z f_j''(z)}{f_j'(z)} \prec \varphi_{\beta_j}(z), \tag{4.5}$$

where $\varphi_{\beta_j} : \mathcal{U} \to \mathbb{C}$ is the convex function defined by

$$\varphi_{\beta_j}(z) = \frac{1 + (1 - 2\beta_j) z}{1 - z}.$$

For $0 \leq \beta_j < 1$, $\varphi_{\beta_j}(\mathcal{U})$ is the half-plane $\operatorname{Re} w > \beta_j$ and hence $\varphi_{\beta_j}(\mathcal{U})$ is a convex domain. Since f_j is a convex function, Lemma 4.2 applied to the subordination (4.5) yields

$$\frac{z_2 z f_j'(z_2 z) - z_1 z f_j'(z_1 z)}{f_j(z_2 z) - f_j(z_1 z)} \prec \varphi_{\beta_j}(z),$$

or equivalently,

$$\operatorname{Re}\left(\frac{z_2 z f_j'(z_2 z) - z_1 z f_j'(z_1 z)}{f_j(z_2 z) - f_j(z_1 z)}\right) > \beta_j. \tag{4.6}$$

A differentiation of (4.1) yields

$$F'(z) = \prod_{j=1}^{n} \left(\frac{f_j(z_2 z) - f_j(z_1 z)}{(z_2 - z_1)z}\right)^{\alpha_j},$$

and differentiating logarithmically shows that

$$1 + \frac{zF''(z)}{F'(z)} = (1 - \sum_{j=1}^{n} \alpha_j) + \sum_{j=1}^{n} \alpha_j \left(\frac{z_2 z f_j'(z_2 z) - z_1 z f_j'(z_1 z)}{f_j(z_2 z) - f_j(z_1 z)}\right). \tag{4.7}$$

It follows from (4.7) by using the inequality (4.6) that $F \in \mathcal{CV}(\gamma)$:

$$\begin{aligned} 1 + \operatorname{Re}\frac{zF''(z)}{F'(z)} &= (1 - \sum_{j=1}^{n} \alpha_j) + \sum_{j=1}^{n} \alpha_j \operatorname{Re}\left(\frac{z_2 z f_j'(z_2 z) - z_1 z f_j'(z_1 z)}{f_j(z_2 z) - f_j(z_1 z)}\right) \\ &> (1 - \sum_{j=1}^{n} \alpha_j) + \sum_{j=1}^{n} \alpha_j \beta_j \\ &= \gamma. \end{aligned}$$

The result that $G \in \mathcal{ST}(\gamma)$ follows from the fact that $zF'(z) = G(z)$ and that $F \in \mathcal{CV}(\gamma)$. ∎

Interesting special cases of Theorem 4.1 are obtained in the following results.

Corollary 4.1 *Let $0 \leq \beta < 1$. For $j = 1, 2, \ldots, n$, let $\alpha_j \geq 0$ and $\gamma := 1 - (1 - \beta)\sum_{j=1}^{n} \alpha_j$. For $f_j \in \mathcal{A}$, let F be given by (4.1). If $f_j \in \mathcal{CV}(\beta)$, then $F \in \mathcal{CV}(\gamma)$*

and $G \in \mathcal{ST}(\gamma)$. In particular, if $\sum_{j=1}^{n} \alpha_j \leq 1$, then $F \in \mathcal{CV}(\beta)$ and $G \in \mathcal{ST}(\beta)$.

Corollary 4.2 *For $j = 1, 2, ..., n$, let $\alpha_j \geq 0$, $0 \leq \beta_j < 1$ and $\gamma := 1 - \sum\limits_{j=1}^{n} \alpha_j (1 - \beta_j)$. For $f_j \in \mathcal{A}$, let $\mathcal{F}_n(f_1, ..., f_n)$ and $\mathcal{G}_n(f_1, ..., f_n)$ be given by (4.3) and (4.4) respectively. If $f_j \in \mathcal{CV}(\beta_j)$, then*

$$\mathcal{F}_n(f_1, ..., f_n) \subset \mathcal{CV}(\gamma) \text{ and } \mathcal{G}_n(f_1, ..., f_n) \subset \mathcal{ST}(\gamma).$$

In particular, if $\sum\limits_{j=1}^{n} \alpha_j (1 - \beta_j) \leq 1$, then

$$\mathcal{F}_n(f_1, ..., f_n) \subset \mathcal{CV} \text{ and } \mathcal{G}_n(f_1, ..., f_n) \subset \mathcal{ST}.$$

Also if $f \in \mathcal{CV}(\alpha)$, $0 \leq \alpha < 1$, then $\mathcal{F}(f) \subset \mathcal{CV}(\alpha)$ and $\mathcal{G}(f) \subset \mathcal{ST}(\alpha)$.

Corollary 4.3 [36, Theorem 2.1, p. 1271 and Theorem 2.4 p. 1273] *Let $0 \leq \alpha < 1$. If $f(z) \in \mathcal{CV}(\alpha)$, then $\mathcal{G}(f) \subset \mathcal{ST}(\alpha)$ and $\mathcal{F}(f) \subset \mathcal{CV}(\alpha)$.*

Corollary 4.4 [121] *If $f \in \mathcal{CV}$, then $\int_0^z (f(t) - f(-t))/(2t)dt \in \mathcal{CV}$.*

4.3 Operators on Subclasses of Ma-Minda Convex Functions

For $j = 1, 2, \ldots, n$, let $\alpha_j \geq 0$, $0 \leq \beta < 1$ and $\sum_{j=1}^{n} \alpha_j = 1$. For $f_j \in \mathcal{A}$, let F be given by (4.1). By Corollary 4.1, if $f_j \in \mathcal{CV}(\beta)$, then $F \in \mathcal{CV}(\beta)$. This result is next proved in a more general setting:

Theorem 4.2 *For $j = 1, 2, \ldots, n$, let $\alpha_j \geq 0$ and $\sum_{j=1}^{n} \alpha_j \leq 1$. Let φ be convex in $\mathcal{U}$ with positive real part, and normalized by $\varphi(0) = 1$. If $f_j \in \mathcal{CV}(\varphi)$, then F given by (4.1) satisfies $F \in \mathcal{CV}(\varphi)$, and G given by (4.2) satifies $G \in \mathcal{ST}(\varphi)$.*

Proof. Let $f_j \in \mathcal{CV}(\varphi)$ so that

$$1 + \frac{z f_j''(z)}{f_j'(z)} \prec \varphi(z).$$

Since φ is a function with positive real part, it follows that

$$1 + \operatorname{Re} \frac{zf_j''(z)}{f_j'(z)} > 0,$$

and hence f_j is a convex function. As shown in the proof of Theorem 4.1, Lemma 4.2 yields

$$\frac{z_2 z f_j'(z_2 z) - z_1 z f_j'(z_1 z)}{f_j(z_2 z) - f_j(z_1 z)} \prec \varphi(z),$$

or for any fixed $z \in \mathcal{U}$,

$$\frac{z_2 z f_j'(z_2 z) - z_1 z f_j'(z_1 z)}{f_j(z_2 z) - f_j(z_1 z)} \in \varphi(\mathcal{U}).$$

Since φ is convex, and $1 = \varphi(0) \in \varphi(\mathcal{U})$, the convex combination of $n+1$ complex numbers

$$1; \frac{z_2 z f_j'(z_2 z) - z_1 z f_j'(z_1 z)}{f_j(z_2 z) - f_j(z_1 z)} \quad (j = 1, 2, \ldots, n),$$

is again in $\varphi(\mathcal{U})$:

$$(1 - \sum_{j=1}^{n} \alpha_j)(1) + \sum_{j=1}^{n} \alpha_j \left(\frac{z_2 z f_j'(z_2 z) - z_1 z f_j'(z_1 z)}{f_j(z_2 z) - f_j(z_1 z)} \right) \in \varphi(\mathcal{U}).$$

Thus it follows that

$$(1 - \sum_{j=1}^{n} \alpha_j) + \sum_{j=1}^{n} \alpha_j \left(\frac{z_2 z f_j'(z_2 z) - z_1 z f_j'(z_1 z)}{f_j(z_2 z) - f_j(z_1 z)} \right) \prec \varphi(z).$$

In view of (4.7), the above subordination becomes

$$1 + \frac{zF''(z)}{F'(z)} \prec \varphi(z),$$

which proves $F \in \mathcal{CV}(\varphi)$. ∎

Corollary 4.5 *Let φ be convex in $\mathcal{U}$ with positive real part, and normalized by*

$\varphi(0) = 1$. If $f \in \mathcal{CV}(\varphi)$, then

$$\int_0^z \frac{f(z_2\zeta) - f(z_1\zeta)}{(z_2 - z_1)\zeta}\, d\zeta \in \mathcal{CV}(\varphi) \quad \textit{and} \quad \frac{f(z_2 z) - f(z_1 z)}{z_2 - z_1} \in \mathcal{ST}(\varphi).$$

Corollary 4.6 *For $j = 1, 2, ..., n$, let $\alpha_j \geq 0$ and $\sum_{j=1}^{n} \alpha_j \leq 1$. Let φ be convex in $\mathcal{U}$ with positive real part, and normalized by $\varphi(0) = 1$. If $f_j \in \mathcal{CV}(\varphi)$, then $\mathcal{F}_n(f_1, ..., f_n) \subset \mathcal{CV}(\varphi)$ and $\mathcal{G}_n(f_1, ..., f_n) \subset \mathcal{ST}(\varphi)$. In particular, if $f \in \mathcal{CV}(\varphi)$, then $\mathcal{F}(f) \subset \mathcal{CV}(\varphi)$ and $\mathcal{G}(f) \subset \mathcal{ST}(\varphi)$.*

4.4 Operators on Subclasses of Starlike and Close-to-Convex Functions

In this section, we shall devote attention to the following special case of the operator F :

$$F_1(z) := \int_0^z \frac{f(z_2\zeta) - f(z_1\zeta)}{(z_2 - z_1)\zeta}\, d\zeta.$$

Theorem 4.3 *Let φ be convex in $\mathcal{U}$ with positive real part, and normalized by $\varphi(0) = 1$. If $f \in \mathcal{ST}(\varphi)$, then $F_1 \in \mathcal{ST}(\varphi)$.*

Proof. Since $f \in \mathcal{ST}(\varphi)$, there exists a function $g \in \mathcal{CV}(\varphi)$ such that $f(z) = zg'(z)$. In fact, such a function g satisfies

$$g(\alpha z) = \int_0^z \frac{f(\alpha\zeta)}{\zeta}\, d\zeta \quad (|\alpha| \leq 1).$$

Using this identity, it follows that

$$F_1(z) = \int_0^z \frac{f(z_2\zeta) - f(z_1\zeta)}{(z_2 - z_1)\zeta}\, d\zeta = \frac{g(z_2 z) - g(z_1 z)}{z_2 - z_1}.$$

Since $g \in \mathcal{CV}(\varphi)$, Corollary 4.5 shows that

$$\frac{g(z_2 z) - g(z_1 z)}{z_2 - z_1} \in \mathcal{ST}(\varphi),$$

and hence $F_1 \in \mathcal{ST}(\varphi)$. ∎

Corollary 4.7 *Let φ be convex in $\mathcal{U}$ with positive real part, and normalized by $\varphi(0) = 1$. If $f \in \mathcal{ST}(\varphi)$, then $\mathcal{F}(f) \subset \mathcal{ST}(\varphi)$.*

Corollary 4.8 [36, Theorem 2.3, p. 1273] *Let $0 \leq \alpha < 1$. If $f(z) \in \mathcal{ST}(\alpha)$, then $\mathcal{F}(f) \subset \mathcal{ST}(\alpha)$.*

Corollary 4.9 [121] *If $f \in \mathcal{ST}$, then $\int_0^z (f(t) - f(-t))/(2t)dt \in \mathcal{ST}$.*

Definition 4.2 *Let φ and ψ be convex functions with positive real part and normalized respectively by $\phi(0) = 1$ and $\psi(0) = 1$. The class $\mathcal{CCV}(\varphi, \psi)$ consists of functions $f \in \mathcal{A}$ satisfying the subordination*

$$\frac{f'(z)}{h'(z)} \prec \varphi(z)$$

where $h \in \mathcal{CV}(\psi)$.

For $0 \leq \alpha, \tau < 1$, let $\varphi_\alpha : \mathcal{U} \to \mathbb{C}$ and $\psi_\tau : \mathcal{U} \to \mathbb{C}$ be defined by

$$\varphi_\alpha(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad \psi_\alpha(z) = \frac{1 + (1 - 2\tau)z}{1 - z}.$$

In this case, the class $\mathcal{CCV}(\varphi, \psi)$ reduces to the familiar class of univalent close-to-convex functions of order α and type τ:

$$\mathcal{CCV}(\alpha, \tau) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{f'(z)}{h'(z)}\right) > \alpha, \ h \in \mathcal{CV}(\tau) \right\}.$$

In this form, the class $\mathcal{CCV}_\alpha$ investigated by Pommerenke [92] becomes a special case of $\mathcal{CCV}(\varphi, \psi)$, that is,

$$\mathcal{CCV}_\alpha = \mathcal{CCV}\left(\left(\frac{1+z}{1-z}\right)^\alpha, \frac{1+z}{1-z}\right).$$

The following closure property for the class $\mathcal{CCV}(\varphi,\psi)$ contains a result of Pommerenke [92].

Theorem 4.4 *If $f \in \mathcal{CCV}(\varphi,\psi)$, then $F_1 \in \mathcal{CCV}(\varphi,\psi)$.*

Proof. If $f \in \mathcal{CCV}(\varphi,\psi)$, then there exists a function $h \in \mathcal{CV}(\psi)$ such that

$$\frac{f'(z)}{h'(z)} \prec \varphi(z).$$

Corollary 4.5 yields

$$H_1(z) := \int_0^z \frac{h(z_2\zeta) - h(z_1\zeta)}{(z_2 - z_1)\zeta}\, d\zeta \in \mathcal{CV}(\psi).$$

Since $\operatorname{Re}\psi(z) > 0$, the function h is convex. It follows from Lemma 4.2 that

$$\frac{f(z_2z) - f(z_1z)}{h(z_2z) - h(z_1z)} \prec \varphi(z).$$

Since

$$\frac{F_1'(z)}{H_1'(z)} = \frac{f(z_2z) - f(z_1z)}{h(z_2z) - h(z_1z)},$$

we deduce that $F_1 \in \mathcal{CCV}(\varphi,\psi)$. ∎

Corollary 4.10 *If $f \in \mathcal{CC}(\varphi,\psi)$, then $\mathcal{F}(f) \subset \mathcal{CC}(\varphi,\psi)$. In particular, for $0 \leq \alpha,\tau < 1$, if $f \in \mathcal{CC}(\alpha,\tau)$, then $\mathcal{F}(f) \subset \mathcal{CC}(\alpha,\tau)$.*

Corollary 4.11 [36, Theorem 2.6, p. 1274] *Let $0 \leq \alpha,\tau < 1$, if $f \in \mathcal{CC}(\alpha,\tau)$, then $\mathcal{F}(f) \subset \mathcal{CC}(\alpha,\tau)$.*

Corollary 4.12 [121] *If $f \in \mathcal{CCV}$ with respect to the convex function h, then $F(f) = \int_0^z (f(t) - f(-t))/(2t)dt \in \mathcal{CCV}$ with respect to the convex function $H(f) = \int_0^z (h(t) - h(-t))/(2t)dt$.*

CHAPTER 5

STARLIKENESS OF INTEGRAL TRANSFORMS VIA DUALITY

5.1 Duality Technique

For $\beta < 1$, the class $\mathcal{R}(\beta)$ defined in Section 1.6, p. 29, consists of functions f satisfying

$$\operatorname{Re}\left(f'(z) + zf''(z)\right) > \beta \quad (z \in \mathcal{U}).$$

Ali [6] conjectured that for $\beta_0 = -(2\log 2 - 1)/2(1 - \log 2) = -0.629$, the class $\mathcal{R}(\beta_0) \subset \mathcal{ST}$, and that β_0 is the best estimate. Fournier and Ruscheweyh [48] proved that Ali's conjecture was true. As the implication

$$\operatorname{Re}\left(f'(z) + zf''(z)\right) > \beta \Rightarrow f \in \mathcal{ST}$$

is equivalent to $V(f) \in \mathcal{ST}$ whenever $\operatorname{Re}\left(f'(z)\right) > \beta$, where $V : \mathcal{A} \to \mathcal{A}$ is given by

$$V(f)(z) = \int_0^1 \frac{f(tz)}{t} dt,$$

Fournier and Ruscheweyh [48] introduced a more general operator $V_\lambda : \mathcal{A} \to \mathcal{A}$ given by

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \tag{5.1}$$

where λ is a non-negative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)dt = 1$. Using the Duality Principle [110, 113], they proved starlikeness of the linear integral transform V_λ over functions f in the class

$$\mathcal{P}(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\left(f'(z) - \beta\right) > 0, \quad z \in \mathcal{U} \right\}.$$

In a recent paper, Miller and Mocanu [76] determined conditions on the kernel function W so that the function f defined by

$$f(z) = \int_0^1 \int_0^1 W(r, s, z) dr ds$$

is starlike. Ali et al. in [7] considered a different kernel W. They considered functions $f \in \mathcal{A}$ given by the double integral operator of the form

$$f(z) = \int_0^1 \int_0^1 G\left(zt^{\mu} s^{\nu}\right) t^{-\mu} s^{-\nu} ds dt.$$

Under this instance, it follows that

$$f'(z) = \int_0^1 \int_0^1 g\left(zt^{\mu} s^{\nu}\right) ds dt,$$

where $G' = g$. Furthermore, this function f satisfies a third-order differential equation of the form

$$f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) = g(z)$$

for appropriate α and γ. For particular cases of a convex function h, Ali et al. [7] investigated starlikeness properties of functions f belonging to the class $\mathcal{R}(\alpha, \gamma, h)$ defined by

$$\mathcal{R}(\alpha, \gamma, h) = \left\{ f \in \mathcal{A} : f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \prec h(z), \quad z \in \mathcal{U} \right\}.$$

In the special case when the function h given by

$$h(z) := h_{\beta}(z) = 1 + \frac{(1 - 2\beta) z}{1 - z} \quad (\beta < 1),$$

the class $\mathcal{R}(\alpha, \gamma, h_\beta)$ reduces to the class

$$\mathcal{R}(\alpha, \gamma, \beta) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \right) > \beta, \quad z \in \mathcal{U} \right\}.$$

For $\alpha \geq 0$, $\gamma \geq 0$ and $\beta < 1$, let

$$\begin{aligned} \mathcal{W}_\beta(\alpha, \gamma) := \Big\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi} \Big((1 - \alpha + 2\gamma) \frac{f(z)}{z} & \\ + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) - \beta \Big) > 0, z \in \mathcal{U} \Big\}. & \end{aligned} \tag{5.2}$$

The class $\mathcal{W}_\beta(\alpha, \gamma)$ is closely related to the class $\mathcal{R}(\alpha, \gamma, \beta)$. It is evident that $f \in \mathcal{R}(\alpha, \gamma, \beta)$ if and only if zf' lies in a subclass of $\mathcal{W}_\beta(\alpha, \gamma)$ where $\phi = 0$. Recently, Kim and Rønning [63] investigated starlikeness property of the integral transform (5.1) for functions f in the class

$$\begin{aligned} \mathcal{P}_\alpha(\beta) := \mathcal{W}_\beta(\alpha, 0) = \{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with} & \\ \operatorname{Re} e^{i\phi} \left((1 - \alpha) \frac{f(z)}{z} + \alpha f'(z) - \beta \right) > 0, z \in \mathcal{U} \Big\}. & \end{aligned}$$

In 2008, Ponnusamy and Rønning [96] discussed this problem for functions f in the class

$$\begin{aligned} \mathcal{R}_\gamma(\beta) := \mathcal{W}_\beta(1 + 2\gamma, \gamma) = \{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with} & \\ \operatorname{Re} e^{i\phi} \left(f'(z) + \gamma z f''(z) - \beta \right) > 0, \ z \in \mathcal{U} \Big\}. & \end{aligned}$$

In this chapter, the Duality Principle is used to investigate the starlikeness of the integral transform $V_\lambda(f)$ over the class $\mathcal{W}_\beta(\alpha, \gamma)$. The results obtained extend earlier works of Fournier and Ruscheweyh [48], Kim and Rønning [63], and Ponnusamy and Rønning [96].

Interestingly the general integral transform $V_\lambda(f)$ in (5.1) reduces to various well-known integral operators for specific choices of λ. For example,

$$\lambda(t) := (1+c)t^c \quad (c > -1),$$

gives the Bernardi integral operator, while the choice

$$\lambda(t) := \frac{(a+1)^p}{\Gamma(p)} t^a (\log \frac{1}{t})^{p-1} \quad (a > -1,\ p \geq 0),$$

gives the Komatu operator [66]. Clearly for $p = 1$ the Komatu operator is in fact the Bernardi operator.

For a certain choice of λ, the integral operator V_λ is the convolution between a function f and the Gaussian hypergeometric function $F(a,b;c;z) := {}_2F_1(a,b;c;z)$, which is related to the general Hohlov operator [49] given by

$$H_{a,b,c}(f) := zF(a,b;c;z) * f(z).$$

In the special case $a = 1$, the operator reduces to the Carlson-Shaffer operator [33]. Here ${}_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function given by the series

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (z \in \mathcal{U}),$$

where the Pochhammer symbol is used to indicate $(a)_n = a(a+1)_{n-1},\ (a)_0 = 1$, and where a, b, c are real parameters with $c \neq 0, -1, -2 \cdots$.

In Section 5.2, the best value of $\beta < 1$ is determined that ensures $V_\lambda(f)$ maps $\mathcal{W}_\beta(\alpha, \gamma)$ into the class of normalized univalent functions $\mathcal{S}$. Additionally, necessary and sufficient conditions are determined that ensure $V_\lambda(f)$ is starlike univalent over the class $\mathcal{W}_\beta(\alpha, \gamma)$. In Section 5.3, easier sufficient conditions for $V_\lambda(f)$ to be starlike is found, and Section 5.4 is devoted to several applications of results obtained for specific choices of the admissible function λ. In particular, the smallest value $\beta < 1$ is obtained that ensures a function f satisfying Re $\big(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)\big) > \beta$ in the unit disk is starlike.

5.2 Univalence and Starlikeness of Integral Transforms

First we introduce two constants $\mu \geq 0$ and $\nu \geq 0$ satisfying

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu\nu = \gamma. \tag{5.3}$$

When $\gamma = 0$, then μ is chosen to be 0, in which case, $\nu = \alpha \geq 0$. When $\alpha = 1+2\gamma$, (5.3) yields $\mu + \nu = 1 + \gamma = 1 + \mu\nu$, or $(\mu - 1)(1 - \nu) = 0$.

(i) For $\gamma > 0$, then choosing $\mu = 1$ gives $\nu = \gamma$.

(ii) For $\gamma = 0$, then $\mu = 0$ and $\nu = \alpha = 1$.

In the sequel, whenever the particular case $\alpha = 1 + 2\gamma$ is considered, the values of μ and ν for $\gamma > 0$ will be taken as $\mu = 1$ and $\nu = \gamma$ respectively, while $\mu = 0$ and $\nu = 1 = \alpha$ in the case $\gamma = 0$.

Next we introduce two auxiliary functions. Let

$$\phi_{\mu,\nu}(z) = 1 + \sum_{n=1}^{\infty} \frac{(n\nu + 1)(n\mu + 1)}{n+1} z^n, \tag{5.4}$$

and

$$\begin{aligned} \psi_{\mu,\nu}(z) = \phi_{\mu,\nu}^{-1}(z) &= 1 + \sum_{n=1}^{\infty} \frac{n+1}{(n\nu + 1)(n\mu + 1)} z^n, \\ &= \int_0^1 \int_0^1 \frac{ds dt}{(1 - t^{\nu} s^{\mu} z)^2}. \end{aligned} \tag{5.5}$$

Here $\phi_{\mu,\nu}^{-1}$ denotes the convolution inverse of $\phi_{\mu,\nu}$ such that $\phi_{\mu,\nu} * \phi_{\mu,\nu}^{-1} = z/(1-z)$. If $\gamma = 0$, then $\mu = 0$, $\nu = \alpha$, and it is clear that

$$\psi_{0,\alpha}(z) = 1 + \sum_{n=1}^{\infty} \frac{n+1}{n\alpha+1} z^n = \int_0^1 \frac{dt}{(1-t^\alpha z)^2}.$$

If $\gamma > 0$, then $\nu > 0$, $\mu > 0$, and making the change of variables $u = t^\nu$, $v = s^\mu$ results in

$$\psi_{\mu,\nu}(z) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu-1} v^{1/\mu-1}}{(1-uvz)^2} dudv.$$

Thus the function ψ can be written as

$$\psi_{\mu,\nu}(z) = \begin{cases} \dfrac{1}{\mu\nu} \displaystyle\int_0^1 \int_0^1 \frac{u^{1/\nu-1} v^{1/\mu-1}}{(1-uvz)^2} dudv & (\gamma > 0), \\ \\ \displaystyle\int_0^1 \frac{dt}{(1-t^\alpha z)^2} & (\gamma = 0,\ \alpha \geq 0). \end{cases} \tag{5.6}$$

Now let g be the solution of the initial value-problem

$$\frac{d}{dt} t^{1/\nu}(1+g(t)) = \begin{cases} \dfrac{2}{\mu\nu} t^{1/\nu-1} \displaystyle\int_0^1 \frac{s^{1/\mu-1}}{(1+st)^2} ds & (\gamma > 0), \\ \\ \dfrac{2}{\alpha} \dfrac{t^{1/\alpha-1}}{(1+t)^2} & (\gamma = 0, \alpha > 0), \end{cases} \tag{5.7}$$

satisfying $g(0) = 1$. It is easily seen that the solution is given by

$$g(t) = \frac{2}{\mu\nu} \int_0^1 \int_0^1 \frac{s^{1/\mu-1} w^{1/\nu-1}}{(1+swt)^2} dsdw - 1 = 2 \sum_{n=0}^{\infty} \frac{(n+1)(-1)^n t^n}{(1+\mu n)(1+\nu n)} - 1. \tag{5.8}$$

In particular,

$$g_\gamma(t) = \frac{1}{\gamma}\int_0^1 s^{1/\gamma-1}\frac{1-st}{1+st}ds \quad (\gamma>0, \quad \alpha = 1+2\gamma),$$
$$g_\alpha(t) = \frac{2}{\alpha}t^{-1/\alpha}\int_0^t \frac{\tau^{1/\alpha-1}}{(1+\tau)^2}d\tau - 1 \quad (\gamma = 0, \quad \alpha>0). \tag{5.9}$$

Functions in the class $\mathcal{W}_\beta(\alpha,\gamma)$ generally are not starlike; indeed, they may not even be univalent. Our central result below provides conditions for univalence and starlikeness.

Theorem 5.1 *Let $\mu \geq 0$, $\nu \geq 0$ satisfy* (5.3), *and let $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)g(t)dt, \tag{5.10}$$

where g is the solution of the initial-value problem (5.7) *given by* (5.8). *If $f \in \mathcal{W}_\beta(\alpha,\gamma)$, then $F = V_\lambda(f) \in \mathcal{W}_0(1,0) \subset \mathcal{S}$.*

Further let

$$\Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}}dx \quad (\nu>0), \tag{5.11}$$

$$\Pi_{\mu,\nu}(t) = \begin{cases} \displaystyle\int_t^1 \Lambda_\nu(x)x^{1/\nu-1-1/\mu}dx & (\gamma>0\ (\mu>0,\ \nu>0)), \\ \\ \Lambda_\alpha(t) & (\gamma=0\ (\mu=0,\ \nu=\alpha>0)), \end{cases} \tag{5.12}$$

and assume that $t^{1/\nu}\Lambda_\nu(t) \to 0$, and $t^{1/\mu}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. Let

$$h(z) = \frac{z(1+\frac{\epsilon-1}{2}z)}{(1-z)^2}, \quad |\epsilon| = 1. \tag{5.13}$$

Then

$$\begin{cases} \operatorname{Re} \displaystyle\int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu - 1} \left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2} \right) dt \geq 0 \ (\gamma > 0) \\ \\ \operatorname{Re} \displaystyle\int_0^1 \Pi_{0,\alpha}(t) t^{1/\alpha - 1} \left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2} \right) dt \geq 0 \ (\gamma = 0). \end{cases} \tag{5.14}$$

if and only if $F(z) = V_\lambda(f)(z)$ *is in* $\mathcal{ST}$*. This conclusion does not hold for smaller values of* β*.*

Proof. Since the case $\gamma = 0$ ($\mu = 0$ and $\nu = \alpha$) corresponds to [63, Theorem 2.1], it is sufficient to consider only the case $\gamma > 0$.

Let

$$H(z) = (1 - \alpha + 2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z).$$

Since $\nu + \mu = \alpha - \gamma$ and $\mu\nu = \gamma$, then

$$\begin{aligned} H(z) &= (1 + \gamma - (\alpha - \gamma)) \frac{f(z)}{z} + (\alpha - \gamma - \gamma) f'(z) + \gamma z f''(z) \\ &= (1 + \mu\nu - \nu - \mu) \frac{f(z)}{z} + (\nu + \mu - \mu\nu) f'(z) + \mu\nu z f''(z) \\ &= \mu\nu \left(\frac{1}{\nu} - 1\right)\left(\frac{1}{\mu} - 1\right) z^{-1} f(z) + \mu\nu \left(\frac{1}{\nu} - 1\right) f'(z) + \nu f'(z) + \mu\nu z f''(z) \\ &= \mu\nu z^{1-1/\mu} \frac{d}{dz}\left(z^{1/\mu - 1/\nu + 1} \left(\left(\frac{1}{\nu} - 1\right) z^{1/\nu - 2} f(z) + z^{1/\nu - 1} f'(z) \right)\right) \\ &= \mu\nu z^{1-1/\mu} \frac{d}{dz}\left(z^{1/\mu - 1/\nu + 1} \frac{d}{dz}\left(z^{1/\nu - 1} f(z) \right)\right). \end{aligned}$$

With $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, it follows from (5.4) that

$$H(z) = 1 + \sum_{n=1}^{\infty} a_{n+1}(n\nu + 1)(n\mu + 1) z^n = f'(z) * \phi_{\mu,\nu}, \tag{5.15}$$

and (5.5) yields

$$f'(z) = H(z) * \psi_{\mu,\nu}(z). \tag{5.16}$$

Let g be given by

$$g(z) = \frac{H(z) - \beta}{1 - \beta}.$$

Since $\mathrm{Re}\, e^{i\phi} g(z) > 0$, from Theorem 1.23, without loss of generality, we may assume that

$$g(z) = \frac{1 + xz}{1 + yz}, \quad |x| = 1, \quad |y| = 1. \tag{5.17}$$

Now (5.16) implies that $f'(z) = [(1 - \beta)g(z) + \beta] * \psi_{\mu,\nu}$, and (5.17) readily gives

$$\frac{f(z)}{z} = \frac{1}{z} \int_0^z \left((1 - \beta) \frac{1 + xw}{1 + yw} + \beta \right) dw * \psi(z), \tag{5.18}$$

where for convenience, we write $\psi := \psi_{\mu,\nu}$.

To show that $F \in \mathcal{S}$, the Noshiro-Warschawski Theorem asserts it is sufficient to prove that $F'(\mathcal{U})$ is contained in a half-plane not containing the origin. Now

$$\begin{aligned} F'(z) &= \int_0^1 \frac{\lambda(t)}{1 - tz} dt * f'(z) = \int_0^1 \frac{\lambda(t)}{1 - tz} dt * \left((1 - \beta) \frac{1 + xz}{1 + yz} + \beta \right) * \psi(z) \\ &= \int_0^1 \lambda(t) \psi(tz) dt * \left((1 - \beta) \frac{1 + xz}{1 + yz} + \beta \right) \\ &= \left(\int_0^1 \lambda(t) \left((1 - \beta) \psi(tz) + \beta \right) dt \right) * \frac{1 + xz}{1 + yz}. \end{aligned}$$

It is known [113, p. 23] that the dual set of functions g given by (5.17) consists of analytic functions q satisfying $q(0) = 1$ and $\mathrm{Re}\, q(z) > 1/2$ in $\mathcal{U}$. Thus

$$\begin{aligned} F' \neq 0 &\Longleftrightarrow \mathrm{Re} \int_0^1 \lambda(t) \left((1 - \beta) \psi(tz) + \beta \right) dt > \frac{1}{2} \\ &\Longleftrightarrow \mathrm{Re}\, (1 - \beta) \left(\int_0^1 \lambda(t) \psi(tz) dt + \frac{\beta}{1 - \beta} - \frac{1}{2(1 - \beta)} \right) > 0. \end{aligned}$$

It follows from (5.10) and (5.6) that the latter condition is equivalent to

$$\mathrm{Re} \int_0^1 \lambda(t) \left(\left(\frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu - 1} v^{1/\mu - 1}}{(1 - uvtz)^2} du dv \right) - \left(\frac{1 + g(t)}{2} \right) \right) dt > 0. \tag{5.19}$$

Now

$$
\begin{aligned}
&\operatorname{Re}\int_0^1 \lambda(t)\left(\left(\frac{1}{\mu\nu}\int_0^1\int_0^1 \frac{u^{1/\nu-1}v^{1/\mu-1}}{(1-uvtz)^2}dudv\right)-\left(\frac{1+g(t)}{2}\right)\right)dt\\
&\geq \operatorname{Re}\int_0^1 \lambda(t)\left(\left(\frac{1}{\mu\nu}\int_0^1\int_0^1 \frac{u^{1/\nu-1}v^{1/\mu-1}}{(1+uvt)^2}dudv\right)-\left(\frac{1+g(t)}{2}\right)\right)dt. \qquad (5.20)
\end{aligned}
$$

The condition (5.8) implies that

$$
\frac{1+g(t)}{2}=\frac{1}{\mu\nu}\int_0^1\int_0^1 \frac{w^{1/\nu-1}s^{1/\mu-1}}{(1+swt)^2}dsdw.
$$

Substituting this value into (5.20) makes the integrand vanishes, and so condition (5.19) holds. Consequently $F'(\mathcal{U}) \subset \operatorname{co} g(\mathcal{U})$ with g given by (5.17) [113, p. 23], [123, Lemma 4, p. 146], which gives $\operatorname{Re} e^{i\theta}F'(z) > 0$ for $z \in \mathcal{U}$. Hence F is close-to-convex, and thus univalent.

If $f \in \mathcal{W}_\beta(\alpha,\gamma)$, a well-known result in [113, p. 94] states that

$$
F \in \mathcal{ST} \Longleftrightarrow \frac{1}{z}(F*h)(z) \neq 0 \quad (z \in \mathcal{U}),
$$

where

$$
h(z)=\frac{z(1+\frac{\epsilon-1}{2}z)}{(1-z)^2}, \quad |\epsilon|=1.
$$

Hence $F \in \mathcal{ST}$ if and only if

$$
\begin{aligned}
0 \neq \frac{1}{z}(V_\lambda(f)(z)*h(z)) &= \frac{1}{z}\left(\int_0^1 \lambda(t)\frac{f(tz)}{t}dt * h(z)\right)\\
&= \int_0^1 \frac{\lambda(t)}{1-tz}dt * \frac{f(z)}{z} * \frac{h(z)}{z}.
\end{aligned}
$$

From (5.18), it follows that

$$
\begin{aligned}
0 &\neq \int_0^1 \frac{\lambda(t)}{1-tz}dt * \left(\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)dw * \psi(z)\right) * \frac{h(z)}{z} \\
&= \int_0^1 \frac{\lambda(t)}{1-tz}dt * \frac{h(z)}{z} * \left(\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)dw\right) * \psi(z) \\
&= \int_0^1 \lambda(t)\frac{h(tz)}{tz}dt * (1-\beta)\left(\frac{1}{z}\int_0^z \frac{1+xw}{1+yw}dw + \frac{\beta}{1-\beta}\right) * \psi(z) \\
&= (1-\beta)\left(\int_0^1 \lambda(t)\frac{h(tz)}{tz}dt + \frac{\beta}{1-\beta}\right) * \frac{1}{z}\int_0^z \frac{1+xw}{1+yw}dw * \psi(z).
\end{aligned}
$$

Hence

$$
0 \neq (1-\beta)\left(\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}dw\right)dt + \frac{\beta}{1-\beta}\right) * \frac{1+xz}{1+yz} * \psi(z)
$$

$$
\begin{aligned}
&\Longleftrightarrow \operatorname{Re}(1-\beta)\left(\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}dw\right)dt + \frac{\beta}{1-\beta}\right) * \psi(z) > \frac{1}{2} \\
&\Longleftrightarrow \operatorname{Re}(1-\beta)\left(\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}dw\right)dt + \frac{\beta}{1-\beta} - \frac{1}{2(1-\beta)}\right) * \psi(z) > 0 \\
&\Longleftrightarrow \operatorname{Re}\left(\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}dw\right)dt + \frac{\beta}{1-\beta} - \frac{1}{2(1-\beta)}\right) * \psi(z) > 0.
\end{aligned}
$$

Using (5.10), the latter condition is equivalent to

$$
\operatorname{Re}\left(\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}dw - \frac{1+g(t)}{2}\right)dt\right) * \psi(z) > 0.
$$

From (5.5), the above inequality is equivalent to

$$\begin{aligned}
0 &< \mathrm{Re} \int_0^1 \lambda(t) \left(\sum_{n=0}^{\infty} \frac{z^n}{(n\nu+1)(n\mu+1)} * \frac{h(tz)}{tz} - \frac{1+g(t)}{2} \right) dt \\
&= \mathrm{Re} \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{d\eta d\zeta}{1 - z\eta^{\nu}\zeta^{\mu}} * \frac{h(tz)}{tz} - \frac{1+g(t)}{2} \right) dt \\
&= \mathrm{Re} \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{h(tz\eta^{\nu}\zeta^{\mu})}{tz\eta^{\nu}\zeta^{\mu}} d\eta d\zeta - \frac{1+g(t)}{2} \right) dt,
\end{aligned}$$

which reduces to

$$\mathrm{Re} \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{1}{\mu\nu} \frac{h(tzuv)}{tzuv} u^{1/\nu-1} v^{1/\mu-1} dv du - \frac{1+g(t)}{2} \right) dt > 0.$$

A change of variable $w = tu$ leads to

$$\mathrm{Re} \int_0^1 \frac{\lambda(t)}{t^{1/\nu}} \left(\int_0^t \int_0^1 \frac{h(wzv)}{wzv} w^{1/\nu-1} v^{1/\mu-1} dv dw - \mu\nu t^{1/\nu} \frac{1+g(t)}{2} \right) dt > 0.$$

Integrating by parts with respect to t and using (5.7) gives the equivalent form

$$\mathrm{Re} \int_0^1 \Lambda_{\nu}(t) \left(\int_0^1 \frac{h(tzv)}{tzv} t^{1/\nu-1} v^{1/\mu-1} dv - t^{1/\nu-1} \int_0^1 \frac{s^{1/\mu-1}}{(1+st)^2} ds \right) dt \geq 0.$$

Making the variable change $w = vt$ and $\eta = st$ reduces the above inequality to

$$\mathrm{Re} \int_0^1 \Lambda_{\nu}(t) t^{1/\nu-1/\mu-1} \left(\int_0^t \frac{h(wz)}{wz} w^{1/\mu-1} dw - \int_0^t \frac{\eta^{1/\mu-1}}{(1+\eta)^2} d\eta \right) dt \geq 0,$$

which after integrating by parts with respect to t yields

$$\mathrm{Re} \int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu-1} \left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2} \right) dt \geq 0.$$

Thus $F \in \mathcal{ST}$ if and only if condition (5.14) holds.

To verify sharpness, let β_0 satisfy

$$\frac{\beta_0}{1-\beta_0} = -\int_0^1 \lambda(t)g(t)dt.$$

Assume that $\beta < \beta_0$ and let $f \in \mathcal{W}_\beta(\alpha,\gamma)$ be the solution of the differential equation

$$(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) = \beta + (1-\beta)\frac{1+z}{1-z}.$$

From (5.15), it follows that

$$f(z) = z + \sum_{n=1}^{\infty} \frac{2(1-\beta)}{(n\nu+1)(n\mu+1)} z^{n+1}.$$

Thus

$$G(z) = V_\lambda(f)(z) = z + \sum_{n=1}^{\infty} \frac{2(1-\beta)\tau_n}{(n\nu+1)(n\mu+1)} z^{n+1},$$

where $\tau_n = \int_0^1 \lambda(t)t^n dt$. Now (5.8) implies that

$$\frac{\beta_0}{1-\beta_0} = -\int_0^1 \lambda(t)g(t)dt = -1 - 2\sum_{n=1}^{\infty} \frac{(n+1)(-1)^n\tau_n}{(1+\mu n)(1+\nu n)}.$$

This means that

$$G'(-1) = 1 + 2(1-\beta)\sum_{n=1}^{\infty} \frac{(n+1)(-1)^n\tau_n}{(1+\mu n)(1+\nu n)} = 1 - \frac{1-\beta}{1-\beta_0} < 0.$$

Hence $G'(z) = 0$ for some $z \in \mathcal{U}$, and so G is not even locally univalent in $\mathcal{U}$. Therefore the value of β in (5.10) is sharp. ▮

Taking $\gamma = 0$ then $\mu = 0$ and $\nu = \alpha$, the particular case $\mathcal{W}_\beta(\alpha,0) = \mathcal{P}_\alpha(\beta)$ is given by Kim and Ronning [63]. Hence the following corollary can be obtained as a special case.

Corollary 5.1 [63, Theorem2.1] *Let $f \in \mathcal{W}_\beta(\alpha, 0) = \mathcal{P}_\alpha(\beta)$, $\alpha > 0$ and $\beta < 1$, with β satisfying* (5.10) *where*

$$g_\alpha(t) = \frac{2}{\alpha} t^{-1/\alpha} \int_0^t \frac{\tau^{1/\alpha - 1}}{(1+\tau)^2} d\tau - 1,$$

and assume that $t^{1/\alpha}\Lambda(t) \to 0$ as $t \to 0^+$. Let for any $\alpha > 0$

$$\Lambda_\alpha(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\alpha}} dx.$$

Then

$$\mathrm{Re} \int_0^1 \Lambda_\alpha(t) t^{1/\alpha - 1}(t) \left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2} \right) dt \geq 0,$$

where $h(z)$ as defined in (5.13)*, if and only if $F(z) = V_\lambda(f)(z)$ is in $\mathcal{ST}$. The conclusion does not hold for smaller values of β.*

Taking $\alpha = 1$ in the above result the following corollary is obtained where $\mathcal{W}_\beta(1, 0) = \mathcal{P}$ is the class considered by Fournier and Ruscheweyh [48].

Corollary 5.2 [48, Theorem2] *Let $f \in \mathcal{W}_\beta(1, 0) = \mathcal{P}_\beta$ and $\beta < 1$, with*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) \frac{1-t}{1+t} dt.$$

and

$$\Lambda(t) = \int_t^1 \frac{\lambda(s)}{s} ds,$$

satisfies $t\Lambda \to 0$ as $t \to 0^+$. Then

$$\mathrm{Re} \int_0^1 \Lambda(t) \left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2} \right) dt \geq 0,$$

where $h(z)$ as defined in (5.13) if and only if $F(z) = V_\lambda(f)(z)$ is in $\mathcal{ST}$. The conclusion does not hold for smaller values of β.

Note that $\mu + \nu = \alpha - \gamma = 1 + 2\gamma - \gamma = 1 + \gamma$ and $\mu\nu = \gamma$, hence $\mu + \nu = 1 + \mu\nu$, or $(\mu - 1)(1 - \nu) = 0$.

For $\gamma > 0$, we have either μ or ν equal to 1, choosing $\mu = 1$ gives $\nu = \gamma$ and

For $\gamma = 0$, we have either μ or ν equal to 0, choosing $\mu = 0$ gives $\nu = \alpha = 1$.

Hence the following corollary is obtained for the class $\mathcal{W}_\beta(1 + 2\gamma, \gamma) = \mathcal{R}_\gamma(\beta)$ considered by Ponnusamy and Ronning [96].

Corollary 5.3 [96, Theorem2.2] *Let $f \in \mathcal{W}_\beta(1 + 2\gamma, \gamma) = \mathcal{R}_\gamma(\beta)$, $\gamma \geq 0$ and $\beta < 1, with$*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g_\gamma(t) dt$$

where

$$g_\gamma(t) = \begin{cases} \frac{1}{\gamma}\int_0^1 s^{1/\gamma - 1} \frac{1-st}{1+st} ds & (\gamma > 0) \\ \frac{1-t}{1+t} & (\gamma = 0). \end{cases}$$

Let

$$\Lambda_\gamma(t) = \int_t^1 \frac{\lambda(s)}{s^{1/\gamma}} ds \quad (\gamma > 0)$$

and

$$\Pi_\gamma(t) = \begin{cases} \int_t^1 \Lambda_\gamma(s) s^{1/\gamma - 2} ds & (\gamma > 0) \\ \\ \Lambda_1(t) = \int_t^1 \frac{\lambda(s)}{s} ds & (\gamma = 0). \end{cases}$$

Then

$$\operatorname{Re}\int_0^1 \Pi_\gamma(t)\left(\frac{h(tz)}{tz}-\frac{1}{(1+t)^2}\right)dt \geq 0,$$

where $h(z)$ *as defined in* (5.13)*, if and only if* $F(z) = V_\lambda(f)(z)$ *is in* $\mathcal{ST}$*. The conclusion does not hold for smaller values of* β*.*

5.3 Sufficient Conditions for Starlikeness of Integral Transforms

An easier sufficient condition for starlikeness of the integral operator (5.1) is given in the following theorem.

Theorem 5.2 *Let* $\Pi_{\mu,\nu}$ *and* Λ_ν *be as given in Theorem* 5.1*. Assume that both* $\Pi_{\mu,\nu}$ *and* Λ_ν *are integrable on* $[0,1]$ *and positive on* $(0,1)$*. Assume further that* $\mu \geq 1$ *and*

$$\frac{\Pi_{\mu,\nu}(t)}{1-t^2} \quad \text{is decreasing on } (0,1). \tag{5.21}$$

If β *satisfies* (5.10)*, and* $f \in \mathcal{W}_\beta(\alpha,\gamma)$*, then* $V_\lambda(f) \in \mathcal{ST}$*.*

Proof. The function $t^{1/\mu-1}$ is decreasing on $(0,1)$ when $\mu \geq 1$. Thus the condition (5.21) along with Theorem 1.26 yield

$$\operatorname{Re}\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(\frac{h(tz)}{tz}-\frac{1}{(1+t^2)}\right)dt \geq 0.$$

The desired conclusion now follows from Theorem 5.1. ∎

Let us scrutinize Theorem 5.2 for helpful conditions to ensure starlikeness of $V_\lambda(f)$. Recall that for $\gamma > 0$,

$$\Pi_{\mu,\nu}(t) = \int_t^1 \Lambda_\nu(y)y^{1/\nu-1-1/\mu}dy \quad \text{and} \quad \Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}}dx.$$

To apply Theorem 5.2, it is sufficient to show that the function

$$p(t) = \frac{\Pi_{\mu,\nu}(t)}{1-t^2}$$

is decreasing in the interval $(0,1)$. Note that $p(t) > 0$ and

$$\frac{p'(t)}{p(t)} = -\frac{\Lambda_\nu(t)}{t^{1+1/\mu-1/\nu}\Pi_{\mu,\nu}(t)} + \frac{2t}{1-t^2}.$$

So it remains to show that $q'(t) \geq 0$ over $(0,1)$, where

$$q(t) := \Pi_{\mu,\nu}(t) - \frac{1-t^2}{2}\Lambda_\nu(t)t^{1/\nu-2-1/\mu}.$$

Since $q(1) = 0$, this will imply that $p'(t) \leq 0$, and p is decreasing on $(0,1)$. Now

$$\begin{aligned}
q'(t) &= \Pi'_{\mu,\nu}(t) - \frac{1}{2}\Big((1-t^2)\Lambda'_\nu(t)t^{1/\nu-2-1/\mu} \\
&\quad + \Lambda_\nu(t)(-2t)t^{1/\nu-2-1/\mu} + \Lambda_\nu(t)(1-t^2)\left(\frac{1}{\nu} - 2 - \frac{1}{\mu}\right)t^{1/\nu-3-1/\mu}\Big) \\
&= \frac{1-t^2}{2}t^{1/\nu-3-1/\mu}\left(\lambda(t)t^{1-1/\nu} - \left(\frac{1}{\nu} - 2 - \frac{1}{\mu}\right)\Lambda_\nu(t)\right).
\end{aligned}$$

So $q'(t) \geq 0$ is equivalent to the condition

$$\Delta(t) := -\lambda(t)t^{1-1/\nu} + \left(\frac{1}{\nu} - 2 - \frac{1}{\mu}\right)\Lambda_\nu(t) \leq 0. \tag{5.22}$$

Since $\lambda(t) \geq 0$ gives $\Lambda_\nu(t) \geq 0$ for $t \in (0,1)$, condition (5.22) holds whenever $1/\nu - 2 - 1/\mu \leq 0$, or $\nu \geq \mu/(2\mu+1)$.

These observations will be used to prove the following theorem.

Theorem 5.3 *Let λ be a non-negative real-valued integrable function on $[0,1]$. Assume that Λ_ν and $\Pi_{\mu,\nu}$ given respectively by (5.11) and (5.12) are both integrable on $[0,1]$, and positive on $(0,1)$. Under the assumptions stated in Theorem 5.1, if*

λ *satisfies*

$$\frac{t\lambda'(t)}{\lambda(t)} \leq \begin{cases} 1+\dfrac{1}{\mu} & (\mu \geq 1\ (\gamma > 0)), \\ 3-\frac{1}{\alpha} & (\gamma = 0,\ \alpha \in (0,1/3] \cup [1,\infty)), \end{cases} \tag{5.23}$$

then $F(z) = V_\lambda(f)(z) \in \mathcal{ST}$*. The conclusion does not hold for smaller values of* β*.*

Proof. Suppose $\mu \geq 1$. In view of (5.22) and Theorem 5.2, the integral transform $V_\lambda(f)(z) \in \mathcal{ST}$ for $\nu \geq \mu/(2\mu+1)$. It remains to find conditions on μ and ν in the range $0 \leq \nu < \mu/(2\mu+1)$ such that for each choice of λ, condition (5.22) is satisfied.

Now $\Delta(t)$ at $t = 1$ in (5.22) reduces to

$$\Delta(1) = -\lambda(1) + \left(\frac{1}{\nu} - 2 - \frac{1}{\mu}\right)\Lambda_\nu(1) = -\lambda(1) \leq 0.$$

Hence to prove condition (5.22), it is enough to show that Δ is an increasing function in $(0,1)$. Now

$$\begin{aligned} \Delta'(t) &= -\lambda'(t)t^{1-1/\nu} - \left(1-\frac{1}{\nu}\right)\lambda(t)t^{-1/\nu} - \left(\frac{1}{\nu} - 2 - \frac{1}{\mu}\right)\frac{\lambda(t)}{t^{1/\nu}} \\ &= -\lambda(t)t^{-1/\nu}\left(\frac{t\lambda'(t)}{\lambda(t)} - \left(1+\frac{1}{\mu}\right)\right), \end{aligned}$$

and this is non-negative when $t\lambda'(t)/\lambda(t) \leq 1 + 1/\mu$.

In the case $\gamma = 0$, then $\mu = 0$, $\nu = \alpha > 0$. Let

$$k(t) := \Lambda_\alpha(t)t^{1/\alpha - 1}, \quad \text{where } \Lambda_\alpha(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\alpha}}dx.$$

To apply Theorem 1 in [48] along with Theorem 5.1 the function $p(t) = k(t)/(1-t^2)$

must be shown to be decreasing on the interval $(0,1)$. This will hold provided

$$q(t) := k(t) + \frac{1-t^2}{2} t^{-1} k'(t) \leq 0.$$

Since $q(1) = 0$, this will certainly hold if q is increasing on $(0,1)$. Now

$$q'(t) = \frac{(1-t^2)}{2} t^{-2} \left(tk''(t) - k'(t)\right),$$

and

$$\begin{aligned} tk''(t) - k'(t) = {} & \Lambda''_\alpha(t) t^{1/\alpha} + 2\left(\frac{1}{\alpha} - 1\right) \Lambda'_\alpha(t) t^{1/\alpha - 1} + \\ & \left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} - 2\right) \Lambda_\alpha(t) t^{1/\alpha - 2} \\ & - \Lambda'_\alpha(t) t^{1/\alpha - 1} - \left(\frac{1}{\alpha} - 1\right) \Lambda_\alpha(t) t^{1/\alpha - 2} \\ = {} & t^{1/\alpha - 2}\left(\Lambda''_\alpha(t) t^2 + \Lambda'_\alpha(t) t \left(\frac{2}{\alpha} - 3\right) + \left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} - 3\right) \Lambda_\alpha(t)\right). \end{aligned}$$

Thus $tk''(t) - k'(t)$ is non-negative if

$$\Lambda''_\alpha(t) t^2 + \Lambda'_\alpha(t) t \left(\frac{2}{\alpha} - 3\right) + \left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} - 3\right) \Lambda_\alpha(t) \geq 0.$$

The latter condition is equivalent to

$$-\lambda'(t) t^{2 - 1/\alpha} + \lambda(t) t^{1 - 1/\alpha} \left(3 - \frac{1}{\alpha}\right) + \left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} - 3\right) \Lambda_\alpha(t) \geq 0. \qquad (5.24)$$

Since $\Lambda_\alpha(t) \geq 0$ and $(1/\alpha - 1)(1/\alpha - 3) \geq 0$ for $\alpha \in (0, 1/3] \cup [1, \infty)$, then $q'(t) \geq 0$ is equivalent to

$$-\lambda'(t) t^{2 - 1/\alpha} + \lambda(t) t^{1 - 1/\alpha} \left(3 - \frac{1}{\alpha}\right) \geq 0 \Longleftrightarrow \frac{t\lambda'(t)}{\lambda(t)} \leq 3 - \frac{1}{\alpha}.$$

Thus (5.22) is satisfied and the proof is complete. ∎

Remark 5.1 *For $\mu < 1$, the conditions obtained will generally be complicated, and for $\mu \geq 1$, the conditions coincide with those given in* [96].

Taking $\alpha = 1 + 2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 5.3 the following corollary can be obtained.

Corollary 5.4 [14, Corollary 3.1] [96, Theorem 3.1] *Let λ be a nonnegative real-valued integrable function on* $[0, 1]$. *Let $f \in \mathcal{W}_\beta(1 + 2\gamma, \gamma)$, $\gamma > 0$ $\beta < 1$ with*

$$\frac{\beta}{1 - \beta} = -\int_0^1 \lambda(t) g_\gamma(t) dt,$$

where g_γ is defined by

$$g_\gamma(t) = \frac{1}{\gamma} \int_0^1 s^{1/\gamma - 1} \frac{1 - st}{1 + st} ds.$$

Assume further that $\Pi_{\gamma,1}$ and Λ_γ are integrable on $[0, 1]$ *and positive on* $(0, 1)$. *If λ satisfies*

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2,$$

then $F(z) = V_\lambda(f)(z) \in \mathcal{ST}$. The conclusion does not hold for smaller values of β.

5.4 Applications to Certain Integral Transforms

In this section, various well-known integral operators are considered, and conditions for starlikeness for $f \in \mathcal{W}_\beta(\alpha, \gamma)$ under these integral operators are obtained. First let λ be defined by

$$\lambda(t) = (1 + c)t^c, \quad c > -1.$$

Then the integral transform

$$F_c(z) = V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)dt, \quad c > -1, \tag{5.25}$$

is the Bernardi integral operator. The classical Alexander and Libera transforms are special cases of (5.25) with $c = 0$ and $c = 1$ respectively. For this special case of λ, the following result holds.

Theorem 5.4 *Let $c > -1$, and $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -(c+1)\int_0^1 t^c g(t)dt,$$

where g is given by (5.8). If $f \in \mathcal{W}_\beta(\alpha,\gamma)$, then the function

$$V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)dt$$

belongs to $\mathcal{ST}$ if

$$c \leq \begin{cases} 1 + \dfrac{1}{\mu} & (\mu \geq 1 \ (\gamma > 0)), \\ 3 - \frac{1}{\alpha} & (\gamma = 0, \ \alpha \in (0, 1/3] \cup [1, \infty)). \end{cases}$$

The value of β is sharp.

Proof. With $\lambda(t) = (1+c)t^c$, then

$$\frac{t\lambda'(t)}{\lambda(t)} = t\frac{c(1+c)t^{c-1}}{(1+c)t^c} = c,$$

and the result now follows from Theorem 5.3. ∎

Taking $\gamma = 0, \ \alpha > 0$ in Theorem 5.4 leads to the following corollary:

Corollary 5.5 *Let* $-1 < c \leq 3 - 1/\alpha$*,* $\alpha \in (0, 1/3] \cup [1, \infty)$*, and* $\beta < 1$ *satisfy*

$$\frac{\beta}{1-\beta} = -(c+1)\int_0^1 t^c g_\alpha(t)dt,$$

where g_α *is given by* (5.9)*. If* $f \in \mathcal{W}_\beta(\alpha, 0) = \mathcal{P}_\alpha(\beta)$*, then the function*

$$V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)dt$$

belongs to $\mathcal{ST}$*. The value of* β *is sharp.*

Taking $\alpha = 1$ in Corollary 5.5 the following result can be obtained.

Corollary 5.6 [48, Corollary 1] *Let* $-1 < c \leq 2$ *and* $\beta < 1$ *be given by*

$$\frac{\beta}{1-\beta} = -(c+1)\int_0^1 t^c \frac{1-t}{1+t} dt,$$

Then for $f \in \mathcal{W}_\beta(1, 0) = \mathcal{P}_\beta$ *the function*

$$V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)dt$$

belongs to $\mathcal{ST}$*. The value of* β *is sharp.*

Taking $\alpha = 1 + 2\gamma, \quad \gamma > 0$ and $\mu = 1$ in Theorem 5.4 the following result is obtained:

Corollary 5.7 [96, Corollary 3.2] *Let* $-1 < c \leq 2$*,* $\gamma > 0$ *and* $\beta < 1$ *given by*

$$\frac{\beta}{1-\beta} = -(c+1)\int_0^1 t^c g_\gamma(t)dt,$$

where

$$g_\gamma(t) = \frac{1}{\gamma}\int_0^1 s^{1/\gamma - 1}\frac{1-st}{1+st}ds.$$

Then for $f \in \mathcal{W}_\beta(1+2\gamma,\gamma) = \mathcal{P}_\gamma(\beta)$, *the function*

$$V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)dt$$

belongs to $\mathcal{ST}$. *The value of* β *is sharp.*

The case $c = 0$ in Theorem 5.4 yields the following interesting result, which we state as a theorem.

Theorem 5.5 *Let* $\alpha \geq \gamma > 0$, *or* $\gamma = 0$, $\alpha \geq 1/3$. *If* $F \in \mathcal{A}$ *satisfies*

$$\operatorname{Re}\left(F'(z) + \alpha z F''(z) + \gamma z^2 F'''(z)\right) > \beta$$

in $\mathcal{U}$, *and* $\beta < 1$ *satisfies*

$$\frac{\beta}{1-\beta} = -\int_0^1 g(t)dt,$$

where g *is given by* (5.8), *then* F *is starlike. The value of* β *is sharp.*

Proof. It is evident that the function $f = zF'$ belongs to the class

$$\mathcal{W}_{\beta,0}(\alpha,\gamma) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) \right.\right.$$
$$\left.\left. + \gamma z f''(z)\right) > \beta,\ z \in \mathcal{U} \right\}.$$

Thus

$$F(z) = \int_0^1 \frac{f(tz)}{t}dt,$$

and the result follows from Theorem 5.4 with $c = 0$ for the ranges $\alpha \geq \gamma > 0$, or $\gamma = 0$, $\alpha \geq 1$. Simple computations show that in fact (5.24) is satisfied in the larger range $\gamma = 0$, $\alpha \geq 1/3$. It is also evident from the proof of sharpness in Theorem 5.1 that indeed the extremal function in $\mathcal{W}_\beta(\alpha,\gamma)$ also belongs to the class $\mathcal{W}_{\beta,0}(\alpha,\gamma)$. ∎

Remark 5.2 *We list two interesting special cases.*

1. *If* $\gamma = 0$, $\alpha \geq 1/3$, *and* $\beta = \kappa/(1+\kappa)$, *where* (5.8) *yields*

$$\kappa = -\int_0^1 g(t)dt = -1 - 2\sum_{n=1}^{\infty}(-1)^n \frac{1}{1+n\alpha} = -\frac{1}{\alpha}\int_0^1 t^{1/\alpha-1}\frac{1-t}{1+t}dt,$$

then

$$\mathrm{Re}\,\big(f'(z) + \alpha z f''(z)\big) > \beta \Rightarrow f \in \mathcal{ST}.$$

This reduces to a result of Fournier and Ruscheweyh [48]*. In particular, if* $\beta = (1-2\ln 2)/(2(1-\ln 2)) = -0.629445$, *then*

$$\mathrm{Re}\,\big(f'(z) + z f''(z)\big) > \beta \Rightarrow f \in \mathcal{ST}.$$

2. *If* $\gamma = 1$, $\alpha = 3$, *then* $\mu = 1 = \nu$. *In this case,* (5.8) *yields* $\beta = (6-\pi^2)/(12-\pi^2) = -1.816378$. *Thus*

$$\mathrm{Re}\,\Big(f'(z) + 3zf''(z) + z^2 f'''(z)\Big) > \beta \Rightarrow f \in \mathcal{ST}.$$

This sharp estimate of β *improves a result of Ali* et al. [7].

Theorem 5.6 *Let* $b > -1$, $a > -1$, *and* $\alpha > 0$. *Let* $\beta < 1$ *satisfy*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)g(t)dt,$$

where g *is given by* (5.8) *and*

$$\lambda(t) = \begin{cases} (a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a} & (b \neq a), \\ (a+1)^2 t^a \log(1/t) & (b = a). \end{cases}$$

If $f \in \mathcal{W}_\beta(\alpha,\gamma)$, *then*

$$\mathcal{G}_f(a,b;z) = \begin{cases} \frac{(a+1)(b+1)}{b-a} \int_0^1 t^{a-1}(1-t^{b-a}) f(tz)dt & (b \neq a), \\ \\ (a+1)^2 \int_0^1 t^{a-1} \log(1/t) f(tz)dt & (b = a), \end{cases}$$

belongs to $\mathcal{ST}$ *if*

$$a \leq \begin{cases} 1 + \dfrac{1}{\mu} & (\gamma > 0\ (\mu \geq 1)) \\ 3 - \frac{1}{\alpha} & (\gamma = 0,\ \alpha \in (0, 1/3] \cup [1, \infty)). \end{cases} \tag{5.26}$$

The value of β *is sharp.*

Proof. It is easily seen that $\int_0^1 \lambda(t)dt = 1$. There are two cases to consider. When $b \neq a$, then

$$\frac{t\lambda'(t)}{\lambda(t)} = a - \frac{(b-a)t^{b-a}}{1-t^{b-a}}.$$

The function λ satisfies (5.23) if

$$a - \frac{(b-a)t^{b-a}}{1-t^{b-a}} \leq \begin{cases} 1 + \dfrac{1}{\mu} & (\gamma > 0), \\ 3 - \frac{1}{\alpha} & (\gamma = 0,\ \alpha \in (0, 1/3] \cup [1, \infty)). \end{cases} \tag{5.27}$$

Since $t \in (0,1)$, the condition $b > a$ implies $(b-a)t^{b-a}/(1-t^{b-a}) > 0$, and so inequality (5.27) holds true whenever a satisfies (5.26). When $b < a$, then $(a-b)/(t^{a-b}-1) < b-a$, and hence $a-(b-a)t^{b-a}/(1-t^{b-a}) < b < a$, and thus inequality (5.27) holds true whenever a satisfies (5.26).

For the case $b = a$, it is seen that

$$\frac{t\lambda'(t)}{\lambda(t)} = a - \frac{1}{\log(1/t)}.$$

Since $t < 1$ implies $1/\log(1/t) \geq 0$, condition (5.23) is satisfied whenever a satisfies (5.26). This completes the proof. ∎

Taking $\gamma = 0,\ \alpha > 0$ in Theorem 5.6 the following corollary can be obtained.

Corollary 5.8 *Let $-1 < a \leq 3 - 1/\alpha,\ \alpha \in (0, 1/3] \bigcup [1, \infty)$ and $b > -1$. If $\beta < 1$ given by*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g_\alpha(t) dt.$$

where λ is defined by

$$\lambda(t) = \begin{cases} (a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a} & (b \neq a,\ b > -1,\ a > -1), \\ (a+1)^2 t^a \log(1/t) & (b = a,\ a > -1), \end{cases}$$

and

$$g_\alpha(t) = \frac{2}{\alpha} t^{-1/\alpha} \int_0^t \frac{\tau^{1/\alpha - 1}}{(1+\tau)^2} d\tau - 1.$$

Then for $f \in \mathcal{W}_\beta(\alpha, 0)$, the function $\mathcal{G}_f(a, b; z)$ defined by

$$\mathcal{G}_f(a,b;z) = \begin{cases} \frac{(a+1)(b+1)}{b-a} \int_0^1 t^{a-1}(1 - t^{b-a}) f(tz) dt & (b \neq a,\ b > -1,\ a > -1), \\ \\ (a+1)^2 \int_0^1 t^{a-1} \log(1/t) f(tz) dt & (b = a,\ a > -1). \end{cases}$$

belongs to $\mathcal{ST}$. The value of β is sharp.

Remark 5.3 *Corollary 5.8 similar to Theorem 2.4 (i) and (ii) obtained in* [13] *for the case $\alpha \in [1/2, 1]$. The condition $b > a$ there resulted in $a \in (-1, 1/\alpha - 1]$. When $\alpha = 1$, the range of a obtained in* [13] *lies in the interval* $(-1, 0]$, *whereas the range of a obtained in Corollary 5.8 lies in $(-1, 2]$, and with the condition $b > a$ removed.*

Taking $\alpha = 1$ in Corollary 5.8 following result can be deduced.

Corollary 5.9 *Let $-1 < a \leq 2$ and $b > -1$. If $\beta < 1$ given by*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)\frac{1-t}{1+t}dt.$$

where λ is defined by

$$\lambda(t) = \begin{cases} (a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a} & (b \neq a,\ b > -1,\ a > -1), \\ \\ (a+1)^2 t^a \log(1/t) & (b = a,\ a > -1), \end{cases}$$

then for $f \in \mathcal{W}_\beta(1, 0)$, the function $\mathcal{G}_f(a, b; z)$ defined by

$$\mathcal{G}_f(a, b; z) = \begin{cases} \frac{(a+1)(b+1)}{b-a} \int_0^1 t^{a-1}(1 - t^{b-a})f(tz)dt & (b \neq a,\ b > -1,\ a > -1), \\ \\ (a+1)^2 \int_0^1 t^{a-1} \log(1/t) f(tz)dt & (b = a,\ a > -1), \end{cases}$$

belongs to $\mathcal{ST}$. The value of β is sharp.

Remark 5.4 *Corollary 5.9 improves Corollary* 3.13 *(i) obtained in* [15] *and Corollary* 3.1 *in* [95]. *Indeed, there the conditions on a and b were $b > a > -1$, whereas in the present situation, it is only require that $b > -1$, $a > -1$.*

Taking $\alpha = 1 + 2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 5.6 we got the following corollary:

Corollary 5.10 *Let $-1 < a < 2,\ \gamma > 0$and $b > -1$. Suppose that $g_\gamma(t)$ defined as*

$$g_\gamma(t) = \frac{1}{\gamma}\int_0^1 s^{1/\gamma - 1}\frac{1-st}{1+st}ds.$$

If $\beta < 1$ given by

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g_\gamma(t)dt.$$

where λ is defined by

$$\lambda(t) = \begin{cases} (a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a} & (b \neq a,\ b > -1,\ a > -1), \\ \\ (a+1)^2 t^a \log(1/t) & (b = a,\ a > -1), \end{cases}$$

then for $f \in \mathcal{W}_\beta(1+2\gamma,\gamma)$, the function $\mathcal{G}_f(a,b;z)$ defined by

$$\mathcal{G}_f(a,b;z) = \begin{cases} \frac{(a+1)(b+1)}{b-a}\int_0^1 t^{a-1}(1-t^{b-a})f(tz)dt & (b \neq a,\ b > -1,\ a > -1), \\ \\ (a+1)^2\int_0^1 t^{a-1}\log(1/t)f(tz)dt & (b = a,\ a > -1). \end{cases}$$

belongs to $\mathcal{ST}$. The value of β is sharp.

Remark 5.5 *Corollary 5.10 improves Theorem 4.1 in [14] in the sense that the condition $b > a > -1$ is now replaced by $b > -1,\ a > -1$.*

For another choice of λ, let it now be given by

$$\lambda(t) = \frac{(1+a)^p}{\Gamma(p)} t^a (\log(1/t))^{p-1} \quad (a > -1,\ p \geq 0).$$

The integral transform V_λ in this case takes the form

$$V_\lambda(f)(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 (\log\left(\frac{1}{t}\right))^{p-1} t^{a-1} f(tz)dt \quad (a > -1,\ p \geq 0).$$

This is the Komatu operator, which reduces to the Bernardi integral operator if $p = 1$. For this λ, the following result holds.

Theorem 5.7 *Let $-1 < a$, $\alpha > 0$, $p \geq 1$, and $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a (\log(1/t))^{p-1} g(t)dt,$$

where g is given by (5.8). If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the function

$$\Phi_p(a;z) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 (\log(1/t))^{p-1} t^{a-1} f(tz)dt$$

belongs to $\mathcal{ST}$ if

$$a \leq \begin{cases} 1 + \dfrac{1}{\mu} & (\gamma > 0\ (\mu \geq 1)), \\ 3 - \frac{1}{\alpha} & (\gamma = 0,\ \alpha \in (0, 1/3] \cup [1, \infty)). \end{cases} \tag{5.28}$$

The value of β is sharp.

Proof. It is evident that

$$\frac{t\lambda'(t)}{\lambda(t)} = a - \frac{(p-1)}{\log(1/t)}.$$

Since $\log(1/t) > 0$ for $t \in (0,1)$, and $p \geq 1$, condition (5.23) is satisfied whenever a satisfies (5.28). ∎

Taking $\gamma = 0,\ \alpha > 0$ in Theorem 5.7 the following corollary can be obtained.

Corollary 5.11 *Let* $-1 < a \leq 3 - 1/\alpha$, $\alpha \in (0, 1/3] \bigcup [1, \infty)$ *and* $p \geq 1$. *If* $\beta < 1$ *given by*

$$\frac{\beta}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a (\log(1/t))^{p-1} g_\alpha(t) dt,$$

where

$$g_\alpha(t) = \frac{2}{\alpha} t^{-1/\alpha} \int_0^t \frac{\tau^{1/\alpha - 1}}{(1+\tau)^2} d\tau - 1.$$

Then for $f \in \mathcal{W}_\beta(0, \gamma)$, *the function* $\Phi_p(a; z) * f(z)$ *defined by*

$$\Phi_p(a; z) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 (\log(1/t))^{p-1} t^{a-1} f(tz) dt$$

belongs to $\mathcal{ST}$. *The value of* β *is sharp.*

Remark 5.6 *Corollary 5.11 similar to Theorem* 2.1 *in* [13] *and Theorem* 2.3 *in* [63] *for the case* $\alpha \in [1/2, 1]$. *When* $\alpha = 1$, *the range of* a *obtained in* [13] *and* [63] *lies in the interval* $(-1, 0]$, *whereas the range of* a *obtained in Corollary 5.8 lies in* $(-1, 2]$.

Taking $\alpha = 1$ in Corollary 5.11 the following corollary can be obtained.

Corollary 5.12 [15, Corollary 3.12 (i)] *Let* $-1 < a \leq 2$, *and* $p \geq 1$. *If* $\beta < 1$ *given by*

$$\frac{\beta}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a (\log(1/t))^{p-1} \frac{1-t}{1+t} dt.$$

Then for $f \in \mathcal{W}_\beta(1, 0)$, *the function* $\Phi_p(a; z) * f(z)$ *defined by*

$$\Phi_p(a; z) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 (\log(1/t))^{p-1} t^{a-1} f(tz) dt$$

belongs to $\mathcal{ST}$. The value of β is sharp.

Taking $\alpha = 1 + 2\gamma, \gamma > 0$ and $\mu = 1$ in Theorem 5.7 the following result can be deduced.

Corollary 5.13 [14, Theorem 4.2 ($\mu = 0$)] *Let $-1 < a \leq 2,\ \gamma > 0$ and $p \geq 1$. If $\beta < 1$ given by*

$$\frac{\beta}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a (\log(1/t))^{p-1} g_\gamma(t) dt.$$

where

$$g_\gamma(t) = \frac{1}{\gamma} \int_0^1 s^{1/\gamma - 1} \frac{1-st}{1+st} ds.$$

Then for $f \in \mathcal{W}_\beta(1+2\gamma, \gamma)$, the function $\Phi_p(a; z) * f(z)$ defined by

$$\Phi_p(a; z) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 (\log(1/t))^{p-1} t^{a-1} f(tz) dt$$

belongs to $\mathcal{ST}$. The value of β is sharp.

Let Φ be defined by $\Phi(1-t) = 1 + \Sigma_{n=1}^{\infty} b_n (1-t)^n$, $b_n \geq 0$ for $n \geq 1$, and

$$\lambda(t) = K t^{b-1} (1-t)^{c-a-b} \Phi(1-t), \tag{5.29}$$

where K is a constant chosen such that $\int_0^1 \lambda(t) dt = 1$. The following result holds in this instance.

Theorem 5.8 *Let $a, b, c, \alpha > 0$, and $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -K \int_0^1 t^{b-1} (1-t)^{c-a-b} \Phi(1-t) g(t) dt,$$

where g is given by (5.8) and K is a constant such that $K \int_0^1 t^{b-1} (1-t)^{c-a-b} \Phi(1-$

$t) = 1$. *If $f \in \mathcal{W}_\beta(\alpha,\gamma)$, then the function*

$$V_\lambda(f)(z) = K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)\frac{f(tz)}{t}dt$$

belongs to $\mathcal{ST}$ provided one of the following conditions hold:

$$\begin{aligned}
&(i) \qquad c < a+b \quad \textit{and} \quad 0 < b \leq 1,\\
&(ii) \qquad c \geq a+b \quad \textit{and} \quad b \leq \begin{cases} 2+\dfrac{1}{\mu} & (\gamma > 0,\ (\mu \geq 1)) \\ 4-\frac{1}{\alpha} & (\gamma = 0,\ \alpha \in (1/4, 1/3] \cup [1,\infty)). \end{cases}
\end{aligned} \tag{5.30}$$

The value of β is sharp.

Proof. For λ given by (5.29),

$$\frac{t\lambda'(t)}{\lambda(t)} = (b-1) - \frac{(c-a-b)t}{1-t} - \frac{t\Phi'(1-t)}{\Phi(1-t)}.$$

For the case $c < a+b$, computing $(b-1)-((c-a-b)t)/(1-t))$ and using the fact that $t\Phi'(1-t)/\Phi(1-t) > 0$ implies condition (5.23) is satisfied whenever $0 < b \leq 1$. For $c \geq a+b$, a similar computation shows that the condition (5.23) is satisfied whenever b satisfies (5.30). Now the result follows by applying Theorem 5.3 for this special λ. ∎

Taking $\gamma = 0,\ \alpha > 0$ in Theorem 5.8 leads to the following corollary:

Corollary 5.14 *Let $a, b, c, \alpha > 0$, and $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)g_\alpha(t)dt,$$

where g_α is given by (5.9), and K is a constant such that $K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-$

$t) = 1$. If $f \in \mathcal{W}_\beta(\alpha, 0) = \mathcal{P}_\alpha(\beta)$, then the function

$$V_\lambda(f)(z) = K \int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)\frac{f(tz)}{t}dt$$

belongs to $\mathcal{ST}$ whenever a, b, c are related by either (i) $c \leq a+b$ and $0 < b \leq 1$, or (ii) $c \geq a+b$ and $b \leq 4 - 1/\alpha$, $\alpha \in (1/4, 1/3] \cup [1, \infty)$, for all $t \in (0,1)$. The value of β is sharp.

Remark 5.7 *For $\alpha = 1$, Corollary 5.14 improves Theorem 3.8 (i) in [15] in the sense that the result now holds not only for $c \geq a+b$ and $0 < b \leq 3$, but also to the range $c \leq a+b$, $0 < b \leq 1$.*

Taking $\alpha = 1 + 2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 5.8 reduces to the following corollary:

Corollary 5.15 *Let $a, b, c > 0$, and let $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -K \int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)g_\gamma(t)dt,$$

where g_γ is given by (5.9), and K is a constant such that $K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t) = 1$. If $f \in \mathcal{W}_\beta(1+2\gamma, \gamma)$, then the function

$$V_\lambda(f)(z) = K \int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)\frac{f(tz)}{t}dt$$

belongs to $\mathcal{ST}$ whenever a, b, c are related by either (i) $c \leq a+b$ and $0 < b \leq 1$, or (ii) $c \geq a+b$ and $0 < b \leq 3$, for all $t \in (0,1)$ and $\gamma > 0$. The value of β is sharp.

Remark 5.8 *Choosing $\Phi(1-t) = F(c-a, 1-a, c-a-b+1; 1-t)$ in Theorem 5.8(ii) gives*

$$K = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}$$

whenever $c - a - b + 1 > 0$*. In this case, the function* $V_\lambda(f)(z)$ *reduces to the Hohlov operator given by*

$$\begin{aligned} V_\lambda(f)(z) &= H_{a,b,c}(f)(z) = zF(a,b;c;z) * f(z) \\ &= K\int_0^1 t^{b-1}(1-t)^{c-a-b}F\left(c-a,1-a,c-a-b+1;1-t\right)\frac{f(tz)}{t}dt, \end{aligned}$$

where $a > 0$*,* $b > 0$*,* $c - a - b + 1 > 0$*. This case of Corollary* 5.14 *was treated in* [13, Theorem 2.2(i), ($\mu = 0$)] *and* [63, Theorem 2.4]*, but the range of* b *provided by Corollary* 5.14(ii) *yields* $0 < b \leq 3$*, which is larger than the range given in* [13] *and* [63] *of* $0 < b \leq 1$*.*

CHAPTER 6

MULTIVALENT STARLIKE AND CONVEX FUNCTIONS ASSOCIATED WITH A PARABOLIC REGION

6.1 Motivation and Preliminaries

For $\delta \geq 0$, Rusheweyh [112] introduced the *δ-neighborhood* $N_\delta(f)$ of a function (cf. Section 1.7, p. 31) $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathcal{U})$ to be the set consisting of all functions $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ satisfying

$$\sum_{k=2}^{\infty} k|a_k - b_k| \leq \delta.$$

He proved among other results that $N_{1/4}(f) \subset \mathcal{ST}$ for $f \in \mathcal{CV}$. In geometric function theory, several authors have investigated the neighborhood characterizations for functions belonging to certain subclasses. For example, Padmanabhan [84] determined the δ-neighborhood of functions in the class $\mathcal{UCV}$. This class consists of uniformly convex functions f satisfying

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathcal{U}),$$

while the analogous class of parabolic starlike functions $\mathcal{PST}$ consists of functions f satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \mathcal{U}).$$

It is clear that $f \in \mathcal{UCV}$ if and only if $zf' \in \mathcal{PST}$ (see pp. 8–9). For a function F_ϵ defined by

$$F_\epsilon = \frac{f(z) + \epsilon z}{1 + \epsilon}, \quad f \in \mathcal{A}$$

Padmanabhan [84] proved that $N_{\delta/2}(f) \subset \mathcal{PST}$ whenever $F_\epsilon \in \mathcal{PST}$ and $|\epsilon| < \delta < 1$. Furthermore, he showed that for $f \in \mathcal{UCV}$, the 1/8-neighborhood $N_{1/8}(f) \subset$

$\mathcal{PST}$.

The δ-neighborhood concept is extended to p-valent functions. For $\delta \geq 0$, and $f \in \mathcal{A}_p$, the *δ-neighborhood* $N_{\delta,p}(f)$ is defined to be the set consisting of all functions $g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$ satisfying

$$\sum_{k=1}^{\infty} \frac{(p+k)}{p} \left| a_{p+k} - b_{p+k} \right| \leq \delta.$$

This chapter investigates the δ-neighborhood of p-valent functions belonging to general subclasses of p-valent parabolic starlike and p-valent parabolic convex functions. For $\alpha > 0$, and $0 < \lambda \leq 1$, the subclass of p-valent parabolic starlike functions of order α and type λ, $\mathcal{SP}_p(\alpha, \lambda)$, consists of functions $f \in \mathcal{A}_p$ satisfying

$$\operatorname{Re}\left(\frac{1}{p}\frac{zf'(z)}{(1-\lambda)z^p + \lambda f(z)}\right) + \alpha > \left|\frac{1}{p}\frac{zf'(z)}{(1-\lambda)z^p + \lambda f(z)} - \alpha\right| \quad (z \in \mathcal{U}). \tag{6.1}$$

By writing $f_\lambda(z) := (1-\lambda)z^p + \lambda f(z)$, the inequality (6.1) can be written as

$$\operatorname{Re}\left(\frac{1}{p}\frac{zf'(z)}{f_\lambda(z)}\right) + \alpha > \left|\frac{1}{p}\frac{zf'(z)}{f_\lambda(z)} - \alpha\right|. \tag{6.2}$$

The corresponding subclass of p-valent parabolic convex functions of order α and type λ, $\mathcal{CP}_p(\alpha, \lambda)$, consists of functions $f \in \mathcal{A}_p$ satisfying

$$\operatorname{Re}\left(\frac{1}{p}\frac{\left(zf'(z)\right)'}{f'_\lambda(z)}\right) + \alpha > \left|\frac{\left(zf'(z)\right)'}{f'_\lambda(z)} - \alpha\right| \quad (z \in \mathcal{U}).$$

Observe that $f \in \mathcal{CP}_p(\alpha, \lambda)$ if and only if $(1/p)(zf') \in \mathcal{SP}_p(\alpha, \lambda)$.

The geometric interpretation of the relation (6.2) is that the class $\mathcal{SP}_p(\alpha, \lambda)$ consists of functions f for which $(1/p)\left(zf'/f_\lambda\right)$ lies in the parabolic region Ω given by

$$\Omega := \{w : |w-\alpha| < \operatorname{Re} w + \alpha\} = \{w = u + iv : v^2 < 4\alpha u\}. \tag{6.3}$$

When $p = 1$, the classes $\mathcal{SP}_p(\alpha, \lambda)$ and $\mathcal{CP}_p(\alpha, \lambda)$ reduce respectively to the classes $\mathcal{SP}(\alpha, \lambda) := \mathcal{SP}_1(\alpha, \lambda)$ and $\mathcal{CP}_1(\alpha, \lambda) := \mathcal{CP}(\alpha, \lambda)$ introduced recently by Ali et al. in [9]. The authors in [9] investigated the δ-neighborhood for functions in $\mathcal{CP}(\alpha, \lambda)$. In addition, if $\lambda = 1$, these classes reduce respectively to the classes $\mathcal{SP}(\alpha) := \mathcal{SP}(\alpha, 1)$ and $\mathcal{CP}(\alpha) := \mathcal{CP}(\alpha, 1)$ investigated in [125] and [138].

This chapter studies neighborhood problems for the two classes $\mathcal{SP}_p(\alpha, \lambda)$ and $\mathcal{CP}_p(\alpha, \lambda)$. Motivated by Ruscheweyh [112], and using the notion of convolution, a new inclusion criterion for the class $\mathcal{SP}_p(\alpha, \lambda)$ will be derived. It will be shown that the classes $\mathcal{SP}_p(\alpha, \lambda)$ and $\mathcal{CP}(\alpha, \lambda)$ are closed under convolution with prestarlike functions in $\mathcal{U}$. A δ-neighborhood description will also be obtained for functions belonging to the class $\mathcal{CP}_p(\alpha, \lambda)$. In particular, it is shown that, under certain conditions, the δ-neighborhood of a p-valent parabolic convex function consists of p-valent parabolic starlike functions for an appropriate positive δ.

6.2 Multivalent Starlike and Convex Functions Associated with a Parabolic Region

For a fixed $\alpha > 0,\ 0 < \lambda \leq 1$ and $t \geq 0$, let $\mathcal{HSP}_p(\alpha, \lambda)$ be the class consisting of functions $\mathcal{H}_{p,t,\lambda}$ of the form

$$\begin{aligned}\mathcal{H}_{p,t,\lambda}(z) &:= \frac{1}{1-(t \pm 2\sqrt{\alpha t} i)} \left(\frac{z^p(1-z+\frac{1}{p}z)}{(1-z)^2} - (t \pm 2\sqrt{\alpha t} i) \frac{z^p(1-(1-\lambda)z)}{(1-z)} \right) \\ &= \frac{z^p}{1-T} \left(\frac{1}{1-z} + \frac{1}{p} \frac{z}{(1-z)^2} - T \left(1 + \frac{\lambda z}{1-z} \right) \right),\end{aligned}$$

where $T = t \pm 2\sqrt{\alpha t}i$. It is easily seen that

$$\mathcal{H}_{p,t,\lambda}(z) = \frac{z^p}{1-T}\left(1 + \sum_{k=1}^{\infty} z^k \left(1 + \frac{k}{p} - \lambda T\right) - T\right) = z^p + \sum_{k=1}^{\infty} \frac{\left(\frac{p+k}{p} - \lambda T\right)}{1-T} z^{p+k}. \tag{6.4}$$

Thus the function $\mathcal{H}_{p,t,\lambda}(z)$ belongs to $\mathcal{A}_p$. Also,

$$\mathcal{H}_{p,t,\lambda}(z) = \frac{1}{1-T}\left(F_1(z) - TF_2(z)\right), \tag{6.5}$$

where

$$F_1(z) = \frac{z^p(1 - z + \frac{1}{p}z)}{(1-z)^2} = z^p + \sum_{k=1}^{\infty} \frac{p+k}{p} z^{p+k},$$

and

$$F_2(z) = \frac{z^p(1-(1-\lambda)z)}{(1-z)} = z^p + \lambda \sum_{k=1}^{\infty} z^{p+k}.$$

Thus for $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$,

$$(F_1 * f)(z) = z^p + \sum_{k=1}^{\infty} \frac{p+k}{p} a_{p+k} z^{p+k} = \frac{1}{p} z f'(z), \tag{6.6}$$

and

$$(F_2 * f)(z) = z^p + \lambda \sum_{k=1}^{\infty} a_{p+k} z^{p+k} = f_\lambda(z). \tag{6.7}$$

The following result yields a new criterion for p-valent functions f to be in the class $\mathcal{SP}_p(\alpha, \lambda)$.

Theorem 6.1 *Let $\alpha > 0$ and $0 < \lambda \leq 1$. A function $f \in \mathcal{SP}_p(\alpha, \lambda)$ if and only if*

$$\frac{1}{z^p}\left(f * \mathcal{H}_{p,t,\lambda}\right)(z) \neq 0 \qquad (z \in \mathcal{U})$$

for all $\mathcal{H}_{p,t,\lambda} \in \mathcal{HSP}_p(\alpha, \lambda)$.

Proof. Let $f \in \mathcal{SP}_p(\alpha, \lambda)$. Then the image of $\mathcal{U}$ under $w = (1/p)\left(zf'/f_\lambda\right)$ lies in the parabolic region Ω given by (6.3). Since $v^2 < 4\alpha u$, this implies that $v < \pm 2\sqrt{\alpha u} \quad (u \geq 0)$, and hence

$$w \neq t \pm 2\sqrt{\alpha t} i \quad (t \geq 0),$$

or

$$\frac{1}{p}\frac{zf'(z)}{f_\lambda(z)} \neq t \pm 2\sqrt{\alpha t} i.$$

Thus $f \in \mathcal{SP}_p(\alpha, \lambda)$ if and only if

$$\frac{\frac{1}{p}zf'(z) - Tf_\lambda(z)}{z^p(1-T)} \neq 0,$$

where $T = t \pm 2\sqrt{\alpha t} i$, or

$$\frac{1}{z^p(1-T)}\left((F_1 * f)(z) - T(F_2 * f)(z)\right) \neq 0.$$

Using (6.5), (6.6) and (6.7), it follows that

$$f \in \mathcal{SP}_p(\alpha, \lambda) \text{ if and only if } \frac{1}{z^p}\left(f(z) * \frac{F_1(z) - TF_2(z)}{1-T}\right) \neq 0,$$

or equivalently,

$$\frac{1}{z^p}\left(f * \mathcal{H}_{p,t,\lambda}\right)(z) \neq 0 \quad (z \in \mathcal{U},\ t \geq 0),$$

for all $\mathcal{H}_{p,t,\lambda} \in \mathcal{HSP}_p(\alpha, \lambda)$. ∎

In order to establish the δ-neighborhood of functions belonging to the class $\mathcal{SP}_p(\alpha,\lambda)$, the following two lemmas are needed.

Lemma 6.1 *Let $\alpha \geq 0$ and $0 < \lambda \leq 1$. If*

$$\mathcal{H}_{p,t,\lambda}(z) := z^p + \sum_{k=1}^{\infty} C_{p+k,\lambda}(t) z^{p+k} \in \mathcal{HSP}_p(\alpha,\lambda), \tag{6.8}$$

then

$$|C_{p+k,\lambda}(t)| \leq \begin{cases} \frac{k+p}{2p\sqrt{\alpha(1-\alpha)}} & (0 < \alpha \leq \frac{1}{2}), \\ \frac{1}{p}(k+p) & (\alpha \geq \frac{1}{2}), \end{cases}$$

for all $t \geq 0$.

Proof. Comparing (6.4) and (6.8)

$$C_{p+k,\lambda}(t) = \frac{\frac{p+k}{p} - \lambda T}{1-T},$$

where $T = t \pm 2\sqrt{\alpha t} i$. Thus for $t \geq 0$ and $0 < \lambda \leq 1$,

$$\begin{aligned} \left|C_{p+k,\lambda}(t)\right|^2 &= \frac{\left(\frac{1}{p}(k+p) - \lambda t\right)^2 + 4\alpha t \lambda^2}{(1-t)^2 + 4\alpha t} \\ &= \lambda^2 + \frac{\left(\frac{1}{p}(k+p) + \lambda - 2\lambda t\right)\left(\frac{1}{p}(k+p) - \lambda\right)}{(1-t)^2 + 4\alpha t} \\ &\leq \lambda^2 + \frac{\left(\frac{1}{p}(k+p) + \lambda\right)\left(\frac{1}{p}(k+p) - \lambda\right)}{(1-t)^2 + 4\alpha t} \\ &\leq \lambda^2 + \frac{\left(\frac{1}{p}(k+p)\right)^2 - \lambda^2}{(1-t)^2 + 4\alpha t}. \end{aligned}$$

For any $t \geq 0$,

$$(1-t)^2 + 4\alpha t \geq \begin{cases} 4\alpha(1-\alpha) & (0 < \alpha \leq \frac{1}{2}), \\ 1 & (\alpha \geq \frac{1}{2}). \end{cases}$$

For $0 < \alpha \le \frac{1}{2}$ and $0 < \lambda \le 1$,

$$\left|C_{p+k,\lambda}(t)\right|^2 \le \lambda^2 + \frac{\left(\frac{1}{p}(k+p)\right)^2 - \lambda^2}{4\alpha(1-\alpha)}.$$

Since

$$\lambda^2 - \frac{\lambda^2}{4\alpha(1-\alpha)} = \frac{-\lambda^2(2\alpha-1)^2}{4\alpha(1-\alpha)} < 0,$$

it follows that

$$\left|C_{p+k,\lambda}(t)\right|^2 \le \frac{\left(\frac{1}{p}(k+p)\right)^2}{4\alpha(1-\alpha)}.$$

For $\alpha \ge \frac{1}{2}$ and $0 < \lambda \le 1$, evidently

$$\left|C_{p+k,\lambda}(t)\right|^2 \le \lambda^2 + \frac{\left(\frac{1}{p}(k+p)\right)^2 - \lambda^2}{1} \le \left(\frac{1}{p}(k+p)\right)^2.$$

This completes the proof. ∎

For each complex number ϵ and $f \in \mathcal{A}_p$, define the function $F_{\epsilon,p}$ by

$$F_{\epsilon,p} := \frac{f(z) + \epsilon z^p}{1+\epsilon}. \tag{6.9}$$

Lemma 6.2 *Let $\alpha > 0,\ 0 < \lambda \le 1$ and for some $\delta > 0$, let $F_{\epsilon,p}$ defined by (6.9) belong to the class $\mathcal{SP}_p(\alpha,\lambda)$ for $|\epsilon| < \delta$. Then*

$$\left|\frac{1}{z^p}\left(f * \mathcal{H}_{p,t,\lambda}\right)(z)\right| > \delta \quad (z \in \mathcal{U})$$

for every $\mathcal{H}_{p,t,\lambda} \in \mathcal{HSP}_p(\alpha,\lambda)$.

Proof. If $F_{\epsilon,p} \in \mathcal{SP}_p(\alpha, \lambda)$ for $|\epsilon| < \delta$, it follows from Theorem 6.1 that

$$\frac{1}{z^p}\left(F_{\epsilon,p} * \mathcal{H}_{p,t,\lambda}\right)(z) \neq 0 \quad (z \in \mathcal{U}),$$

which is equivalent to

$$\frac{1}{z^p}\left(\frac{\left(f * \mathcal{H}_{p,t,\lambda}\right)(z) + \epsilon z^p}{1+\epsilon}\right) \neq 0,$$

and hence

$$\frac{1}{z^p}\left(f * \mathcal{H}_{p,t,\lambda}\right)(z) \neq -\epsilon,$$

for every $|\epsilon| < \delta$. Therefore

$$\left|\frac{1}{z^p}\left(f * \mathcal{H}_{p,t,\lambda}\right)(z)\right| > \delta. \qquad \blacksquare$$

Theorem 6.2 *Let $\alpha \geq 0,\ 0 < \lambda \leq 1,\ f \in \mathcal{A}_p$ and $\delta > 0$. For a complex number ϵ with $|\epsilon| < \delta$, let $F_{\epsilon,p}$ defined by* (6.9) *belong to $\mathcal{SP}_p(\alpha, \lambda)$. Then $N_{\delta',p}(f) \subset \mathcal{SP}_p(\alpha, \lambda)$ for*

$$\delta' := \begin{cases} 2\delta\sqrt{\alpha(1-\alpha)} & (0 < \alpha \leq \frac{1}{2}), \\ \delta & (\alpha \geq \frac{1}{2}). \end{cases}$$

Proof. Let $g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \in N_{\delta',p}(f)$. For any $\mathcal{H}_{p,t,\lambda} \in \mathcal{HSP}_p(\alpha, \lambda)$

$$\begin{aligned} \left|\frac{1}{z^p}\left(g * \mathcal{H}_{p,t,\lambda}\right)(z)\right| &= \left|\frac{1}{z^p}\left((g - f + f) * \mathcal{H}_{p,t,\lambda}\right)(z)\right| \\ &\geq \left|\frac{1}{z^p}\left(f * \mathcal{H}_{p,t,\lambda}\right)(z)\right| - \left|\frac{1}{z^p}\left((g - f) * \mathcal{H}_{p,t,\lambda}\right)(z)\right|. \end{aligned}$$

Lemma 6.2 now gives

$$\begin{aligned}\left|\frac{1}{z^p}\left(g*\mathcal{H}_{p,t,\lambda}\right)(z)\right| &> \delta - \left|\frac{1}{z^p}\sum_{k=1}^{\infty}\left(b_{p+k}-a_{p+k}\right)C_{p+k}(t)z^{p+k}\right| \\ &> \delta - \sum_{k=1}^{\infty}\left|b_{p+k}-a_{p+k}\right|\left|C_{p+k}(t)\right|.\end{aligned}$$

Since $g \in N_{\delta',p}$, using Lemma 6.1, it follows that

$$\begin{aligned}\left|\frac{1}{z^p}\left(g*\mathcal{H}_{p,t,\lambda}\right)(z)\right| &> \delta - \sum_{k=1}^{\infty}\frac{(p+k)}{p}\left|b_{p+k}-a_{p+k}\right|\frac{\left|C_{p+k,\lambda}(t)\right|}{p+k}p \\ &> \begin{cases} \delta - \frac{\delta'}{2\sqrt{\alpha(1-\alpha)}} & (0<\alpha\le\frac{1}{2}), \\ \delta-\delta' & (\alpha\ge\frac{1}{2}).\end{cases}\end{aligned}$$

Hence $\left|\frac{1}{z^p}\left(g*\mathcal{H}_{p,t,\lambda}\right)(z)\right| \neq 0$ for all $\mathcal{H}_{p,t,\lambda}\in\mathcal{HSP}_p(\alpha,\lambda)$ provided that

$$\delta' := \begin{cases} 2\delta\sqrt{\alpha(1-\alpha)} & (0<\alpha\le\frac{1}{2}), \\ \delta & (\alpha\ge\frac{1}{2}).\end{cases}$$

Using Theorem 6.1, $g\in\mathcal{SP}_p(\alpha,\lambda)$ and therefore $N_{\delta',p}(f)\subset\mathcal{SP}_p(\alpha,\lambda)$. ∎

The following result shows that the class $\mathcal{SP}_p(\alpha,\lambda)$ is closed under convolution with prestarlike functions belonging to the class $\mathcal{R}_\gamma$ defined on p. 15.

Theorem 6.3 *Let $f,g\in\mathcal{A}_p$. If $f/z^{p-1}\in\mathcal{R}_\gamma,\ 0\le\gamma\le1,\ g\in\mathcal{SP}_p(\alpha,\lambda)$ and $g_\lambda/z^{p-1}\in\mathcal{ST}(\gamma)$, then $f*g\in\mathcal{SP}_p(\alpha,\lambda)$.*

Proof. It is evident that $(f(z)*g(z))_\lambda = f(z)*g_\lambda(z)$, hence

$$\frac{\frac{1}{p}z\left(f(z)*g(z)\right)'}{\left(f(z)*g(z)\right)_\lambda} = \frac{\frac{f(z)}{z^{p-1}}*\frac{g_\lambda(z)}{z^{p-1}}\cdot\frac{1}{p}\frac{zg'(z)}{g_\lambda(z)}}{\frac{f(z)}{z^{p-1}}*\frac{g_\lambda(z)}{z^{p-1}}}.$$

Taking $F=(1/p)(zg'/g_\lambda)$, and taking into account that $f/z^{p-1}\in\mathcal{R}_\gamma$ and

$g_\lambda/z^{p-1} \in \mathcal{ST}(\gamma)$, Theorem 3.3, page 50, gives

$$\frac{\frac{f(z)}{z^{p-1}} * \frac{g_\lambda(z)}{z^{p-1}}.F(z)}{\frac{f(z)}{z^{p-1}} * \frac{g_\lambda(z)}{z^{p-1}}} \subset \overline{co}F(\mathcal{U}) = \overline{co}\left(\frac{1}{p}\frac{zg'(z)}{g_\lambda(z)}\right),$$

or

$$\frac{\frac{1}{p}z\left(f(z)*g(z)\right)'}{\left(f(z)*g(z)\right)_\lambda} \subset \overline{co}\left(\frac{1}{p}\frac{zg'(z)}{g_\lambda(z)}\right).$$

Since $g \in \mathcal{SP}_p(\alpha,\lambda)$, it follows that $(1/p)\left(zg'/g_\lambda\right)$ lies in the parabolic region Ω given by (6.3), and hence so does $(1/p)\left(z(f*g)'/(f*g)_\lambda\right)$. Thus $f*g \in \mathcal{SP}_p(\alpha,\lambda)$. ∎

To obtain a similar result for the class $\mathcal{CP}_p(\alpha,\lambda)$, the following lemma will be required.

Lemma 6.3 *Let $f \in \mathcal{A}_p$ and $f \in \mathcal{ST}_p\left(1-(1-\gamma)/p\right)$. Then $f/z^{p-1} \in \mathcal{ST}(\gamma)$.*

Proof. Now $f \in \mathcal{ST}_p\left(1-(1-\gamma)/p\right)$ yields

$$\mathrm{Re}\left(\frac{1}{p}\frac{zf'(z)}{f(z)}\right) > 1 - \frac{(1-\gamma)}{p}.$$

If $G(z) = f(z)/z^{p-1}$, then

$$\mathrm{Re}\,\frac{zG'(z)}{G(z)} = \mathrm{Re}\,\frac{zf'(z)}{f(z)} - p + 1 > p - (1-\gamma) - p + 1 > \gamma,$$

and hence $G \in \mathcal{ST}(\gamma)$. ∎

Using the above lemma, the following result shows that the class $\mathcal{CP}_p(\alpha,\lambda)$ is closed under convolution with prestarlike functions.

Theorem 6.4 *Let $f, g \in \mathcal{A}_p$. If $f/z^{p-1} \in \mathcal{R}_\gamma, 0 \le \gamma \le 1, g \in \mathcal{CP}_p(\alpha,\lambda)$ and $g_\lambda \in \mathcal{CV}_p\left(1-(1-\gamma)/p\right)$, then $f*g \in \mathcal{CP}_p(\alpha,\lambda)$.*

Proof. Using Alexander's relation, $g_\lambda \in \mathcal{CV}_p\left(1-(1-\gamma)/p\right)$ if and only if $(1/p)(zg'_\lambda) \in \mathcal{ST}_p\left(1-(1-\gamma)/p\right)$, and hence Lemma 6.3 gives $(1/p)\left(zg'_\lambda/z^{p-1}\right) \in \mathcal{ST}(\gamma)$. Also

$$g \in \mathcal{CP}_p(\alpha,\lambda) \Leftrightarrow \frac{1}{p}zg'(z) \in \mathcal{SP}_p(\alpha,\lambda),$$

and since $f/z^{p-1} \in \mathcal{R}_\gamma$, Theorem 6.3 yields

$$f(z) * \frac{1}{p}zg'(z) \in \mathcal{SP}_p(\alpha,\lambda).$$

This is equivalent to

$$\frac{1}{p}z(f*g)'(z) \in \mathcal{SP}_p(\alpha,\lambda),$$

and hence $f * g \in \mathcal{CP}_p(\alpha,\lambda)$. ∎

To investigate the δ-neighborhood of functions belonging to the class $\mathcal{CP}_p(\alpha,\lambda)$, the following result will be required. First we recall the class of p-valent prestarlike functions of order β as given on p. 18. This class $\mathcal{R}_p(\beta)$ consists of functions $h \in \mathcal{A}_p$ satisfying

$$h(z) * \frac{z^p}{(1-z)^{2p(1-\beta)}} \in \mathcal{ST}_p(\beta).$$

Theorem 6.5 *Let* $0 \le \gamma \le 1,\ 0 < \lambda \le 1$ *and* $\alpha \ge 0$. *Further assume that the function* $h_p(z) = z^p(1-\rho z)/(1-z) \in \mathcal{R}_p(\beta)$ *where* $\rho = \epsilon/(1+\epsilon)$, *and* $\beta = 1-(1-\gamma)/p$. *If* $f \in \mathcal{CP}_p(\alpha,\lambda)$ *and* $f_\lambda \in \mathcal{CV}_p\left(1-(1-\gamma)/p\right)$, *then* $F_{\epsilon,p}(z)$ *defined by* (6.9) *belongs to the class* $\mathcal{SP}_p(\alpha,\lambda)$ *for every complex number* ϵ.

Proof. If $f \in \mathcal{CP}_p(\alpha, \lambda)$, then

$$\begin{aligned} F_{\epsilon,p}(z) &= \frac{f(z) + \epsilon z^p}{1+\epsilon} \\ &= z^p + \frac{a_{p+1}}{1+\epsilon} z^{p+1} + \dots \\ &= f(z) * \left(z^p + \left(1 - \frac{\epsilon}{1+\epsilon}\right) z^{p+1} + \dots \right) \\ &= f(z) * \left(z^p + (1-\rho)\, z^{p+1} + \dots \right) \\ &= f(z) * h_p(z), \end{aligned}$$

where

$$h_p(z) = z^p + (1-\rho) z^{p+1} + \dots = \frac{z^p(1-\rho z)}{1-z} \qquad (z \in \mathcal{U}).$$

Let $g \in \mathcal{A}_p$ be given by

$$g(z) = z^p + \frac{p}{p+1} z^{p+1} + \frac{p}{p+2} z^{p+2} + \dots .$$

It is evident that

$$p \int_0^z \frac{h_p(t)}{t} dt = h_p(z) * g(z), \tag{6.10}$$

and

$$\frac{1}{p} z f'(z) * g(z) = f(z).$$

Thus

$$F_{\epsilon,p}(z) = f(z) * h_p(z) = \frac{1}{p} z f'(z) * g(z) * h_p(z).$$

Since $f \in \mathcal{CP}_p(\alpha, \lambda)$, it follows that $(1/p)(zf') \in \mathcal{SP}_p(\alpha, \lambda)$. Also $f_\lambda \in \mathcal{CV}_p\left(1 - (1-\gamma)/p\right)$ yields $(1/p)(zf'_\lambda) \in \mathcal{ST}_p\left(1 - (1-\gamma)/p\right)$. The relation $(1/p)\left(zf'\right)_\lambda = (1/p)\left(zf'_\lambda\right)$ and Lemma 6.3 implies that $(1/p)\left(zf'(z)\right)_\lambda / z^{p-1} \in \mathcal{ST}(\gamma)$. Now, to use Theorem

6.3, it remains to show that $\left(g * h_p\right)/z^{p-1} \in \mathcal{R}_\gamma$.

The function $h_p \in \mathcal{R}_p(\beta)$ if and only if

$$h_p(z) * \frac{z^p}{(1-z)^{2p(1-\beta)}} \in \mathcal{ST}_p(\beta).$$

Since the class $\mathcal{ST}_p(\beta)$ is closed under Alexander's transform it follows that

$$p\int_0^z \left(\frac{h_p(t)}{t} * \frac{t^{p-1}}{(1-t)^{2p(1-\beta)}}\right) dt \ \in \mathcal{ST}_p(\beta).$$

For any two p-valent functions $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^k$ and $g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^k$, it is evident that

$$p\int_0^z (f * g)(t)dt = p\int_0^z f(t)dt * zg(z).$$

Hence

$$p\int_0^z \frac{h_p(t)}{t} dt * \frac{z^p}{(1-z)^{2p(1-\beta)}} \ \in \mathcal{ST}_p(\beta).$$

Equation (6.10) now gives

$$g(z) * h_p(z) * \frac{z^p}{(1-z)^{2p(1-\beta)}} \in \mathcal{ST}_p(\beta).$$

Using Lemma 6.3, and taking $\beta = 1-(1-\gamma)/p,$ or $2p(1-\beta) = 2(1-\gamma)$, it follows that

$$\frac{g(z) * h_p(z)}{z^{p-1}} * \frac{z}{(1-z)^{2(1-\gamma)}} \in \mathcal{ST}(\gamma),$$

and hence

$$\frac{g(z) * h_p(z)}{z^{p-1}} \in \mathcal{R}_\gamma.$$

Now it follows from Theorem 6.3 that $F_{\epsilon,p}(z) = (1/p)(zf') * g * h_p \in \mathcal{SP}_p(\alpha, \lambda)$. ∎

Combining Theorem 6.5 and Theorem 6.2 yield the following δ-neighborhood description for functions in the class $\mathcal{CP}_p(\alpha,\lambda)$.

Theorem 6.6 *Let* $0 \leq \gamma \leq 1,\ 0 < \lambda \leq 1$ *and* $\alpha \geq 0$. *Further assume that for every complex number* ϵ, *the function* $h_p(z) = z^p(1-\rho z)/(1-z) \in \mathcal{R}_p(\beta)$, *where* $\rho = \epsilon/(1+\epsilon)$, *and* $\beta = 1-(1-\gamma)/p$. *If* $f \in \mathcal{CP}_p(\alpha,\lambda)$ *and* $f_\lambda \in \mathcal{CV}_p\left(1-(1-\gamma)/p\right)$, *then* $N_{\delta',p}(f) \subset \mathcal{SP}_p(\alpha,\lambda)$ *for*

$$\delta' := \begin{cases} 2\delta\sqrt{\alpha(1-\alpha)} & (0 < \alpha \leq \frac{1}{2}), \\ \delta & (\alpha \geq \frac{1}{2}). \end{cases}$$

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