DIFFERENTIAL SUBORDINATION AND COEFFICIENTS PROBLEMS OF CERTAIN ANALYTIC FUNCTIONS

SHAMANI A/P SUPRAMANIAM

UNIVERSITI SAINS MALAYSIA

2014

DIFFERENTIAL SUBORDINATION AND COEFFICIENTS PROBLEMS OF CERTAIN ANALYTIC FUNCTIONS

by

SHAMANI A/P SUPRAMANIAM

Thesis Submitted in fulfilment of the requirements for the Degree of Doctor of Philosophy in Mathematics

September 2014

ACKNOWLEDGEMENT

First and foremost, I am very grateful to God, for this thesis would not have been possible without His grace and blessings.

I am most indebted to my supervisor, Dr. Lee See Keong, for his continuous support and encouragement. Without his guidance on all aspects of my research, I could not have completed my dissertation.

I would like to express my special thanks to my co-supervisor and the head of the Research Group in Complex Function Theory USM, Prof. Dato' Rosihan M. Ali, for his valuable suggestions, encouragement and guidance throughout my studies. I also deeply appreciate his financial assistance for this research through generous grants, which supported overseas conference as well as my living expenses during the term of my study.

I express my sincere gratitude to my field-supervisor, Prof. V. Ravichandran, Professor at the Department of Mathematics, University of Delhi, for his constant guidance and support to complete the writing of this thesis as well as the challenging research that lies behind it. I am also thankful to Prof. K. G. Subramaniam and to other members of the Research Group in Complex Function Theory at USM for their help and support especially my friends Abeer, Chandrashekar, Mahnaz, Maisarah and Najla.

Also, my sincere appreciation to Prof. Ahmad Izani Md. Ismail, the Dean of the School of Mathematical Sciences USM, and the entire staff of the school and the authorities of USM for providing excellent facilities to me.

My research is supported by MyBrain (MyPhD) programme of the Ministry of Higher Education, Malaysia and it is gratefully acknowledged.

Last but not least, I express my love and gratitude to my beloved family and husband, for their understanding, encouragement and endless love. Not forgetting Shakthie and Sarvhesh for always cheering up my dreary days.

TABLE OF CONTENTS

SYMBOLS

Symbol	Description	Page
$\mathcal{A}_{p,n}$	Class of all analytic functions f of the form $f(z)=z^p+a_{n+p}z^{n+p}+a_{n+p+1}z^{n+p+1}+\dots \quad (z\in\mathbb{D})$	99
$\mathcal{A}_p$	Class of all p-valent analytic functions f of the form $f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k \quad (z\in\mathbb{D})$	2
$\mathcal{A}_{n,b}$	Class of all functions $f(z)=z+bz^{n+1}+a_{n+2}z^{n+2}+\cdots$ where b is a fixed non-negative real number.	60
$\mathcal{A}$	Class of analytic functions f of the form $f(z)=z+\sum_{k=2}^{\infty}a_kz^k \quad (z\in\mathbb{D})$	1
$\mathbb{C}$	Complex plane	1
$\mathcal{CV}$	Class of convex functions in $\mathcal{A}$	4
$\mathcal{CV}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$	5
$\mathcal{CV}(\varphi)$	$\left\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec\varphi(z)\right\}$	8
$\mathcal{CCV}$	Class of close-to-convex functions in $\mathcal{A}$	6
$\mathbb{D}$	Open unit disk $\{z\in\mathbb{C}:\lvert z\rvert<1\}$	1
$\mathbb{D}^*$	Open punctured unit disk $\{z\in\mathbb{C}:0<\lvert z\rvert<1\}$	6
$\partial\mathbb{D}$	Boundary of unit disk $\mathbb{D}$	9
$\mathcal{G}_\alpha(\varphi)$	$\left\{f\in\mathcal{A}:(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\varphi(z)\right\}$	54
$\mathcal{H}(\mathbb{D})$	Class of all analytic functions in $\mathbb{D}$	1
$\mathcal{H}[a,n]=\mathcal{H}_n(a)$	Class of all analytic functions f in $\mathbb{D}$ of the form $f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots$	1

$\mathcal{H} := \mathcal{H}[1,1]$	Class of analytic functions f in $\mathbb{D}$ of the form $f(z) = 1 + a_1 z + a_2 z^2 + \cdots$	1
$\mathcal{H}_{\mu,n}$	Class of analytic functions p on $\mathbb{D}$ of the form $p(z) = 1 + \mu z^n + p_{n+1} z^{n+1} + \cdots$	60
$\mathcal{H}_{\mathcal{SB}}(\varphi)$	$\{f \in \mathcal{SB} : f'(z) \prec \varphi(z)$ and $g'(w) \prec \varphi(w),\ g(w) := f^{-1}(w)\}$	34
$H_q(n)$	Hankel determinants of functions $f \in \mathcal{A}$	36
$\mathcal{L}(\alpha,\varphi)$	$\left\{f \in \mathcal{S} : \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z)\right\}$	13
$\mathcal{M}(\alpha,\varphi)$	$\left\{f \in \mathcal{S} : (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z)\right\}$	13
$\mathcal{P}$	$\{p \in \mathcal{H} : \text{with } \operatorname{Re} p(z) > 0,\ z \in \mathbb{D}\}$	6
$\mathcal{P}(\alpha)$	$\{p \in \mathcal{H} : \text{with } \operatorname{Re} p(z) > \alpha,\ z \in \mathbb{D}\}$	6
$\mathbb{R}$	Set of all real numbers	2
Re	Real part of a complex number	4
$\mathcal{R}^{\tau}_{\gamma}(\varphi)$	$\left\{f \in \mathcal{A} : 1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) \prec \varphi(z)\right\}$	49
$\mathcal{S}$	Class of all normalized univalent functions f in $\mathcal{A}$	2
$\mathcal{ST}$	Class of starlike functions in $\mathcal{A}$	5
$\mathcal{ST}(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$	5
$\mathcal{ST}(\varphi)$	$\left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z)\right\}$	8
$\mathcal{ST}(\alpha,\varphi)$	$\left\{f \in \mathcal{S} : \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec \varphi(z)\right\}$	13
Σ	Class of all meromorphic functions f of the form $f(z) = f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$	6
$\Sigma_{n,b}$	Class of all meromorphic functions f of the form	73

	$f(z) = \frac{1}{z} + bz^n + a_{n+1}z^{n+1} + \cdots \quad (b \leq 0)$	
$\prec$	Subordinate to	6
$\mathcal{SB}$	Class of bi-univalent functions	12
$\Psi_n[\Omega, q]$	Class of admissible functions for differential subordination	9
$\Psi_{\mu,n}[\Omega]$	Class of admissible functions for fixed second coefficient	60

SUBORDINASI PEMBEZA DAN MASALAH PEKALI UNTUK FUNGSI-FUNGSI ANALISIS

ABSTRAK

Lambangkan $\mathcal{A}$ sebagai kelas fungsi analisis ternormal pada cakera unit $\mathbb{D}$ berbentuk $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Fungsi f dalam $\mathcal{A}$ adalah univalen jika fungsi tersebut ialah pemetaan satu ke satu. Tesis ini mengkaji lima masalah penyelidikan.

Fungsi $f \in \mathcal{A}$ dikatakan dwi univalen dalam $\mathbb{D}$ jika kedua-dua fungsi f dan songsangannya f^{-1} adalah univalen dalam $\mathbb{D}$. Anggaran pekali awal, $|a_2|$ dan $|a_3|$, fungsi dwi univalen akan dikaji untuk f dan f^{-1} yang masing-masing terkandung di dalam subkelas fungsi univalen tertentu. Seterusnya, batas penentu Hankel kedua $H_2(2) = a_2a_4 - a_3^2$ untuk fungsi analisis f dengan $zf'(z)/f(z)$ dan $1 + zf''(z)/f'(z)$ subordinat kepada suatu fungsi analisis tertentu diperoleh.

Bermotivasikan kerja terdahulu dalam subordinasi pembeza peringkat kedua untuk fungsi analisis dengan pekali awal tetap, syarat cukup bak-bintang dan univalen untuk suatu subkelas fungsi berpekali kedua tetap ditentukan. Kemudian, syarat cukup cembung untuk fungsi yang pekali keduanya tidak ditetapkan dan yang memenuhi ketaksamaan pembeza peringkat kedua dan ketiga tertentu diperoleh.

Akhir sekali, subkelas fungsi multivalen yang memenuhi syarat bak-bintang dan hampir cembung dikaji.

Beberapa aspek permasalahan dalam teori fungsi univalen dibincangkan dalam tesis ini dan hasil-hasil menarik diperoleh.

DIFFERENTIAL SUBORDINATION AND COEFFICIENTS PROBLEMS OF CERTAIN ANALYTIC FUNCTIONS

ABSTRACT

Let $\mathcal{A}$ be the class of normalized analytic functions f on the unit disk $\mathbb{D}$, in the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. A function f in $\mathcal{A}$ is univalent if it is a one-to-one mapping. This thesis discussed five research problems.

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and its inverse f^{-1} are univalent in $\mathbb{D}$. Estimates on the initial coefficients, $|a_2|$ and $|a_3|$, of bi-univalent functions f are investigated when f and f^{-1} respectively belong to some subclasses of univalent functions. Next, the bounds for the second Hankel determinant $H_2(2) = a_2a_4 - a_3^2$ of analytic function f for which $zf'(z)/f(z)$ and $1 + zf''(z)/f'(z)$ is subordinate to certain analytic function are obtained.

Motivated by the earlier work on second order differential subordination for analytic functions with fixed initial coefficient, the sufficient conditions for starlikeness and univalence for a subclass of functions with fixed second coefficient are obtained. Then, without fixing the second coefficient, the sufficient condition for convexity of these functions satisfying certain second order and third order differential inequalities are determined.

Lastly, the close-to-convexity and starlikeness of a subclass of multivalent functions are investigated.

A few aspects of problems in univalent function theory is discussed in this thesis and some interesting results are obtained.

CHAPTER 1

INTRODUCTION

1.1 Univalent function

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. A function f is analytic at a point $z_0 \in D$ if it is differentiable in some neighborhood of z_0 and it is analytic in a domain D if it is analytic at all points in domain D. An analytic function f is said to be univalent in a domain if it provides a one-to-one mapping onto its image: $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$. Geometrically, this means that different points in the domain will be mapped into different points on the image domain. An analytic function f is locally univalent at a point $z_0 \in D$ if it is univalent in some neighborhood of z_0. The well known Riemann Mapping Theorem states that every simply connected domain (which is not the whole complex plane $\mathbb{C}$), can be mapped conformally onto the unit disk $\mathbb{D}$.

Theorem 1.1 (Riemann Mapping Theorem) [29, p. 11] *Let D be a simply connected domain which is a proper subset of the complex plane. Let ζ be a given point in D. Then there is a unique univalent analytic function f which maps D onto the unit disk $\mathbb{D}$ satisfying $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

In view of this theorem, the study of analytic univalent functions on a simply connected domain can be restricted to the open unit disk $\mathbb{D}$.

Let $\mathcal{H}(\mathbb{D})$ be the class of analytic functions defined on $\mathbb{D}$. Let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(\mathbb{D})$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$$

with $\mathcal{H} \equiv \mathcal{H}[1, 1]$.

Also let $\mathcal{A}$ denote the class of all functions f analytic in the open unit disk $\mathbb{D}$, and normalized by $f(0) = 0$, and $f'(0) = 1$. A function $f \in \mathcal{A}$ has the Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}).$$

For a fixed $p \in \mathbb{N} := \{1, 2, \dots\}$, let $\mathcal{A}_p$ be the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p},$$

that are p-valent (multivalent) in the open unit disk, with $\mathcal{A} := \mathcal{A}_1$.

The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. The function k given by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n \quad (z \in \mathbb{D})$$

is called the Koebe function, which maps $\mathbb{D}$ onto the complex plane except for a slit along the half-line $(-\infty, -1/4]$, and is univalent. It plays a very important role in the study of the class $\mathcal{S}$. The Koebe function and its rotations $e^{-i\beta}k(e^{i\beta}z)$, for $\beta \in \mathbb{R}$, are the only extremal functions for various problem in the class $\mathcal{S}$. In 1916, Bieberbach [19] conjectured that for $f \in \mathcal{S}$, $|a_n| \leq n$, $(n \geq 2)$. He proved only for the case when $n = 2$.

Theorem 1.2 (Bieberbach's Conjecture) [19] *If $f \in \mathcal{S}$, then $|a_n| \leq 2$ $(n \geq 2)$ with equality if and only if f is the rotation of the Koebe function k.*

For the cases $n = 3$, and $n = 4$ the conjecture was proved by Lowner [58] and Garabedian and Schiffer [34], respectively. Later, Pederson and Schiffer [98] proved the conjecture for $n = 5$, and for $n = 6$, it was proved by Pederson [97] and Ozawa [95], independently. In 1985, Louis de Branges [20], proved the Bieberbach's conjecture for all the coefficients n.

Theorem 1.3 (de Branges Theorem or Bieberbach's Theorem) [20] *If* $f \in \mathcal{S}$, *then*

$$|a_n| \leq n \quad (n \geq 2),$$

with equality if and only if f *is the Koebe function* k *or one of its rotations.*

Bieberbach's theorem has many important properties in univalent functions. These include the well known covering theorem: If $f \in \mathcal{S}$, then the image of $\mathbb{D}$ under f contains a disk of radius 1/4.

Theorem 1.4 (Koebe One-Quarter Theorem) [29, p. 31] *The range of every function* $f \in \mathcal{S}$ *contains the disk* $\{w \in \mathbb{C} : |w| < 1/4\}$.

The Distortion theorem, being another consequence of the Bieberbach theorem gives sharp upper and lower bounds for $|f'(z)|$.

Theorem 1.5 (Distortion Theorem) [29, p. 32] *For each* $f \in \mathcal{S}$,

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} \quad (|z| = r < 1).$$

The distortion theorem can be used to obtain sharp upper and lower bounds for $|f(z)|$ which is known as the Growth theorem.

Theorem 1.6 (Growth Theorem) [29, p. 33] *For each* $f \in \mathcal{S}$,

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2} \quad (|z| = r < 1).$$

Another consequence of the Bieberbach theorem is the Rotation theorem.

Theorem 1.7 (Rotation Theorem) [29, p. 99] *For each* $f \in \mathcal{S}$,

$$|\arg f'(z)| \leq \begin{cases} 4sin^{-1}r, & r \leq \frac{1}{\sqrt{2}}, \\ \pi + \log \frac{r^2}{1-r^2}, & r \geq \frac{1}{\sqrt{2}}, \end{cases}$$

where $|z| = r < 1$. *The bound is sharp.*

The Fekete-Szego coefficient functional also arises in the investigation of univalency of analytic functions.

Theorem 1.8 (Fekete-Szego Theorem) [29, p. 104] *For each* $f \in \mathcal{S}$*,*

$$|a_3 - \alpha a_2^2| \leq 1 + 2e^{-2\alpha/(1-\alpha)}, \quad (0 < \alpha < 1).$$

1.2 Subclasses of univalent functions

The long gap between the Bieberbach's conjecture in 1916 and its proof by de Branges in 1985 motivated researchers to consider classes defined by geometric conditions. Notable among them are the classes of convex functions, starlike functions and close-to-convex functions.

A set D in the complex plane is called *convex* if for every pair of points w_1 and w_2 lying in the interior of D, the line segment joining w_1 and w_2 also lies in the interior of D, i.e.

$$tw_1 + (1-t)w_2 \ \in D \quad \text{for} \quad 0 \leq t \leq 1.$$

If a function $f \in \mathcal{A}$ maps $\mathbb{D}$ onto a convex domain, then f is a *convex function.* The class of all convex functions in $\mathcal{A}$ is denoted by $\mathcal{CV}$. An analytic description of the class $\mathcal{CV}$ is given by

$$\mathcal{CV} := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}.$$

Let w_0 be an interior point of D. A set D in the complex plane is called *starlike* with respect to w_0 if the line segment joining w_0 to every other point $w \in D$ lies in the interior of D, i.e.

$$(1-t)w + tw_0 \in D \quad \text{for} \quad 0 \leq t \leq 1.$$

If a function $f \in \mathcal{A}$ maps $\mathbb{D}$ onto a starlike domain, then f is a *starlike function.* The class of starlike functions with respect to origin is denoted by $\mathcal{ST}$. Analytically,

$$\mathcal{ST} := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\}.$$

In 1936, Robertson [105] introduced the concepts of convex functions of order α and starlike functions of order α for $0 \leq \alpha < 1$. A function $f \in \mathcal{A}$ is said to be *convex of order* α if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{D}),$$

and *starlike of order* α if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{D}).$$

These classes are respectively denoted by $\mathcal{CV}(\alpha)$ and $\mathcal{ST}(\alpha)$.

Note that $\mathcal{CV}(0) = \mathcal{CV}$ and $\mathcal{ST}(0) = \mathcal{ST}$. An important relationship between convex and starlike functions was first observed by Alexander [1] in 1915 and known later as Alexander's theorem.

Theorem 1.9 (Alexander's Theorem) [29, p. 43] *Let $f \in \mathcal{A}$. Then $f \in \mathcal{CV}$ if and only if $zf' \in \mathcal{ST}$.*

From this, it is easily proven that $f \in \mathcal{CV}(\alpha)$ if and only if $zf' \in \mathcal{ST}(\alpha)$.

Another subclass of $\mathcal{S}$ that has an important role in the study of univalent functions is the class of close-to-convex functions introduced by Kaplan [45] in 1952. A function $f \in \mathcal{A}$ is *close-to-convex* in $\mathbb{D}$ if there is a convex function g and a real number θ, $-\pi/2 < \theta < \pi/2$, such that

$$\operatorname{Re}\left(e^{i\theta}\frac{f'(z)}{g'(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

The class of all such functions is denoted by $\mathcal{CCV}$. The subclasses of $\mathcal{S}$, namely convex, starlike and close-to-convex functions are related as follows:

$$\mathcal{CV} \subset \mathcal{ST} \subset \mathcal{CCV} \subset \mathcal{S}.$$

The well known Noshiro-Warschawski theorem states that a function $f \in \mathcal{A}$ with positive derivative in $\mathbb{D}$ is univalent.

Theorem 1.10 [82, 131] *For some real α, if a function f is analytic in a convex domain D and*

$$\operatorname{Re}\left(e^{i\alpha}f'(z)\right) > 0,$$

then f is univalent in D.

Kaplan [45] applied Noshiro-Warschawski theorem to prove that every close-to-convex function is univalent.

The class of meromorphic functions is yet another subclass of univalent functions. Let Σ denote the class of normalized *meromorphic* functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

that are analytic in the punctured unit disk $\mathbb{D}^* := \{z : 0 < |z| < 1\}$ except for a simple pole at 0.

A function f is said to be subordinate to F in $\mathbb{D}$, written $f(z) \prec F(z)$, if there exists a Schwarz function w, analytic in $\mathbb{D}$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = F(w(z))$. If the function F is univalent in $\mathbb{D}$, then $f \prec F$ if $f(0) = F(0)$ and $f(U) \subseteq F(U)$.

Let $\mathcal{P}$ be the class of all analytic functions p of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n$$

with

$$\operatorname{Re} p(z) > 0 \quad (z \in \mathbb{D}). \tag{1.1}$$

Any function in $\mathcal{P}$ is called a function with positive real part, also known as Caratheodory function. The following lemma is known for functions in $\mathcal{P}$.

Lemma 1.1 [29] *If the function $p \in \mathcal{P}$ is given by the series*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots ,$$

then the following sharp estimate holds:

$$|p_n| \leq 2 \quad (n = 1, 2, \dots).$$

The above fact will be used often in the thesis especially in Chapters 2 and 3. More generally, for $0 \leq \alpha < 1$, we denote by $\mathcal{P}(\alpha)$ the class of analytic functions $p \in \mathcal{P}$ with

$$\operatorname{Re} p(z) > \alpha \quad (z \in \mathbb{D}).$$

In terms of subordination, the analytic condition (1.1) can be written as

$$p(z) \prec \frac{1+z}{1-z} \quad (z \in \mathbb{D}).$$

This follows since the mapping $q(z) = (1+z)/(1-z)$ maps $\mathbb{D}$ onto the right-half plane.

Ma and Minda [59] have given a unified treatment of various subclasses consisting of starlike and convex functions by replacing the superordinate function

$q(z) = (1+z)/(1-z)$ by a more general analytic function. For this purpose, they considered an analytic function φ with positive real part on $\mathbb{D}$ with $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathbb{D}$ onto a region starlike with respect to 1, symmetric with respect to the real axis. The class of Ma-Minda starlike functions denoted by $\mathcal{ST}(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \varphi(z)$$

and similarly the class of Ma-Minda convex functions denoted by $\mathcal{CV}(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z), \quad (z \in \mathbb{D}).$$

respectively.

1.3 Differential subordination

Recall that a function f is said to be subordinate to F in $\mathbb{D}$, written $f(z) \prec F(z)$, if there exists a Schwarz function w, analytic in $\mathbb{D}$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = F(w(z))$. If the function F is univalent in $\mathbb{D}$, then $f \prec F$ if $f(0) = F(0)$ and $f(U) \subseteq F(U)$.

The basic definitions and theorems in the theory of subordination and certain applications of differential subordinations are stated in this section. The theory of differential subordination were developed by Miller and Mocanu [61].

Let $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ and let h be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfies the second order differential subordination

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \tag{1.2}$$

then p is called a *solution of the differential subordination.* The univalent function q is called a *dominant* of the solution of the differential subordination or more simply dominant, if $p \prec q$ for all p satisfying (1.2). A dominant q_1 satisfying $q_1 \prec q$ for all dominants q of (1.2) is said to be the *best dominant* of (1.2). The best dominant is unique up to a rotation of $\mathbb{D}$.

If $p \in \mathcal{H}[a, n]$, then p will be called an (a, n)-*solution*, q an (a, n)-*dominant*, and q_1 the *best* (a, n)-*dominant.* Let $\Omega \subset \mathbb{C}$ and let (1.2) be replaced by

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega, \text{ for all } z \in \mathbb{D}, \tag{1.3}$$

where Ω is a simply connected domain containing $h(\mathbb{D})$. Even though this is a differential inclusion and $\psi\left(p(z), zp'(z), z^2p''(z); z\right)$ may not be analytic in $\mathbb{D}$, the condition in (1.3) shall also be referred as a *second order differential subordination,* and the same definition of solution, dominant and best dominant as given above can be extended to this generalization. The monograph [61] by Milller and Mocanu provides more detailed information on the theory of differential subordination.

Denote by $\mathcal{Q}$ the set of functions q that are analytic and injective on $\bar{\mathbb{D}}\backslash E(q)$, where

$$E(q) = \{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty\}$$

and $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D}\backslash E(q)$.

The subordination methodology is applied to an appropriate class of admissible functions. The following class of admissible functions was given by Miller and Mocanu [61].

Definition 1.1 [61, Definition 2.3a, p. 27] *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and m be a positive integer. The class of admissible functions $\Psi_m[\Omega, q]$ consists of functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfying the admissibility condition $\psi(r, s, t; z) \notin \Omega$ whenever*

$r = q(\zeta), s = k\zeta q'(\zeta)$ *and*

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in \mathbb{D}, \zeta \in \partial\mathbb{D} \backslash E(q)$ *and* $k \geq m$. *In particular,* $\Psi[\Omega, q] := \Psi_1[\Omega, q]$.

The next theorem is the foundation result in the theory of first and second-order differential subordinations.

Theorem 1.11 [61, Theorem 2.3b, p. 28] *Let* $\psi \in \Psi_m[\Omega, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{H}[a, n]$ *satisfies*

$$\psi\big(p(z), zp'(z), z^2 p''(z); z\big) \in \Omega,$$

then $p \prec q$.

1.4 Scope of thesis

This thesis will discuss five research problems. In Chapter 2, estimates on the initial coefficients for bi-univalent functions f in the open unit disk with f and its inverse $g = f^{-1}$ satisfying the conditions that $zf'(z)/f(z)$ and $zg'(z)/g(z)$ are both subordinate to a univalent function whose range is symmetric with respect to the real axis. Several related classes of functions are also considered, and connections to earlier known results are made.

In Chapter 3, the bounds for the second Hankel determinant $a_2a_4 - a_3^2$ of analytic function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ for which either $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to certain analytic function are investigated. The problem is also investigated for two other related classes defined by subordination. The classes introduced by subordination naturally include several well known classes of univalent functions and the results for some of these special classes are indicated. In particular, the estimates for the Hankel determinant of strongly starlike, parabolic starlike, lemniscate starlike functions are obtained.

In Chapter 4, several well known results for subclasses of univalent functions was extended to functions with fixed initial coefficient by using the theory of differential subordination. Further applications of this subordination theory is given. In particular, several sufficient conditions related to starlikeness, meromorphic starlikeness and univalence of normalized analytic functions are derived.

In Chapter 5, the convexity conditions for analytic functions defined in the open unit disk satisfying certain second-order and third-order differential inequalities are obtained. As a consequence, conditions are also determined for convexity of functions defined by following integral operators

$$f(z) = \int_0^1 \int_0^1 W(r,s,z)drds, \quad \text{and} \quad f(z) = \int_0^1 \int_0^1 \int_0^1 W(r,s,t,z)drdsdt.$$

In the final chapter, several sufficient conditions for close-to-convexity and starlikeness of a subclass of multivalent functions are investigated. Relevant connections with previously known results are indicated.

CHAPTER 2

COEFFICIENTS FOR BI-UNIVALENT FUNCTIONS

2.1 Introduction and preliminaries

For functions $f \in \mathcal{S}$, let f^{-1} be its inverse function. The Koebe one-quarter theorem (Theorem 1.4) ensures the existence of f^{-1}, that is, every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, $(z \in \mathbb{D})$ and

$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq 1/4) .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let $\mathcal{SB}$ denote the class of bi-univalent functions defined in $\mathbb{D}$. Examples of functions in the class $\mathcal{SB}$ are $z/(1-z)$ and $-\log(1-z)$.

In 1967, Lewin [51] introduced this class $\mathcal{SB}$ and proved that the bound for the second coefficients of every $f \in \mathcal{SB}$ satisfies the inequality $|a_2| \leq 1.51$. He also investigated $\mathcal{SB}_1 \subset \mathcal{SB}$, the class of all functions $f = \phi \circ \psi^{-1}$, where ϕ and ψ map $\mathbb{D}$ onto domains containing $\mathbb{D}$ and $\phi'(0) = \psi'(0)$. For an example that shows $\mathcal{SB} \neq \mathcal{SB}_1$, see [23]. In 1969, Suffridge [122] showed that a function in $\mathcal{SB}_1$ satisfies $a_2 = 4/3$ and thus conjectured that $|a_2| \leq 4/3$ for all functions in $\mathcal{SB}$. Netanyahu [69], in the same year, proved this conjecture for a subclass of $\mathcal{SB}_1$. In 1981, Styer and Wright [121] showed that $a_2 > 4/3$ for some function in $\mathcal{SB}$, thus disproved the conjecture of Suffridge. For bi-univalent polynomial $f(z) = z + a_2 z^2 + a_3 z^3$ with real coefficients, Smith [114] showed that $|a_2| \leq 2/\sqrt{27}$ and $|a_3| \leq 4/27$ and the latter inequality being the best possible. He also showed that if $z + a_n z^n$ is bi-univalent, then $|a_n| \leq (n-1)^{n-1}/n^n$ with equality best possible for $n = 2, 3$. Kędzierawski and Waniurski [47] proved the conjecture of Smith [114] for $n = 3, 4$ in the case of bi-univalent polynomial of degree n. Extending the results of Srivasta *et al.* [118], Frasin and Aouf [33] obtained estimate of $|a_2|$ and $|a_3|$ for bi-univalent function f for which

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \quad \text{and} \quad (1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \quad (g = f^{-1})$$

belongs to a sector in the half plane. Tan [125] improved Lewin's result to $|a_2| \leq 1.485$. For $0 \leq \alpha < 1$, a function $f \in \mathcal{SB}$ is bi-starlike of order α or bi-convex of order α if both f and f^{-1} are respectively starlike or convex of order α. These classes were introduced by Brannan and Taha [22]. They obtained estimates on the initial coefficients for functions in these classes. For some open problems and survey, see [35,115]. Bounds for the initial coefficients of several classes of functions were also investigated in [7,8,24–26,33,39,48,60,64,67,108,117–120,126,133,134].

2.2 Kędzierawski type results

In 1985, Kędzierawski [46] considered functions f belonging to certain subclasses of univalent functions while its inverse f^{-1} belongs to some other subclasses of univalent functions. Among other results, he obtained the following.

Theorem 2.1 [46] *Let $f \in \mathcal{SB}$ with Taylor series $f(z) = z + a_2 z^2 + \cdots$ and $g = f^{-1}$. Then*

$$|a_2| \leq \begin{cases} 1.5894 & \text{if } f \in \mathcal{S},\ g \in \mathcal{S}, \\ \sqrt{2} & \text{if } f \in \mathcal{ST},\ g \in \mathcal{ST}, \\ 1.507 & \text{if } f \in \mathcal{ST},\ g \in \mathcal{S}, \\ 1.224 & \text{if } f \in \mathcal{CV},\ g \in \mathcal{S}. \end{cases}$$

Consider the following classes investigated in [7,8,14].

Definition 2.1 *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic and $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$. For $\alpha \geq 0$, let*

$$\mathcal{M}(\alpha, \varphi) := \left\{ f \in \mathcal{S} \,:\, (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\},$$

$$\mathcal{L}(\alpha,\varphi) := \left\{ f \in \mathcal{S} \,:\, \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \right\},$$

$$\mathcal{ST}(\alpha,\varphi) := \left\{ f \in \mathcal{S} \,:\, \frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \right\}.$$

Suppose that f is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2.1}$$

then it is known that $g = f^{-1}$ has the expression

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots .$$

Motivated by Theorem 2.1, we will consider the following cases and then will obtain the estimates for the second and third coefficients of functions f:

1. $f \in \mathcal{ST}(\alpha,\varphi)$ and $g \in \mathcal{ST}(\beta,\psi)$, or $g \in \mathcal{M}(\beta,\psi)$, or $g \in \mathcal{L}(\beta,\psi)$,

2. $f \in \mathcal{M}(\alpha,\varphi)$ and $g \in \mathcal{M}(\beta,\psi)$, or $g \in \mathcal{L}(\beta,\psi)$,

3. $f \in \mathcal{L}(\alpha,\varphi)$ and $g \in \mathcal{L}(\beta,\psi)$,

where φ and ψ are analytic functions of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots , \quad (B_1 > 0) \tag{2.2}$$

and

$$\psi(z) = 1 + D_1 z + D_2 z^2 + D_3 z^3 + \cdots , \quad (D_1 > 0). \tag{2.3}$$

2.3 Second and third coefficients of functions f when $f \in \mathcal{ST}(\alpha,\varphi)$ and $g \in \mathcal{ST}(\beta,\psi)$, or $g \in \mathcal{M}(\beta,\psi)$, or $g \in \mathcal{L}(\beta,\psi)$

We begin with the cases for $f \in \mathcal{ST}(\alpha,\varphi)$.

Theorem 2.2 *Let $f \in \mathcal{SB}$ and $g = f^{-1}$. If $f \in \mathcal{ST}(\alpha, \varphi)$ and $g \in \mathcal{ST}(\beta, \psi)$, then*

$$|a_2| \leq \frac{B_1 D_1 \sqrt{B_1(1+3\beta) + D_1(1+3\alpha)}}{\sqrt{|\rho B_1^2 D_1^2 - (1+2\alpha)^2(1+3\beta)(B_2 - B_1)D_1^2 - (1+2\beta)^2(1+3\alpha)(D_2 - D_1)B_1^2|}} \tag{2.4}$$

and

$$2\rho|a_3| \leq B_1(3+10\beta) + D_1(1+2\alpha) + (3+10\beta)|B_2 - B_1| + \frac{(1+2\beta)^2 B_1^2 |D_2 - D_1|}{D_1^2(1+2\alpha)} \tag{2.5}$$

where $\rho := 2 + 7\alpha + 7\beta + 24\alpha\beta$.

Proof. Since $f \in \mathcal{ST}(\alpha, \varphi)$ and $g \in \mathcal{ST}(\beta, \psi)$, there exist analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \frac{\beta w^2 g''(w)}{g(w)} = \psi(v(w)). \tag{2.6}$$

Define the functions p_1 and p_2 by

$$p_1(z) := \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad \text{and} \quad p_2(z) := \frac{1+v(z)}{1-v(z)} = 1 + b_1 z + b_2 z^2 + \cdots,$$

or, equivalently,

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right) \tag{2.7}$$

and

$$v(z) = \frac{p_2(z)-1}{p_2(z)+1} = \frac{1}{2}\left(b_1 z + \left(b_2 - \frac{b_1^2}{2}\right) z^2 + \cdots\right). \tag{2.8}$$

Then p_1 and p_2 are analytic in $\mathbb{D}$ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \mathbb{D} \to \mathbb{D}$, the functions p_1 and p_2 have positive real part in $\mathbb{D}$, and thus $|b_i| \leq 2$ and $|c_i| \leq 2$

(Lemma 1.1). In view of (2.6), (2.7) and (2.8), it is clear that

$$\frac{zf'(z)}{f(z)}+\frac{\alpha z^2 f''(z)}{f(z)}=\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \quad \text{and} \quad \frac{wg'(w)}{g(w)}+\frac{\beta w^2 g''(w)}{g(w)}=\psi\left(\frac{p_2(w)-1}{p_2(w)+1}\right). \tag{2.9}$$

Using (2.7) and (2.8) together with (2.2) and (2.3), it is evident that

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=1+\frac{1}{2}B_1c_1z+\left(\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right)z^2+\cdots \tag{2.10}$$

and

$$\psi\left(\frac{p_2(w)-1}{p_2(w)+1}\right)=1+\frac{1}{2}D_1b_1w+\left(\frac{1}{2}D_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}D_2b_1^2\right)w^2+\cdots. \tag{2.11}$$

Since

$$\frac{zf'(z)}{f(z)}+\frac{\alpha z^2 f''(z)}{f(z)}=1+a_2(1+2\alpha)z+\Big(2(1+3\alpha)a_3-(1+2\alpha)a_2^2\Big)z^2+\cdots$$

and

$$\frac{wg'(w)}{g(w)}+\frac{\beta w^2 g''(w)}{g(w)}=1-(1+2\beta)a_2w+\Big((3+10\beta)a_2^2-2(1+3\beta)a_3\Big)w^2+\cdots,$$

it follows from (2.9), (2.10) and (2.11) that

$$a_2(1+2\alpha)=\frac{1}{2}B_1c_1, \tag{2.12}$$

$$2(1+3\alpha)a_3-(1+2\alpha)a_2^2=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2, \tag{2.13}$$

$$-(1+2\beta)a_2=\frac{1}{2}D_1b_1 \tag{2.14}$$

and

$$(3+10\beta)a_2^2 - 2(1+3\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2 b_1^2. \tag{2.15}$$

It follows from (2.12) and (2.14) that

$$b_1 = -\frac{B_1(1+2\beta)}{D_1(1+2\alpha)}c_1. \tag{2.16}$$

Multiplying (2.13) with $(1+3\beta)$ and (2.15) with $(1+3\alpha)$, and adding the results give

$$\begin{aligned} a_2^2((1+3\alpha)(3+10\beta) - (1+3\beta)(1+2\alpha)) &= \frac{1}{2}B_1(1+3\beta)c_2 + \frac{1}{2}D_1(1+3\alpha)b_2 \\ + \frac{1}{4}c_1^2(1+3\beta)(B_2 - B_1) + \frac{1}{4}b_1^2(1+3\alpha)(D_2 - D_1). \end{aligned}$$

Substituting c_1 from (2.12) and b_1 from (2.16) in the above equation give

$$\begin{aligned} &a_2^2((1+3\alpha)(3+10\beta) - (1+3\beta)(1+2\alpha)) \\ &\quad - a_2^2\left(\frac{(1+3\beta)(1+2\alpha)^2(B_2 - B_1)}{B_1^2} + \frac{(1+2\beta)^2(1+3\alpha)(D_2 - D_1)}{D_1^2}\right) \\ &= \frac{1}{2}B_1(1+3\beta)c_2 + \frac{1}{2}D_1(1+3\alpha)b_2 \end{aligned}$$

which lead to

$$a_2^2 = \frac{B_1^2 D_1^2[B_1(1+3\beta)c_2 + D_1(1+3\alpha)b_2]}{2[\rho B_1^2 D_1^2 - (1+2\alpha)^2(1+3\beta)(B_2 - B_1)D_1^2 - (1+2\beta)^2(1+3\alpha)(D_2 - D_1)B_1^2]},$$

where $\rho := 2 + 7\alpha + 7\beta + 24\alpha\beta$, which, in view of $|b_2| \leq 2$ and $|c_2| \leq 2$, gives us the desired estimate on $|a_2|$ as asserted in (2.4).

Multiplying (2.13) with $(3+10\beta)$ and (2.15) with $(1+2\alpha)$, and adding the results give

$$2((1+3\alpha)(3+10\beta)-(1+3\beta)(1+2\alpha))a_3 = \frac{1}{2}B_1(3+10\beta)c_2 + \frac{1}{2}D_1(1+2\alpha)b_2 + \frac{c_1^2}{4}(3+10\beta)(B_2-B_1) + \frac{b_1^2}{4}(D_2-D_1)(1+2\alpha).$$

Substituting b_1 from (2.16) in the above equation lead to

$$2\rho a_3 = \frac{1}{2}\left[B_1(3+10\beta)c_2 + D_1(1+2\alpha)b_2\right] + \frac{c_1^2}{4}\left[(3+10\beta)(B_2-B_1) + \frac{(1+2\beta)^2B_1^2(D_2-D_1)}{D_1^2(1+2\alpha)}\right],$$

and this yields the estimate given in (2.5). ∎

Remark 2.1 *When* $\alpha=\beta=0$ *and* $B_1=B_2=2$, $D_1=D_2=2$, *inequality* (2.4) *reduces to the second result in Theorem 2.1.*

In the case when $\beta=\alpha$ and $\psi=\varphi$, Theorem 2.2 reduces to the following corollary.

Corollary 2.1 *Let* f *given by* (2.1) *and* $g=f^{-1}$. *If* $f, g \in \mathcal{ST}(\alpha,\varphi)$, *then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2(1+4\alpha)+(B_1-B_2)(1+2\alpha)^2|}}, \tag{2.17}$$

and

$$|a_3| \le \frac{B_1+|B_2-B_1|}{(1+4\alpha)}. \tag{2.18}$$

For φ given by

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^\gamma = 1+2\gamma z+2\gamma^2z^2+\cdots \quad (0<\gamma\le 1),$$

we have $B_1=2\gamma$ and $B_2=2\gamma^2$. Hence, when $\alpha=0$, the inequality (2.17) reduces to the following result.

Corollary 2.2 [22, Theorem 2.1] *Let f given by (2.1) be in the class of strongly bi-starlike functions of order γ, $0 < \gamma \leq 1$. Then*

$$|a_2| \leq \frac{2\gamma}{\sqrt{1+\gamma}}.$$

On the other hand, when $\alpha = 0$ and

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots$$

so that $B_1 = B_2 = 2(1-\beta)$, the inequalities in (2.17) and (2.18) reduce to the following result.

Corollary 2.3 [22, Theorem 3.1] *Let f given by (2.1) be in the class of bi-starlike functions of order β, $0 < \beta \leq 1$. Then*

$$|a_2| \leq \sqrt{2(1-\beta)} \quad \text{and} \quad |a_3| \leq 2(1-\beta).$$

Theorem 2.3 *Let $f \in \mathcal{SB}$ and $g = f^{-1}$. If $f \in \mathcal{ST}(\alpha, \varphi)$ and $g \in \mathcal{M}(\beta, \psi)$, then*

$$|a_2| \leq \frac{B_1 D_1 \sqrt{B_1(1+2\beta) + D_1(1+3\alpha)}}{\sqrt{|\rho B_1^2 D_1^2 - (1+2\alpha)^2(1+2\beta)(B_2 - B_1)D_1^2 - (1+\beta)^2(1+3\alpha)(D_2 - D_1)B_1^2|}} \tag{2.19}$$

and

$$2\rho|a_3| \leq B_1(3+5\beta) + D_1(1+2\alpha) + (3+5\beta)|B_2 - B_1| + \frac{(1+\beta)^2 B_1^2 |D_2 - D_1|}{D_1^2(1+2\alpha)} \tag{2.20}$$

where $\rho := 2 + 7\alpha + 3\beta + 11\alpha\beta$.

Proof. Let $f \in \mathcal{ST}(\alpha, \varphi)$ and $g \in \mathcal{M}(\beta, \psi)$, $g = f^{-1}$. Then there exist analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, such that

$$\frac{zf'(z)}{f(z)}+\frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad (1-\beta)\frac{wg'(w)}{g(w)}+\beta\left(1+\frac{wg''(w)}{g'(w)}\right) = \psi(v(w)), \tag{2.21}$$

Since

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1+2\alpha)z + (2(1+3\alpha)a_3 - (1+2\alpha)a_2^2)z^2 + \cdots$$

and

$$(1-\beta)\frac{wg'(w)}{g(w)}+\beta\left(1+\frac{wg''(w)}{g'(w)}\right) = 1-(1+\beta)a_2w+((3+5\beta)a_2^2-2(1+2\beta)a_3)w^2+\cdots,$$

equations (2.10), (2.11) and (2.21) yield

$$a_2(1+2\alpha) = \frac{1}{2}B_1c_1, \tag{2.22}$$

$$2(1+3\alpha)a_3 - (1+2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \tag{2.23}$$

$$-(1+\beta)a_2 = \frac{1}{2}D_1b_1 \tag{2.24}$$

and

$$(3+5\beta)a_2^2 - 2(1+2\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2b_1^2. \tag{2.25}$$

It follows from (2.22) and (2.24) that

$$b_1 = -\frac{B_1(1+\beta)}{D_1(1+2\alpha)}c_1. \tag{2.26}$$

Multiplying (2.23) with $(1+2\alpha)$ and (2.25) with $(1+3\alpha)$, and adding the results give

$$
\begin{aligned}
a_2^2(2+7\alpha+3\beta+11\alpha\beta) &= \frac{B_1}{2}(1+2\beta)c_2+\frac{D_1}{2}(1+3\alpha)b_2 \\
&+\frac{c_1^2}{4}(1+2\beta)(B_2-B_1)+\frac{b_1^2}{4}(1+3\alpha)(D_2-D_1)
\end{aligned}
$$

Substituting c_1 from (2.22) and b_1 from (2.26) in the above equation give

$$
\begin{aligned}
&a_2^2(2+7\alpha+3\beta+11\alpha\beta) \\
&-\frac{a_2^2(1+2\alpha)^2}{B_1^2}\left((1+2\beta)(B_2-B_1)+\frac{(1+3\alpha)(D_2-D_1)(1+\beta)^2B_1^2}{(1+2\alpha)^2D_1^2}\right) \\
&=\frac{B_1}{2}(1+2\beta)c_2+\frac{D_1}{2}(1+3\alpha)b_2
\end{aligned}
$$

which lead to

$$
a_2^2=\frac{B_1^2D_1^2[B_1(1+2\beta)c_2+D_1(1+3\alpha)b_2]}{2[\rho B_1^2D_1^2-(1+2\alpha)^2(1+2\beta)(B_2-B_1)D_1^2-(1+\beta)^2(1+3\alpha)(D_2-D_1)B_1^2]},
$$

which gives us the desired estimate on $|a_2|$ as asserted in (2.19) when $|b_2|\leq 2$ and $|c_2|\leq 2$.

Multiplying (2.23) with $(3+5\beta)$ and (2.25) with $(1+2\alpha)$, and adding the results give

$$
\begin{aligned}
2a_3(2+7\alpha+3\beta+11\alpha\beta) &= \frac{B_1}{2}(3+5\beta)c_2+\frac{D_1}{2}(1+2\alpha)b_2 \\
&+\frac{c_1^2}{4}(3+5\beta)(B_2-B_1)+\frac{b_1^2}{4}(1+2\alpha)(D_2-D_1)
\end{aligned}
$$

Substituting b_1 from (2.26) in the above equation give

$$
\begin{aligned}
2a_3(2+7\alpha+3\beta+11\alpha\beta) &= \frac{B_1}{2}(3+5\beta)c_2+\frac{D_1}{2}(1+2\alpha)b_2 \\
&+\frac{c_1^2}{4}\left((3+5\beta)(B_2-B_1)+\frac{(1+\beta)^2(D_2-D_1)B_1^2}{D_1^2(1+2\alpha)}\right)
\end{aligned}
$$

which lead to

$$\begin{aligned}2\rho a_3 &= \frac{1}{2}[B_1(3+5\beta)c_2 + D_1(1+2\alpha)b_2]\\ &+ \frac{c_1^2}{4}\left[(3+5\beta)(B_2-B_1) + \frac{(1+\beta)^2B_1^2(D_2-D_1)}{D_1^2(1+2\alpha)}\right],\end{aligned}$$

where $\rho = 2+7\alpha+3\beta+11\alpha\beta$ and this yields the estimate given in (2.20). ∎

Theorem 2.4 *Let $f \in \mathcal{SB}$ and $g = f^{-1}$. If $f \in \mathcal{ST}(\alpha,\varphi)$ and $g \in \mathcal{L}(\beta,\psi)$, then*

$$|a_2| \leq \frac{B_1D_1\sqrt{2[B_1(3-2\beta)+D_1(1+3\alpha)]}}{\sqrt{|\rho B_1^2D_1^2 - 2(1+2\alpha)^2(3-2\beta)(B_2-B_1)D_1^2 - 2(2-\beta)^2(1+3\alpha)(D_2-D_1)B_1^2|}} \tag{2.27}$$

and

$$\begin{aligned}|\rho a_3| &\leq \frac{1}{2}B_1(\beta^2-11\beta+16) + D_1(1+2\alpha) + \frac{1}{2}(\beta^2-11\beta+16)|B_2-B_1|\\ &+ \frac{(2-\beta)^2B_1^2|D_2-D_1|}{D_1^2(1+2\alpha)}\end{aligned} \tag{2.28}$$

where $\rho := 10+36\alpha-7\beta-25\alpha\beta+\beta^2+3\alpha\beta^2$.

Proof. Let $f \in \mathcal{ST}(\alpha,\varphi)$ and $g \in \mathcal{L}(\beta,\psi)$. Then there are analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad \left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\beta} = \psi(v(w)), \tag{2.29}$$

Using

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1+2\alpha)z + (2(1+3\alpha)a_3 - (1+2\alpha)a_2^2)z^2 + \cdots,$$

$$\left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\beta}$$

$$= 1-(2-\beta)a_2 w + \Big((8(1-\beta)+\frac{1}{2}\beta(\beta+5))a_2^2 - 2(3-2\beta)a_3\Big)w^2 + \cdots,$$

and equations (2.10), (2.11) and (2.29) will yield

$$a_2(1+2\alpha) = \frac{1}{2}B_1 c_1, \tag{2.30}$$

$$2(1+3\alpha)a_3 - (1+2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2, \tag{2.31}$$

$$-(2-\beta)a_2 = \frac{1}{2}D_1 b_1 \tag{2.32}$$

and

$$[8(1-\beta)+\frac{\beta}{2}(\beta+5)]a_2^2 - 2(3-2\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2 b_1^2. \tag{2.33}$$

It follows from (2.30) and (2.32) that

$$b_1 = -\frac{B_1(2-\beta)}{D_1(1+2\alpha)}c_1. \tag{2.34}$$

Multiplying (2.31) with $(3-2\beta)$ and (2.33) with $(1+3\alpha)$, and adding the results give

$$a_2^2\left(5 - \frac{7\beta}{2} + 18\alpha - \frac{25\alpha\beta}{2} + \frac{\beta^2}{2} + \frac{3\alpha\beta^2}{2}\right) = \frac{B_1}{2}(3-2\beta)c_2 + \frac{D_1}{2}(1+3\alpha)b_2$$
$$+ \frac{c_1^2}{4}(3-2\beta)(B_2-B_1) + \frac{b_1^2}{4}(1+3\alpha)(D_2-D_1)$$

Substituting c_1 from (2.30) and b_1 from (2.34) in the above equation give

$$a_2^2\left(5 - \frac{7\beta}{2} + 18\alpha - \frac{25\alpha\beta}{2} + \frac{\beta^2}{2} + \frac{3\alpha\beta^2}{2}\right)$$
$$- \frac{a^2(1+2\alpha)^2}{B_1^2}\left((3-2\beta)(B_2-B_1) + \frac{B_1^2(2-\beta)^2}{D_1^2(1+2\alpha)^2}(1+3\alpha)(D_2-D_1)\right)$$

$$= \frac{B_1}{2}(3-2\beta)c_2 + \frac{D_1}{2}(1+3\alpha)b_2$$

which lead to

$$a_2^2 = \frac{B_1^2 D_1^2[B_1(3-2\beta)c_2 + D_1(1+3\alpha)b_2]}{\rho B_1^2 D_1^2 - 2(1+2\alpha)^2(3-2\beta)(B_2-B_1)D_1^2 - 2(2-\beta)^2(1+3\alpha)(D_2-D_1)B_1^2},$$

which again in view of $|b_2| \leq 2$ and $|c_2| \leq 2$ gives the desired estimate on $|a_2|$ as asserted in (2.27). Multiplying (2.31) with $[8(1-\beta) + \frac{\beta}{2}(\beta+5)]$ and (2.33) with $(1+2\alpha)$, and adding the results give

$$a_3(10+36\alpha-7\beta-25\alpha\beta+\beta^2+3\alpha\beta^2) = \frac{B_1}{4}(\beta^2-11\beta+16)c_2 + \frac{D_1}{2}(1+2\alpha)b_2$$
$$+\frac{c_1^2}{4}\left[8(1-\beta)+\frac{\beta}{2}(\beta+5)\right](B_2-B_1) + \frac{b_1^2}{4}(1+2\alpha)(D_2-D_1)$$

Substituting b_1 from (2.34) in the above equation give

$$a_3(10+36\alpha-7\beta-25\alpha\beta+\beta^2+3\alpha\beta^2) = \frac{B_1}{4}(\beta^2-11\beta+16)c_2 + \frac{D_1}{2}(1+2\alpha)b_2$$
$$+\frac{c_1^2}{4}\left[8(1-\beta)+\frac{\beta}{2}(\beta+5)\right](B_2-B_1) + \frac{c_1^2(2-\beta)^2B_1^2(D_2-D_1)}{4D_1^2(1+2\alpha)}$$

which lead to

$$\begin{aligned}\rho a_3 = {} & \frac{B_1}{4}(\beta^2-11\beta+16)c_2 + \frac{D_1}{2}(1+2\alpha)b_2 \\ & + \frac{c_1^2}{4}\left[\frac{1}{2}(\beta^2-11\beta+16)(B_2-B_1) + \frac{(2-\beta)^2B_1^2(D_2-D_1)}{D_1^2(1+2\alpha)}\right]\end{aligned}$$

where $\rho := 10+36\alpha-7\beta-25\alpha\beta+\beta^2+3\alpha\beta^2$ and this yields the estimate given in (2.28). ∎

2.4 Second and third coefficients of functions f when $f \in \mathcal{M}(\alpha, \varphi)$ and $g \in \mathcal{M}(\beta, \psi)$, or $g \in \mathcal{L}(\beta, \psi)$

Now we consider the case where $f \in \mathcal{M}(\alpha, \varphi)$ and its inverse g is either in $\mathcal{M}(\beta, \psi)$ or $\mathcal{L}(\beta, \psi)$. By using the similar technique, we determine the bound of second and third coefficients of f.

Theorem 2.5 *Let $f \in \mathcal{SB}$ and $g = f^{-1}$. If $f \in \mathcal{M}(\alpha, \varphi)$, $g \in \mathcal{M}(\beta, \psi)$, then*

$$|a_2| \leq \frac{B_1 D_1 \sqrt{B_1(1+2\beta) + D_1(1+2\alpha)}}{\sqrt{|\rho B_1^2 D_1^2 - (1+\alpha)^2(1+2\beta)(B_2 - B_1)D_1^2 - (1+\beta)^2(1+2\alpha)(D_2 - D_1)B_1^2|}} \tag{2.35}$$

and

$$2\rho|a_3| \leq B_1(3+5\beta) + D_1(1+3\alpha) + (3+5\beta)|B_2 - B_1| + \frac{(1+\beta)^2(1+3\alpha)B_1^2|D_2 - D_1|}{D_1^2(1+\alpha)^2} \tag{2.36}$$

where $\rho := 2 + 3\alpha + 3\beta + 4\alpha\beta$.

Proof. For $f \in \mathcal{M}(\alpha, \varphi)$ and $g \in \mathcal{M}(\beta, \psi)$, there exist analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\begin{aligned} (1-\alpha)\tfrac{zf'(z)}{f(z)} + \alpha\left(1 + \tfrac{zf''(z)}{f'(z)}\right) &= \varphi(u(z)), \\ (1-\beta)\tfrac{wg'(w)}{g(w)} + \beta\left(1 + \tfrac{wg''(w)}{g'(w)}\right) &= \psi(v(w)). \end{aligned} \tag{2.37}$$

Since

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + (1+\alpha)a_2 z + (2(1+2\alpha)a_3 - (1+3\alpha)a_2^2)z^2 + \cdots$$

and

$$(1-\beta)\frac{wg'(w)}{g(w)} + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 - (1+\beta)a_2 w + ((3+5\beta)a_2^2 - 2(1+2\beta)a_3)w^2 + \cdots,$$

then (2.10), (2.11) and (2.37) yield

$$a_2(1+\alpha) = \frac{1}{2}B_1c_1, \tag{2.38}$$

$$2(1+2\alpha)a_3 - (1+3\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \tag{2.39}$$

$$-(1+\beta)a_2 = \frac{1}{2}D_1b_1 \tag{2.40}$$

and

$$(3+5\beta)a_2^2 - 2(1+2\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2b_1^2. \tag{2.41}$$

$$b_1 = -\frac{B_1(1+\beta)}{D_1(1+\alpha)}c_1. \tag{2.42}$$

Multiplying (2.39) with $(1+2\beta)$ and (2.41) with $(1+2\alpha)$, and adding the results give

$$\begin{aligned} a_2^2(2+3\alpha+3\beta+4\alpha\beta) &= \frac{B_1}{2}(1+2\beta)c_2 + \frac{D_1}{2}(1+2\alpha)b_2 \\ + \frac{c_1^2}{4}(1+2\beta)(B_2-B_1) &+ \frac{b_1^2}{4}(1+2\alpha)(D_2-D_1) \end{aligned}$$

Substituting c_1 from (2.38) and b_1 from (2.42) in the above equation give

$$\begin{aligned} a_2^2(2+3\alpha+3\beta+4\alpha\beta) &= \frac{B_1}{2}(1+2\beta)c_2 + \frac{D_1}{2}(1+2\alpha)b_2 \\ + \frac{(1+\alpha)^2a_2^2}{B_1^2}&\left((1+2\beta)(B_2-B_1) + \frac{(1+\beta)^2(1+2\alpha)(D_2-D_1)B_1^2}{(1+\alpha)^2D_1^2}\right) \end{aligned}$$

which lead to

$$a_2^2 = \frac{B_1^2D_1^2[B_1(1+2\beta)c_2 + D_1(1+2\alpha)b_2]}{2\rho B_1^2D_1^2 - 2(1+\alpha)^2(1+2\beta)(B_2-B_1)D_1^2 - 2(1+\beta)^2(1+2\alpha)(D_2-D_1)B_1^2},$$

which, in view of $|b_2| \le 2$ and $|c_2| \le 2$ gives the desired estimate on $|a_2|$ as asserted in (2.35).

Multiplying (2.39) with $(3+5\beta)$ and (2.41) with $(1+3\alpha)$, and adding the results give

$$\begin{aligned}2\rho a_3 = {} & \frac{B_1}{2}(3+5\beta)c_2 + \frac{D_1}{2}(1+3\alpha)b_2 + \frac{c_1^2}{4}(3+5\beta)(B_2-B_1) \\ & + \frac{b_1^2}{4}(1+3\alpha)(D_2-D_1)\end{aligned}$$

Substituting b_1 from (2.42) in the above equation give

$$\begin{aligned}2\rho a_3 = {} & \frac{B_1}{2}(3+5\beta)c_2 + \frac{D_1}{2}(1+3\alpha)b_2 + \frac{c_1^2}{4}(3+5\beta)(B_2-B_1) \\ & + \frac{(1+\beta)^2(1+3\alpha)B_1^2(D_2-D_1)}{D_1^2(1+\alpha)^2}\Big]\end{aligned}$$

where $\rho := 2+3\alpha+3\beta+4\alpha\beta$ and the estimate in (2.36) is obtained. ∎

When $\beta = \alpha$ and $\psi = \varphi$, Theorem 2.5 reduces to the following corollary.

Corollary 2.4 *Let f be given by* (2.1) *and $g = f^{-1}$. If $f, g \in \mathcal{M}(\alpha, \varphi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{(1+\alpha)|B_1^2 + (1+\alpha)(B_1 - B_2)|}} \tag{2.43}$$

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{1+\alpha}. \tag{2.44}$$

For $\alpha = 0$, Corollary 2.4 gives the coefficient estimates for Ma-Minda bi-starlike functions, while for $\alpha = 1$, it gives the following estimates for Ma-Minda bi-convex functions.

Corollary 2.5 *Let f given by* (2.1) *be in the class of Ma-Minda bi-convex func-*

tions. Then

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2|B_1^2 + 2B_1 - 2B_2|}} \quad \textit{and} \quad |a_3| \leq \frac{1}{2}(B_1 + |B_2 - B_1|).$$

For φ given by

$$\varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + \cdots ,$$

evidently $B_1 = B_2 = 2(1 - \beta)$, and thus when $\alpha = 1$ (bi-convex functions), Corollary 2.4 reduces to the following result.

Corollary 2.6 [22, Theorem 4.1] *Let f given by* (2.1) *be in the class of bi-convex functions of order β, $0 < \beta \leq 1$. Then*

$$|a_2| \leq \sqrt{1 - \beta} \quad \textit{and} \quad |a_3| \leq 1 - \beta.$$

Theorem 2.6 *Let $f \in \mathcal{SB}$ and $g = f^{-1}$. If $f \in \mathcal{M}(\alpha, \varphi)$ and $g \in \mathcal{L}(\beta, \psi)$, then*

$$|a_2| \leq \frac{B_1 D_1 \sqrt{2[B_1(3 - 2\beta) + D_1(1 + 2\alpha)]}}{\sqrt{|\rho B_1^2 D_1^2 - 2(1 + \alpha)^2(3 - 2\beta)(B_2 - B_1)D_1^2 - 2(2 - \beta)^2(1 + 2\alpha)(D_2 - D_1)B_1^2|}} \tag{2.45}$$

and

$$\begin{aligned} |\rho a_3| \leq & \frac{B_1}{2}(\beta^2 - 11\beta + 16) + D_1(1 + 3\alpha) + \frac{1}{2}(\beta^2 - 11\beta + 16)|B_2 - B_1| \\ & + \frac{(2 - \beta)^2(1 + 3\alpha)B_1^2|D_2 - D_1|}{D_1^2(1 + \alpha)^2} \end{aligned} \tag{2.46}$$

where $\rho := 10 + 14\alpha - 7\beta + \beta^2 + 2\alpha\beta^2 - 10\alpha\beta$.

Proof. Let $f \in \mathcal{M}(\alpha, \varphi)$ and $g \in \mathcal{L}(\beta, \psi)$. Then there are analytic functions

$u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$
\begin{aligned}
&(1-\alpha)\tfrac{zf'(z)}{f(z)} + \alpha\left(1 + \tfrac{zf''(z)}{f'(z)}\right) = \varphi(u(z)), \\
&\left(\tfrac{wg'(w)}{g(w)}\right)^{\beta}\left(1 + \tfrac{wg''(w)}{g'(w)}\right)^{1-\beta} = \psi(v(w)).
\end{aligned}
\tag{2.47}
$$

Since

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + (1+\alpha)a_2 z + (2(1+2\alpha)a_3 - (1+3\alpha)a_2^2)z^2 + \cdots$$

and

$$
\begin{aligned}
&\left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\beta} \\
&= 1 - (2-\beta)a_2 w + \left((8(1-\beta) + \frac{1}{2}\beta(\beta+5))a_2^2 - 2(3-2\beta)a_3\right)w^2 + \cdots,
\end{aligned}
$$

equations (2.10), (2.11) and (2.47) yield

$$a_2(1+\alpha) = \frac{1}{2}B_1 c_1, \tag{2.48}$$

$$2(1+2\alpha)a_3 - (1+3\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2, \tag{2.49}$$

$$-(2-\beta)a_2 = \frac{1}{2}D_1 b_1 \tag{2.50}$$

and

$$[8(1-\beta) + \frac{\beta}{2}(\beta+5)]a_2^2 - 2(3-2\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2 b_1^2. \tag{2.51}$$

It follows from (2.48) and (2.50) that

$$b_1 = -\frac{B_1(2-\beta)}{D_1(1+\alpha)}c_1. \tag{2.52}$$

Multiplying (2.49) with $(3-2\beta)$ and (2.51) with $(1+2\alpha)$, and adding the results give

$$\frac{a_2^2}{2}(10+14\alpha-7\beta+\beta^2+2\alpha\beta^2-10\alpha\beta)=\frac{B_1}{2}(3-2\beta)c_2+\frac{D_1}{2}(1+2\alpha)b_2$$
$$+\frac{c_1^2}{4}(3-2\beta)(B_2-B_1)+\frac{b_1^2}{4}(1+2\alpha)(D_2-D_1)$$

Substituting c_1 from (2.48) and b_1 from (2.52) in the above equation give

$$a_2^2\left(\frac{1}{2}(10+14\alpha-7\beta+\beta^2+2\alpha\beta^2-10\alpha\beta)-\frac{(1+\alpha)^2(3-2\beta)(B_2-B_1)}{B_1^2}\right.$$
$$\left.-\frac{(2-\beta)^2(1+2\alpha)(D_2-D_1)}{D_1^2}\right)=\frac{B_1}{2}(3-2\beta)c_2+\frac{D_1}{2}(1+2\alpha)b_2$$

which lead to

$$a_2^2=\frac{B_1^2D_1^2[B_1(3-2\beta)c_2+D_1(1+2\alpha)b_2]}{\rho B_1^2D_1^2-2(1+\alpha)^2(3-2\beta)(B_2-B_1)D_1^2-2(2-\beta)^2(1+2\alpha)(D_2-D_1)B_1^2}.$$

Since $|b_2|\leq 2$ and $|c_2|\leq 2$ the desired estimate on $|a_2|$ as asserted in (2.45) is obtained.

Multiplying (2.49) with $[8(1-\beta)+\frac{\beta}{2}(\beta+5)]$ and (2.51) with $(1+3\alpha)$, and adding the results give

$$\rho a_3=\frac{B_1}{4}(\beta^2-11\beta+16)c_2+\frac{D_1}{2}(1+3\alpha)b_2+\frac{c_1^2}{4}(\beta^2-11\beta+16)(B_2-B_1)$$
$$+\frac{b_1^2}{4}(1+3\alpha)(D_2-D_1)$$

Substituting b_1 from (2.52) in the above equation give

$$\rho a_3=\frac{B_1}{4}(\beta^2-11\beta+16)c_2+\frac{D_1}{2}(1+3\alpha)b_2+\frac{c_1^2}{4}\Big[(\beta^2-11\beta+16)(B_2-B_1)$$
$$+\frac{(2-\beta)^2(1+3\alpha)B_1^2(D_2-D_1)}{D_1^2(1+\alpha)^2}\Big]$$

where $\rho := 10 + 14\alpha - 7\beta + \beta^2 + 2\alpha\beta^2 - 10\alpha\beta$ and this yields the estimate given in (2.46). ∎

2.5 Second and third coefficients of functions f when $f \in \mathcal{L}(\alpha, \varphi)$ and $g \in \mathcal{L}(\beta, \psi)$

Theorem 2.7 *Let $f \in \mathcal{SB}$ and $g = f^{-1}$. If $f \in \mathcal{L}(\alpha, \varphi)$ and $g \in \mathcal{L}(\beta, \psi)$, then*

$$|a_2| \leq \frac{B_1 D_1 \sqrt{2[B_1(3-2\beta) + D_1(3-2\alpha)]}}{\sqrt{|\rho B_1^2 D_1^2 - 2(2-\alpha)^2(3-2\beta)(B_2 - B_1)D_1^2 - 2(2-\beta)^2(3-2\alpha)(D_2 - D_1)B_1^2|}} \tag{2.53}$$

and

$$\begin{aligned} 2|\rho a_3| \leq\ & B_1(\beta^2 - 11\beta + 16) + D_1(8 - 5\alpha - \alpha^2) + (\beta^2 - 11\beta + 16)|B_2 - B_1| \\ & + \frac{(2-\beta)^2(\alpha^2 + 5\alpha - 8)B_1^2|D_2 - D_1|}{D_1^2(2-\alpha)^2} \end{aligned} \tag{2.54}$$

where $\rho := 24 + 3\alpha^2 + 3\beta^2 - 17\alpha - 17\beta - 2\beta\alpha^2 - 2\alpha\beta^2 + 12\alpha\beta$.

Proof. Let $f \in \mathcal{L}(\alpha, \varphi)$ and $g \in \mathcal{L}(\beta, \psi)$. Then there are analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\begin{aligned} \left(\tfrac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \tfrac{zf''(z)}{f'(z)}\right)^{1-\alpha} &= \varphi(u(z)), \\ \left(\tfrac{wg'(w)}{g(w)}\right)^{\beta} \left(1 + \tfrac{wg''(w)}{g'(w)}\right)^{1-\beta} &= \psi(v(w)). \end{aligned} \tag{2.55}$$

Since

$$\begin{aligned} & \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \\ & = 1 + (2-\alpha)a_2 z + \left(2(3-2\alpha)a_3 + \frac{(\alpha-2)^2 - 3(4-3\alpha)}{2} a_2^2\right) z^2 + \cdots \end{aligned}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\beta}$$
$$= 1-(2-\beta)a_2w+\Big((8(1-\beta)+\frac{1}{2}\beta(\beta+5))a_2^2-2(3-2\beta)a_3\Big)w^2+\cdots,$$

equations (2.10), (2.11) and (2.55) yield

$$a_2(2-\alpha)=\frac{1}{2}B_1c_1, \tag{2.56}$$

$$2(3-2\alpha)a_3+\frac{1}{2}[(\alpha-2)^2-3(4-3\alpha)]a_2^2=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2, \tag{2.57}$$

$$-(2-\beta)a_2=\frac{1}{2}D_1b_1 \tag{2.58}$$

and

$$[8(1-\beta)+\frac{\beta}{2}(\beta+5)]a_2^2-2(3-2\beta)a_3=\frac{1}{2}D_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}D_2b_1^2. \tag{2.59}$$

It follows from (2.56) and (2.58) that

$$b_1=-\frac{B_1(2-\beta)}{D_1(2-\alpha)}c_1. \tag{2.60}$$

Multiplying (2.57) with $(3-2\beta)$ and (2.59) with $(3-2\alpha)$, and adding the results give

$$\frac{1}{2}a_2^2(24+3\alpha^2+3\beta^2-17\alpha-17\beta-2\beta\alpha^2-2\alpha\beta^2+12\alpha\beta)$$
$$=\frac{B_1}{2}(3-2\beta)c_2+\frac{D_1}{2}(3-2\alpha)b_2+\frac{c_1^2}{4}(3-2\beta)(B_2-B_1)+\frac{b_1^2}{4}(3-2\alpha)(D_2-D_1)$$

Substituting c_1 from (2.56) and b_1 from (2.60) in the above equation give

$$\begin{aligned}&\frac{1}{2}a_2^2(24+3\alpha^2+3\beta^2-17\alpha-17\beta-2\beta\alpha^2-2\alpha\beta^2+12\alpha\beta)\\&=\frac{B_1}{2}(3-2\beta)c_2+\frac{D_1}{2}(3-2\alpha)b_2+\frac{(2-\alpha)^2a_2^2}{B_1^2}\Big((3-2\beta)(B_2-B_1)\\&\quad+\frac{(2-\beta)^2B_1^2(3-2\alpha)(D_2-D_1)}{(2-\alpha)^2D_1^2}\Big)\end{aligned}$$

and so

$$a_2^2=\frac{B_1^2D_1^2[B_1(3-2\beta)c_2+D_1(3-2\alpha)b_2]}{\rho B_1^2D_1^2-2(2-\alpha)^2(3-2\beta)(B_2-B_1)D_1^2-2(2-\beta)^2(3-2\alpha)(D_2-D_1)B_1^2},$$

which, again by using $|b_2|\leq 2$ and $|c_2|\leq 2$ gives the estimate on $|a_2|$ as asserted in (2.53).

Multiplying (2.57) with $(\beta^2-11\beta+16)$ and (2.59) with $(\alpha^2+5\alpha-8)$, and adding the results give

$$\begin{aligned}2\rho a_3=&\frac{B_1}{2}(\beta^2-11\beta+16)c_2+\frac{D_1}{2}(8-5\alpha-\alpha^2)b_2+\frac{c_1^2}{4}(\beta^2-11\beta+16)(B_2-B_1)\\&+\frac{b_1^2}{4}(\alpha^2+5\alpha-8)(D_2-D_1)\end{aligned}$$

Substituting b_1 from (2.60) in the above equation give

$$\begin{aligned}2\rho a_3=&\frac{B_1}{2}(\beta^2-11\beta+16)c_2+\frac{D_1}{2}(8-5\alpha-\alpha^2)b_2+\frac{c_1^2}{4}\Big[(\beta^2-11\beta+16)(B_2-B_1)\\&+\frac{(2-\beta)^2(\alpha^2+5\alpha-8)B_1^2(D_2-D_1)}{D_1^2(2-\alpha)^2}\Big]\end{aligned}$$

where $\rho:=24+3\alpha^2+3\beta^2-17\alpha-17\beta-2\beta\alpha^2-2\alpha\beta^2+12\alpha\beta$ and this yields the estimate given in (2.54). ∎

When $\beta=\alpha$ and $\psi=\varphi$, Theorem 2.7 reduces to the following corollary.

Corollary 2.7 *Let f be given by (2.1) and $g = f^{-1}$. If $f, g \in \mathcal{L}(\alpha, \varphi)$ then*

$$|a_2| \leq \frac{2B_1\sqrt{B_1}}{\sqrt{|2(\alpha^2 - 3\alpha + 4)B_1^2 + 4(\alpha - 2)^2(B_1 - B_2)|}} \tag{2.61}$$

and

$$|a_3| \leq \frac{2(3 - 2\alpha)\Big(B_1 + |B_1 - B_2|\Big)}{|(3 - 2\alpha)(\alpha^2 - 3\alpha + 4)|}. \tag{2.62}$$

2.6 Second and third coefficients of functions f when $f \in \mathcal{H}_{\mathcal{SB}}(\varphi)$

A function $f \in \mathcal{A}$ with $\mathrm{Re}(f'(z)) > 0$ is known to be univalent (Theorem 1.10). This motivates the following class of functions studied in [118].

Definition 2.2 *A function $f \in \mathcal{SB}$ is said to be in the class $\mathcal{H}_{\mathcal{SB}}(\varphi)$ if the following subordinations hold:*

$$f'(z) \prec \varphi(z) \quad \textit{and} \quad g'(w) \prec \varphi(w),$$

where φ is as given in Definition 2.1

For functions in the class $\mathcal{H}_{\mathcal{SB}}(\varphi)$, we have the following coefficient bounds.

Theorem 2.8 *If $f \in \mathcal{H}_{\mathcal{SB}}(\varphi)$ is given by (2.1), then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|3B_1^2 - 4B_2 + 4B_1|}} \quad \textit{and} \quad |a_3| \leq \left(\frac{1}{3} + \frac{B_1}{4}\right) B_1. \tag{2.63}$$

Proof. Let $f \in \mathcal{H}_{\mathcal{SB}}(\varphi)$ and $g = f^{-1}$. Similar computations as in earlier theorems give

$$2a_2 = \frac{1}{2}B_1c_1, \tag{2.64}$$

$$3a_3 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \tag{2.65}$$

$$-2a_2 = \frac{1}{2}B_1 b_1 \tag{2.66}$$

and

$$3(2a_2^2 - a_3) = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2. \tag{2.67}$$

From (2.64) and (2.66), it follows that

$$c_1 = -b_1. \tag{2.68}$$

Now (2.65), (2.66), (2.67) and (2.68) yield

$$a_2^2 = \frac{B_1^3(b_2 + c_2)}{4(3B_1^2 - 4B_2 + 4B_1)},$$

which, in view of $|b_2| \leq 2$ and $|c_2| \leq 2$ gives the desired estimate on $|a_2|$ in (2.63).

By subtracting (2.67) from (2.65), further computation using (2.64) and (2.68) leads to

$$a_3 = \frac{1}{12}B_1(c_2 - b_2) + \frac{1}{16}B_1^2 c_1^2,$$

and this yields the estimate given in (2.63). ∎

For φ given by

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha, \quad 0 < \alpha \leq 1,$$

the inequalities in (2.63) reduce to the following result.

Corollary 2.8 [118, Theorem 1, p. 3] *Let* $f \in \mathcal{H}_\Sigma^\alpha := \mathcal{H}_{\mathcal{SB}}((\frac{1+z}{1-z})^\alpha)$. *Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+2}} \quad \textit{and} \quad |a_3| \leq \frac{\alpha(3\alpha+2)}{3}.$$

In the case when

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z}, \quad 0 \leq \beta < 1,$$

the inequalities in (2.63) reduce to the following result.

Corollary 2.9 [118, Theorem 2, p.4] *Let* $f \in \mathcal{H}_{\Sigma}(\beta) := \mathcal{H}_{\mathcal{SB}}(\frac{1+(1-2\beta)z}{1-z})$. *Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}} \quad \textit{and} \quad |a_3| \leq \frac{(1-\beta)(5-3\beta)}{3}.$$

CHAPTER 3

BOUNDS FOR THE SECOND HANKEL DETERMINANT OF UNIVALENT FUNCTIONS

3.1 Introduction and preliminaries

Recall that $\mathcal{A}$ denote the class of all analytic functions defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{3.1}$$

The *Hankel determinants* $H_q(n)$, $(n = 1, 2, \ldots,\ q = 1, 2, \ldots)$ of the function f are defined by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1).$$

In general, Hankel determinants are useful in the study of the singularities and in the study of power series with integral coefficients. For example, in showing that a function of bounded characteristic in $\mathbb{D}$, i.e., a function which is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral coefficients, is rational [27]. For the use of Hankel determinant in the study of meromorphic functions, see [132]. Various properties of these determinants can be found in [129, Chapter 4]. In 1966, Pommerenke [99] investigated the Hankel determinant of areally mean p-valent functions, univalent functions as well as for starlike functions. In [100], he proved that the Hankel determinants of univalent functions satisfy

$$|H_q(n)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}} \quad (n = 1, 2, \dots, \ q = 2, 3, \dots),$$

where $\beta > 1/4000$ and K depends only on q. Later, Hayman [43] proved that $|H_2(n)| < An^{1/2}$, ($n = 1, 2, \dots$; A an absolute constant) for areally mean univalent functions. In [71–73], the estimates for Hankel determinant for areally mean p-valent functions were investigated. ElHosh obtained bounds for Hankel determinants of univalent functions with positive Hayman index α [30] and of k-fold symmetric and close-to-convex functions [31]. For bounds on the Hankel determinants of close-to-convex functions, see [78, 79, 81]. Noor and Al-Bany studied the Hankel determinant of Bazilevic functions in [77] and of functions with bounded boundary rotation in [74–76, 80]. The Hankel determinant $H_2(1) = a_3 - a_2^2$ is the well known Fekete-Szegö functional. For results related to this functional, see [7,8].

The second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2a_4 - a_3^2$. In the recent years, several authors have investigated bounds for the second Hankel determinant of functions belonging to various subclasses of univalent and multivalent functions [16, 40–42, 44, 63, 65, 66].

In this chapter, the bounds for $H_2(2)$ are obtained for functions belonging to subclasses of Ma-Minda starlike and convex functions in Section 2. In Section 3, the problem is investigated for two other related classes defined by subordination. In proving our results, we do not assume the univalence or starlikeness of φ as they were required only in obtaining the distortion, growth estimates and the convolution theorems. The classes introduced by subordination naturally include several well known classes of univalent functions and the results for some of these special classes are indicated as corollaries.

Recall the class $\mathcal{P}$ of *functions with positive real part* consisting of all analytic functions $p : \mathbb{D} \to \mathbb{C}$ satisfying $p(0) = 1$ and $\operatorname{Re} p(z) > 0$.

Lemma 3.1 [37] *If the function* $p \in \mathcal{P}$ *is given by the series* $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, *then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{3.2}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{3.3}$$

for some x, z *with* $|x| \leq 1$ *and* $|z| \leq 1$.

Another result that will be required is the optimal value of a quadratic expression. Standard computations show that

$$\max_{0 \leq t \leq 4}(Pt^2 + Qt + R) = \begin{cases} R, & Q \leq 0,\ P \leq -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \geq 0,\ P \geq -\frac{Q}{8} \text{ or } Q \leq 0,\ P \geq -\frac{Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0,\ P \leq -\frac{Q}{8}. \end{cases} \tag{3.4}$$

3.2 Second Hankel determinant of Ma-Minda starlike/convex functions

Various subclasses of starlike functions are characterized by the quantity $zf'(z)/f(z)$ lying in some domain in the right half-plane. For example, f is strongly starlike of order β if $zf'(z)/f(z)$ lies in a sector $|\arg w| < \beta\pi/2$ while it is starlike of order α if $zf'(z)/f(z)$ lies in the half-plane $\operatorname{Re} w > \alpha$. The various subclasses of starlike functions were unified by subordination in [59]. The following definition of the class of Ma-Minda starlike functions is the same as the one in [59] except for the omission of starlikeness assumption of φ.

Definition 3.1 *Let* $\varphi : \mathbb{D} \to \mathbb{C}$ *be analytic and the Maclaurin series of* φ *is given by*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{3.5}$$

The class $\mathcal{ST}(\varphi)$ *of* Ma-Minda starlike functions with respect to φ *consists of functions* $f \in \mathcal{A}$ *satisfying the subordination*

$$\frac{zf'(z)}{f(z)} \prec \varphi(z).$$

Several choices of φ would reduce the class $\mathcal{ST}(\varphi)$ to some known subclasses. For the function φ given by $\varphi_\alpha(z) := (1+(1-2\alpha)z)/(1-z)$, $0 < \alpha \leq 1$, the class $\mathcal{ST}(\alpha) := \mathcal{ST}(\varphi_\alpha)$ is the well-known class of starlike functions of order α. Let

$$\varphi(z) = \varphi_{PAR}(z) := 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2.$$

Then the class

$$\mathcal{ST}_P := \mathcal{ST}(\varphi_{PAR}) = \left\{f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|\right\}$$

is the *parabolic starlike* functions introduced by Rønning [106]. For a survey of parabolic starlike functions and the related class of uniformly convex functions, see [11]. For $0 < \beta \leq 1$, the class

$$\mathcal{ST}_\beta := \mathcal{ST}\left(\left(\frac{1+z}{1-z}\right)^\beta\right) = \left\{f \in \mathcal{A} : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2}\right\}$$

is the familiar class of *strongly starlike functions of order* β. The class

$$\mathcal{ST}_L := \mathcal{ST}(\sqrt{1+z}) = \left\{f \in \mathcal{A} : \left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1\right\}$$

is the class of *lemniscate starlike* functions studied in [116].

Now we determine the bound for the second Hankel determinant for the class $\mathcal{ST}(\varphi)$.

Theorem 3.1 *Let the function* $f \in \mathcal{ST}(\varphi)$ *be given by* (3.1).

1. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| \leq B_1, \quad |4B_1B_3 - B_1^4 - 3B_2^2| - 3B_1^2 \leq 0,$$

then the second Hankel determinant of f satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{4}.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| \geq B_1, \quad |4B_1B_3 - B_1^4 - 3B_2^2| - B_1|B_2| - 2B_1^2 \geq 0,$$

or the conditions

$$|B_2| \leq B_1, \quad |4B_1B_3 - B_1^4 - 3B_2^2| - 3B_1^2 \geq 0,$$

then the second Hankel determinant of f satisfies

$$|a_2a_4 - a_3^2| \leq \frac{1}{12}|4B_1B_3 - B_1^4 - 3B_2^2|.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| > B_1, \quad |4B_1B_3 - B_1^4 - 3B_2^2| - B_1|B_2| - 2B_1^2 \leq 0,$$

then the second Hankel determinant of f satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{12}\left(\frac{3|4B_1B_3 - B_1^4 - 3B_2^2| - 4B_1|B_2| - 4B_1^2 - B_2^2}{|4B_1B_3 - B_1^4 - 3B_2^2| - 2B_1|B_2| - B_1^2}\right).$$

Proof. Since $f \in \mathcal{ST}(\varphi)$, there exists an analytic function w with $w(0) = 0$ and

$|w(z)| < 1$ in $\mathbb{D}$ such that

$$\frac{zf'(z)}{f(z)} = \varphi(w(z)). \tag{3.6}$$

Define the functions p_1 by

$$p_1(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots$$

or equivalently,

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right). \tag{3.7}$$

Then p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$ and has positive real part in $\mathbb{D}$. By using (3.7) together with (3.5), it is evident that

$$\varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) z^2 + \cdots. \tag{3.8}$$

Since

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (-a_2^2 + 2a_3)z^2 + (3a_4 - 3a_2 a_3 + a_2^3)z^3 + \cdots, \tag{3.9}$$

it follows by (3.6), (3.8) and (3.9) that

$$\begin{aligned}
a_2 &= \frac{B_1 c_1}{2}, \\
a_3 &= \frac{1}{8}\left[(B_1^2 - B_1 + B_2)c_1^2 + 2B_1 c_2\right], \\
a_4 &= \frac{1}{48}[(-4B_2 + 2B_1 + B_1^3 - 3B_1^2 + 3B_1 B_2 + 2B_3)c_1^3 \\
&\quad + 2(3B_1^2 - 4B_1 + 4B_2)c_1 c_2 + 8B_1 c_3].
\end{aligned}$$

Hence

$$a_2a_4 - a_3^2 = \frac{B_1}{96}\left[c_1^4\left(-\frac{B_1^3}{2} + \frac{B_1}{2} - B_2 + 2B_3 - \frac{3B_2^2}{2B_1}\right)\right.$$
$$\left. +2c_2c_1^2(B_2 - B_1) + 8B_1c_1c_3 - 6B_1c_2^2\right].$$

Let

$$\begin{aligned} &d_1 = 8B_1, \quad d_2 = 2(B_2 - B_1), \\ &d_3 = -6B_1, \quad d_4 = -\tfrac{B_1^3}{2} + \tfrac{B_1}{2} - B_2 + 2B_3 - \tfrac{3B_2^2}{2B_1}, \\ &T = \tfrac{B_1}{96}. \end{aligned} \tag{3.10}$$

Then

$$|a_2a_4 - a_3^2| = T|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|. \tag{3.11}$$

Since the function $p(e^{i\theta}z)$ $(\theta \in \mathbb{R})$ is in the class $\mathcal{P}$ for any $p \in \mathcal{P}$, there is no loss of generality in assuming $c_1 > 0$. Write $c_1 = c$, $c \in [0, 2]$. Substituting the values of c_2 and c_3 respectively from (3.2) and (3.3) into (3.11), gives

$$|a_2a_4 - a_3^2| = \frac{T}{4}\left|c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4 - c^2)(d_1 + d_2 + d_3)\right.$$
$$\left. +(4 - c^2)x^2(-d_1c^2 + d_3(4 - c^2)) + 2d_1c(4 - c^2)(1 - |x|^2)z\right|.$$

Replacing $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (3.10), yield

$$\begin{aligned} |a_2a_4 - a_3^2| &\le \frac{T}{4}\Big[c^4\left|-2B_1^3 + 8B_3 - 6\frac{B_2^2}{B_1}\right| + 4|B_2|\mu c^2(4 - c^2) \\ &\quad + \mu^2(4 - c^2)(2B_1c^2 + 24B_1) + 16B_1c(4 - c^2)(1 - \mu^2)\Big] \\ &= T\Big[\frac{c^4}{4}\left|-2B_1^3 + 8B_3 - 6\frac{B_2^2}{B_1}\right| + 4B_1c(4 - c^2) + |B_2|(4 - c^2)\mu c^2 \\ &\quad + \frac{B_1}{2}\mu^2(4 - c^2)(c - 6)(c - 2)\Big] \end{aligned} \tag{3.12}$$

$$\equiv F(c,\mu).$$

Note that for $(c,\mu) \in [0,2] \times [0,1]$, differentiating $F(c,\mu)$ in (3.12) partially with respect to μ yields

$$\frac{\partial F}{\partial \mu} = T\left[|B_2|(4-c^2)c^2 + B_1\mu(4-c^2)(c-2)(c-6)\right]. \tag{3.13}$$

Then for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$, it is clear from (3.13) that $\frac{\partial F}{\partial \mu} > 0$, that is, $F(c,\mu)$ is an increasing function of μ. Hence for fixed $c \in [0,2]$, the maximum of $F(c,\mu)$ occurs at $\mu = 1$, and

$$\max F(c,\mu) = F(c,1) \equiv G(c),$$

where

$$G(c) = \frac{B_1}{96}\left[\frac{c^4}{4}\left(\left|-2B_1^3 + 8B_3 - 6\frac{B_2^2}{B_1}\right| - 4|B_2| - 2B_1\right) + 4c^2(|B_2| - B_1) + 24B_1\right].$$

Let

$$\begin{aligned} P &= \frac{1}{4}\left(\left|-2B_1^3 + 8B_3 - 6\frac{B_2^2}{B_1}\right| - 4|B_2| - 2B_1\right), \\ Q &= 4(|B_2| - B_1), \\ R &= 24B_1. \end{aligned} \tag{3.14}$$

Then $G(c) = \frac{B_1}{96}(Pt^2 + Qt + R)$, where $t = c^2$. Using (3.4) we have

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{96}\begin{cases} R, & Q \leq 0,\ P \leq -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \geq 0,\ P \geq -\frac{Q}{8} \text{ or } Q \leq 0,\ P \geq -\frac{Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0,\ P \leq -\frac{Q}{8} \end{cases}$$

where P, Q, R are given by (3.14).

Hence for $|B_2| \leq B_1$, $|4B_1B_3 - B_1^4 - 3B_2^2| - 3B_1^2 \leq 0$, then

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{4}.$$

For $|B_2| \geq B_1$, $|4B_1B_3 - B_1^4 - 3B_2^2| - B_1|B_2| - 2B_1^2 \geq 0$, or for $|B_2| \leq B_1$, $|4B_1B_3 - B_1^4 - 3B_2^2| - 3B_1^2 \geq 0$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{12}|4B_1B_3 - B_1^4 - 3B_2^2|.$$

And for $|B_2| > B_1$, $|4B_1B_3 - B_1^4 - 3B_2^2| - B_1|B_2| - 2B_1^2 \leq 0$, then

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{12}\left(\frac{3|4B_1B_3 - B_1^4 - 3B_2^2| - 4B_1|B_2| + 4B_1^2 - B_2^2}{|4B_1B_3 - B_1^4 - 3B_2^2| - 2B_1|B_2| - B_1^2}\right).$$

This completes the proof. ∎

In the case when $B_1 = B_2 = B_3 = 2$, Theorem 3.1 reduces to the following corollary.

Corollary 3.1 [44, Theorem 3.1] *Let $f \in \mathcal{ST}$. Then $|a_2a_4 - a_3^2| \leq 1$.*

Corollary 3.2

1. *If $f \in \mathcal{ST}(\alpha)$, then $|a_2a_4 - a_3^2| \leq (1-\alpha)^2$.*

2. *If $f \in \mathcal{ST}_L$, then* $|a_2a_4 - a_3^2| \leq 1/16 = 0.0625$.

3. *If $f \in \mathcal{ST}_P$, then* $|a_2a_4 - a_3^2| \leq 16/\pi^4 \approx 0.164255$.

4. *If $f \in \mathcal{ST}_\beta$, then* $|a_2a_4 - a_3^2| \leq \beta^2$.

Proof. 1. If $f \in \mathcal{ST}(\alpha)$, then $\varphi(z) = (1+(1-2\alpha)z)/(1-z)$, and so $B_1 = B_2 = B_3 = 2(1-\alpha)$. Hence $|a_2a_4 - a_3^2| \leq (1-\alpha)^2$.

2. If $f \in \mathcal{ST}_L$, then $\varphi(z) = \sqrt{1+z}$, and so $B_1 = 1/2$, $B_2 = -1/8$ and $B_3 = 1/16$. Hence $|a_2a_4 - a_3^2| \leq 1/16$.

3. If $f \in \mathcal{ST}_P$, then $\varphi(z) = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$, and so $B_1 = 8/\pi^2$, $B_2 = 16/(3\pi^2)$ and $B_3 = 184/(45\pi^2)$. Hence $|a_2a_4 - a_3^2| \leq 16/\pi^4$.

4. If $f \in \mathcal{ST}_\beta$, then $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\beta$, and so $B_1 = 2\beta$, $B_2 = 2\beta^2$ and $B_3 = 2\beta(1+2\beta^2)/3$. Hence $|a_2a_4 - a_3^2| \leq \beta^2$. ∎

After considering the Ma-Minda starlike functions, now we consider the Ma-Minda convex functions with respect to φ and its bound for the second Hankel determinant can be determined similarly.

Definition 3.2 *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic and is given as in* (3.5). *The class $\mathcal{CV}(\varphi)$ of* Ma-Minda convex functions with respect to φ *consists of functions f satisfying the subordination*

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z).$$

Theorem 3.2 *Let the function $f \in \mathcal{CV}(\varphi)$ be given by* (3.1).

1. *If B_1, B_2 and B_3 satisfy the conditions*

$$B_1^2 + 4|B_2| - 2B_1 \leq 0, \quad |6B_1B_3 + B_1^2B_2 - B_1^4 - 4B_2^2| - 4B_1^2 \leq 0,$$

then the second Hankel determinant of f satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{36}.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$B_1^2+4|B_2|-2B_1 \geq 0, \quad 2|6B_1B_3+B_1^2B_2-B_1^4-4B_2^2|-B_1^3-4B_1|B_2|-6B_1^2 \geq 0,$$

or the conditions

$$B_1^2 + 4|B_2| - 2B_1 \leq 0, \quad |6B_1B_3 + B_1^2B_2 - B_1^4 - 4B_2^2| - 4B_1^2 \geq 0,$$

then the second Hankel determinant of f satisfies

$$|a_2a_4 - a_3^2| \leq \frac{1}{144}|6B_1B_3 + B_1^2B_2 - B_1^4 - 4B_2^2|.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$B_1^2+4|B_2|-2B_1 > 0, \quad 2|6B_1B_3+B_1^2B_2-B_1^4-4B_2^2|-B_1^3-4B_1|B_2|-6B_1^2 \leq 0,$$

then the second Hankel determinant of f satisfies

$$\begin{aligned} &|a_2a_4 - a_3^2| \\ &\leq \frac{B_1^2}{576}\left(\frac{\begin{array}{r}16|6B_1B_3 + B_1^2B_2 - B_1^4 - 4B_2^2| - 12B_1^3 - 48B_1|B_2| - 36B_1^2 \\ - B_1^4 - 8B_1^2|B_2| - 16B_2^2\end{array}}{|6B_1B_3 + B_1^2B_2 - B_1^4 - 4B_2^2| - B_1^3 - 4B_1|B_2| - 2B_1^2}\right). \end{aligned}$$

Proof. Since $f \in \mathcal{CV}(\varphi)$, there exists an analytic function w with $w(0) = 0$ and

$|w(z)| < 1$ in $\mathbb{D}$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \varphi(w(z)). \tag{3.15}$$

Since

$$1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2 z + (-4a_2^2 + 6a_3)z^2 + (8a_2^3 - 18a_2 a_3 + 12a_4)z^3 + \cdots, \tag{3.16}$$

equations (3.8), (3.15) and (3.16) yield

$$\begin{aligned}
a_2 &= \frac{B_1 c_1}{4}, \\
a_3 &= \frac{1}{24}\left[(B_1^2 - B_1 + B_2)c_1^2 + 2B_1 c_2\right], \\
a_4 &= \frac{1}{192}[(-4B_2 + 2B_1 + B_1^3 - 3B_1^2 + 3B_1 B_2 + 2B_3)c_1^3 \\
&\quad + 2(3B_1^2 - 4B_1 + 4B_2)c_1 c_2 + 8B_1 c_3].
\end{aligned}$$

Therefore

$$\begin{aligned}
a_2 a_4 - a_3^2 = \frac{B_1}{768}\Bigg[c_1^4\left(-\frac{4}{3}B_2 + \frac{2}{3}B_1 - \frac{1}{3}B_1^3 - \frac{1}{3}B_1^2 + \frac{1}{3}B_1 B_2 + 2B_3 - \frac{4}{3}\frac{B_2^2}{B_1}\right) \\
+\frac{2}{3}c_2 c_1^2 (B_1^2 - 4B_1 + 4B_2) + 8B_1 c_1 c_3 - \frac{16}{3}B_1 c_2^2\Bigg].
\end{aligned}$$

By writing

$$\begin{aligned}
&d_1 = 8B_1, \quad d_2 = \tfrac{2}{3}(B_1^2 - 4B_1 + 4B_2), \\
&d_3 = -\tfrac{16}{3}B_1, \quad d_4 = -\tfrac{4}{3}B_2 + \tfrac{2}{3}B_1 - \tfrac{1}{3}B_1^3 - \tfrac{1}{3}B_1^2 + \tfrac{1}{3}B_1 B_2 + 2B_3 - \tfrac{4}{3}\tfrac{B_2^2}{B_1}, \qquad (3.17) \\
&T = \tfrac{B_1}{768},
\end{aligned}$$

we have

$$|a_2 a_4 - a_3^2| = T|d_1 c_1 c_3 + d_2 c_1^2 c_2 + d_3 c_2^2 + d_4 c_1^4|.$$

Similar as in Theorems 3.1, it follows from (3.2) and (3.3) that

$$|a_2a_4 - a_3^2| = \frac{T}{4}\Big|c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4 - c^2)(d_1 + d_2 + d_3)$$
$$+(4 - c^2)x^2(-d_1c^2 + d_3(4 - c^2)) + 2d_1c(4 - c^2)(1 - |x|^2)z\Big|.$$

Replacing $|x|$ by μ and then substituting the values of d_1, d_2, d_3 and d_4 from (3.17) yield

$$\begin{aligned}
&|a_2a_4 - a_3^2| \\
&\leq \frac{T}{4}\Big[c^4\left|-\frac{4}{3}B_1^3 + \frac{4}{3}B_1B_2 + 8B_3 - \frac{16}{3}\frac{B_2^2}{B_1}\right| + 2\mu c^2(4 - c^2)\left(\frac{2}{3}B_1^2 + \frac{8}{3}|B_2|\right) \\
&\quad + \mu^2(4 - c^2)\left(\frac{8}{3}B_1c^2 + \frac{64}{3}B_1\right) + 16B_1c(4 - c^2)(1 - \mu^2)\Big] \\
&= T\Big[\frac{c^4}{3}\left|-B_1^3 + B_1B_2 + 6B_3 - 4\frac{B_2^2}{B_1}\right| + 4B_1c(4 - c^2) + \frac{1}{3}\mu c^2(4 - c^2)(B_1^2 + 4|B_2|) \\
&\quad + \frac{2B_1}{3}\mu^2(4 - c^2)(c - 4)(c - 2)\Big] \qquad (3.18) \\
&\equiv F(c, \mu).
\end{aligned}$$

Again, differentiating $F(c, \mu)$ in (3.18) partially with respect to μ yield

$$\frac{\partial F}{\partial \mu} = T\left[\frac{c^2}{3}(4 - c^2)(B_1^2 + 4|B_2|) + \frac{4B_1}{3}\mu(4 - c^2)(c - 4)(c - 2)\right]. \qquad (3.19)$$

It is clear from (3.19) that $\frac{\partial F}{\partial \mu} > 0$. Thus $F(c, \mu)$ is an increasing function of μ for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$. So the maximum of $F(c, \mu)$ occurs at $\mu = 1$ and

$$\max F(c, \mu) = F(c, 1) \equiv G(c).$$

Note that

$$G(c) = T\left[\frac{c^4}{3}\left(\left|-B_1^3 + B_1B_2 + 6B_3 - 4\frac{B_2^2}{B_1}\right| - B_1^2 - 4|B_2| - 2B_1\right)\right.$$
$$\left. + \frac{4}{3}c^2(B_1^2 + 4|B_2| - 2B_1) + \frac{64}{3}B_1\right].$$

Let

$$\begin{aligned} P &= \frac{1}{3}\left(\left|-B_1^3 + B_1B_2 + 6B_3 - 4\frac{B_2^2}{B_1}\right| - B_1^2 - 4|B_2| - 2B_1\right), \\ Q &= \frac{4}{3}(B_1^2 + 4|B_2| - 2B_1), \\ R &= \frac{64}{3}B_1, \end{aligned} \tag{3.20}$$

By using (3.4), we have

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{768}\begin{cases} R, & Q \leq 0,\ P \leq -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \geq 0,\ P \geq -\frac{Q}{8} \text{ or } Q \leq 0,\ P \geq -\frac{Q}{4}; \\ \frac{4PR-Q^2}{4P}, & Q > 0,\ P \leq -\frac{Q}{8} \end{cases}$$

where P, Q, R are given in (3.20). ∎

For the choice of $\varphi(z) = (1+z)/(1-z)$, Theorem 3.2 reduces to the following corollary.

Corollary 3.3 [44, Theorem 3.2] *Let* $f \in \mathcal{CV}$. *Then* $|a_2a_4 - a_3^2| \leq 1/8$.

3.3 Further results on the second Hankel determinant

In this section, we determine the bound for the second Hankel determinant of functions in two more classes related to subordination.

A function f is in the class $\mathcal{R}_\gamma^\tau(A, B)$, $-1 \leq B < A \leq 1$ if it satisfies $1 +$

$\frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) \prec \frac{1+Az}{1+Bz}$. This class $\mathcal{R}_\gamma^\tau(A, B)$, is essentially motivated by Swaminathan [124] and introduced by Bansal [17]. Bansal [18] also obtained upper bound of second Hankel determinant for functions belonging to this class. Now we define the following class.

Definition 3.3 *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic and be given as in* (3.5)*. Let $0 \leq \gamma \leq 1$ and $\tau \in \mathbb{C}\setminus\{0\}$. A function $f \in \mathcal{A}$ is in the class $\mathcal{R}_\gamma^\tau(\varphi)$ if it satisfies the following subordination:*

$$1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) \prec \varphi(z).$$

The following theorem provides a bound for the second Hankel determinant of the functions in the class $\mathcal{R}_\gamma^\tau(\varphi)$.

Theorem 3.3 *Let $0 \leq \gamma \leq 1$, $\tau \in \mathbb{C} \setminus \{0\}$ and suppose the function f as in* (3.1) *is in the class $\mathcal{R}_\gamma^\tau(\varphi)$. Also, let*

$$p = \frac{8}{9}\frac{(1+\gamma)(1+3\gamma)}{(1+2\gamma)^2}.$$

1. *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2|(1-p) + B_1(1-2p) \leq 0, \quad |B_1B_3 - pB_2^2| - pB_1^2 \leq 0,$$

then the second Hankel determinant of f satisfies

$$|a_2a_4 - a_3^2| \leq \frac{|\tau|^2 B_1^2}{9(1+2\gamma)^2}.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2|(1-p) + B_1(1-2p) \geq 0, \quad 2|B_1B_3 - pB_2^2| - 2(1-p)B_1|B_2| - B_1 \geq 0,$$

or the conditions

$$2|B_2|(1-p)+B_1(1-2p)\leq 0,\quad |B_1B_3-pB_2^2|-B_1^2\geq 0,$$

then the second Hankel determinant of f satisfies

$$|a_2a_4-a_3^2|\leq \frac{|\tau|^2}{8(1+\gamma)(1+3\gamma)}|B_3B_1-pB_2^2|.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2|(1-p)+B_1(1-2p)>0,\quad 2|B_1B_3-pB_2^2|-2(1-p)B_1|B_2|-B_1^2\leq 0,$$

then the second Hankel determinant of f satisfies

$$\begin{aligned}&|a_2a_4-a_3^2|\\ &\leq \frac{|\tau|^2B_1^2}{32(1+\gamma)(1+3\gamma)}\left(\frac{\begin{array}{r}4p|B_3B_1-pB_2^2|-4(1-p)B_1[|B_2|(3-2p)+B_1]\\ -4B_2^2(1-p)^2-B_1^2(1-2p)^2\end{array}}{|B_3B_1-pB_2^2|-(1-p)B_1(2|B_2|+B_1)}\right).\end{aligned}$$

Proof. For $f\in\mathcal{R}_{\gamma}^{\tau}(\varphi)$, there exists an analytic function w with $w(0)=0$ and $|w(z)|<1$ in $\mathbb{D}$ such that

$$1+\frac{1}{\tau}(f'(z)+\gamma zf''(z)-1)=\varphi(w(z)). \tag{3.21}$$

Since f has the Maclaurin series given by (3.1), a computation shows that

$$1+\frac{1}{\tau}(f'(z)+\gamma zf''(z)-1)=1+\frac{2a_2(1+\gamma)}{\tau}z+\frac{3a_3(1+2\gamma)}{\tau}z^2+\frac{4a_4(1+3\gamma)}{\tau}z^3+\cdots. \tag{3.22}$$

It follows from (3.21), (3.8) and (3.22) that

$$
\begin{aligned}
a_2 &= \frac{\tau B_1 c_1}{4(1+\gamma)},\\
a_3 &= \frac{\tau B_1}{12(1+2\gamma)}\left[2c_2 + c_1^2\left(\frac{B_2}{B_1}-1\right)\right],\\
a_4 &= \frac{\tau}{32(1+3\gamma)}[B_1(4c_3 - 4c_1c_2 + c_1^3) + 2B_2c_1(2c_2 - c_1^2) + B_3c_1^3].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
a_2a_4 - a_3^2 &= \frac{\tau^2 B_1 c_1}{128(1+\gamma)(1+3\gamma)}\left[B_1(4c_3 - 4c_1c_2 + c_1^3) + 2B_2c_1(2c_2 - c_1^2) + B_3c_1^3\right]\\
&\quad - \frac{\tau^2 B_1^2}{144(1+2\gamma)^2}\left[4c_2^2 + c_1^4\left(\frac{B_2}{B_1}-1\right)^2 + 4c_2c_1^2\left(\frac{B_2}{B_1}-1\right)\right]\\
&= \frac{\tau^2 B_1^2}{128(1+\gamma)(1+3\gamma)}\left\{\left[(4c_1c_3 - 4c_1^2c_2 + c_1^4) + \frac{2B_2c_1^2}{B_1}(2c_2 - c_1^2) + \frac{B_3}{B_1}c_1^4\right]\right.\\
&\quad \left. -\frac{8}{9}\frac{(1+\gamma)(1+3\gamma)}{(1+2\gamma)^2}\left[4c_2^2 + c_1^4\left(\frac{B_2}{B_1}-1\right)^2 + 4c_2c_1^2\left(\frac{B_2}{B_1}-1\right)\right]\right\},
\end{aligned}
$$

which yields

$$
\begin{aligned}
|a_2a_4 - a_3^2| = T\,\Bigg|&4c_1c_3 + c_1^4\left[1 - 2\frac{B_2}{B_1} - p\left(\frac{B_2}{B_1}-1\right)^2 + \frac{B_3}{B_1}\right] - 4pc_2^2\\
&-4c_1^2c_2\left[1 - \frac{B_2}{B_1} + p\left(\frac{B_2}{B_1}-1\right)\right]\Bigg|,
\end{aligned}
\tag{3.23}
$$

where

$$
T = \frac{\tau^2 B_1^2}{128(1+\gamma)(1+3\gamma)} \quad \text{and} \quad p = \frac{8}{9}\frac{(1+\gamma)(1+3\gamma)}{(1+2\gamma)^2}.
$$

It can be easily verified that $p \in \left[\frac{64}{81}, \frac{8}{9}\right]$ for $0 \le \gamma \le 1$.

Let

$$
\begin{aligned}
&d_1 = 4, \quad d_2 = -4\left[1 - \tfrac{B_2}{B_1} + p\left(\tfrac{B_2}{B_1}-1\right)\right],\\
&d_3 = -4p, \quad d_4 = 1 - 2\tfrac{B_2}{B_1} - p\left(\tfrac{B_2}{B_1}-1\right)^2 + \tfrac{B_3}{B_1}.
\end{aligned}
\tag{3.24}
$$

Then (3.23) becomes

$$|a_2a_4 - a_3^2| = T|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|.$$

It follows that

$$\begin{aligned}|a_2a_4 - a_3^2| = \frac{T}{4}\Big|&c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4 - c^2)(d_1 + d_2 + d_3)\\ &+(4 - c^2)x^2(-d_1c^2 + d_3(4 - c^2)) + 2d_1c(4 - c^2)(1 - |x|^2)z\Big|.\end{aligned}$$

An application of triangle inequality, replacement of $|x|$ by μ and substitution of the values of d_1, d_2, d_3 and d_4 from (3.24) yield

$$\begin{aligned}|a_2a_4 - a_3^2| &\leq \frac{T}{4}\Big[4c^4\left|\frac{B_3}{B_1} - p\frac{B_2^2}{B_1^2}\right| + 8\left|\frac{B_2}{B_1}\right|\mu c^2(4 - c^2)(1 - p)\\ &\quad + (4 - c^2)\mu^2(4c^2 + 4p(4 - c^2)) + 8c(4 - c^2)(1 - \mu^2)\Big]\\ &= T\Big[c^4\left|\frac{B_3}{B_1} - p\frac{B_2^2}{B_1^2}\right| + 2c(4 - c^2) + 2\mu\left|\frac{B_2}{B_1}\right|c^2(4 - c^2)(1 - p)\\ &\quad + \mu^2(4 - c^2)(1 - p)(c - \alpha)(c - \beta)\Big]\\ &\equiv F(c, \mu)\end{aligned}$$

where $\alpha = 2$ and $\beta = 2p/(1 - p) > 2$.

Similarly as in the previous proofs, it can be shown that $F(c, \mu)$ is an increasing function of μ for $0 < \mu < 1$. So for fixed $c \in [0, 2]$, let

$$\max F(c, \mu) = F(c, 1) \equiv G(c),$$

which is

$$G(c) = T\left\{c^4\left[\left|\frac{B_3}{B_1} - p\frac{B_2^2}{B_1^2}\right| - (1 - p)\left(2\left|\frac{B_2}{B_1}\right| + 1\right)\right]\right.$$

$$+4c^2\left[2\left|\frac{B_2}{B_1}\right|(1-p)+1-2p\right]+16p\right\}.$$

Let

$$\begin{aligned}
P &= \left|\frac{B_3}{B_1}-p\frac{B_2^2}{B_1^2}\right|-(1-p)\left(2\left|\frac{B_2}{B_1}\right|+1\right),\\
Q &= 4\left[2\left|\frac{B_2}{B_1}\right|(1-p)+1-2p\right], \qquad (3.25)\\
R &= 16p.
\end{aligned}$$

Using (3.4), we have

$$|a_2a_4-a_3^2|\leq T\begin{cases} R, & Q\leq 0,\ P\leq -\frac{Q}{4};\\ 16P+4Q+R, & Q\geq 0,\ P\geq -\frac{Q}{8} \text{ or } Q\leq 0,\ P\geq -\frac{Q}{4};\\ \frac{4PR-Q^2}{4P}, & Q>0,\ P\leq -\frac{Q}{8}\end{cases}$$

where P, Q, R are given in (3.25). ∎

For the choice $\varphi(z) := (1+Az)/(1+Bz)$ with $-1\leq B<A\leq 1$, Theorem 3.3 reduces to the following corollary.

Corollary 3.4 [18, Theorem 2.1] *Let the function f be in the class $\mathcal{R}_\gamma^\tau(A,B)$ where $0\leq\gamma\leq 1$, $\tau\in\mathbb{C}\setminus\{0\}$ and $-1\leq B<A\leq 1$, then*

$$|a_2a_4-a_3^2|\leq\frac{|\tau|^2(A-B)^2}{9(1+2\gamma)^2}.$$

In [2], Al-Amiri and Reade introduced the class $\mathcal{G}_\alpha := \mathcal{G}_\alpha((1+z)/(1-z))$ and showed that $\mathcal{G}_\alpha\subset\mathcal{S}$ for $\alpha<0$. Univalence of the functions in the class $\mathcal{G}_\alpha$ was also investigated in [111, 112]. Verma *et al.* [130] obtained the bound for the second Hankel determinant of functions in $\mathcal{G}_\alpha$. Now we define the following class.

Definition 3.4 *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic and as given in* (3.5)*. For a fixed real number α, a function $f \in \mathcal{A}$ is in the class $\mathcal{G}_\alpha(\varphi)$ if it satisfies the following subordination:*

$$(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z).$$

The following theorem provides a bound for the second Hankel determinant of the functions in the class $\mathcal{G}_\alpha(\varphi)$.

Theorem 3.4 *Let the function f given by* (3.1) *be in the class $\mathcal{G}_\alpha(\varphi)$, $0 \leq \alpha \leq 1$. Also, let*

$$p = \frac{8}{9}\frac{(1+2\alpha)}{(1+\alpha)}.$$

1. *If B_1, B_2 and B_3 satisfy the conditions*

$$B_1^2\alpha(3-2p) + 2|B_2|(1+\alpha-p) + B_1(1+\alpha-2p) \leq 0,$$

$$|B_1^4\alpha(2\alpha-1-p\alpha) + \alpha B_1^2 B_2(3-2p) + (\alpha+1)B_1B_3 - pB_2^2| - pB_1^2 \leq 0,$$

then the second Hankel determinant of f satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{9(1+\alpha)^2}.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$B_1^2\alpha(3-2p) + 2|B_2|(1+\alpha-p) + B_1(1+\alpha-2p) \geq 0,$$

$$2|B_1^4\alpha(2\alpha-1-p\alpha) + \alpha B_1^2 B_2(3-2p) + (\alpha+1)B_1B_3 - pB_2^2| - B_1^3\alpha(3-2p)$$
$$-2(1+\alpha-p)B_1|B_2| - (\alpha+1)B_1^2 \geq 0,$$

or

$$B_1^2\alpha(3-2p) + 2|B_2|(1+\alpha-p) + B_1(1+\alpha-2p) \leq 0,$$

$$|B_1^4\alpha(2\alpha-1-p\alpha)+\alpha B_1^2B_2(3-2p)+(\alpha+1)B_1B_3-pB_2^2|-pB_1^2\geq 0,$$

then the second Hankel determinant of f satisfies

$$|a_2a_4-a_3^2|\leq\frac{|B_1^4\alpha(2\alpha-1-p\alpha)+\alpha B_1^2B_2(3-2p)+(\alpha+1)B_1B_3-pB_2^2|}{8(1+\alpha)(1+2\alpha)}.$$

3. If B_1, B_2 and B_3 satisfy the conditions

$$B_1^2\alpha(3-2p)+2|B_2|(1+\alpha-p)+B_1(1+\alpha-2p)>0,$$

$$2|B_1^4\alpha(2\alpha-1-p\alpha)+\alpha B_1^2B_2(3-2p)+(\alpha+1)B_1B_3-pB_2^2|-B_1^3\alpha(3-2p)$$
$$-2(1+\alpha-p)B_1|B_2|-(\alpha+1)B_1^2\leq 0,$$

then the second Hankel determinant of f satisfies

$$|a_2a_4-a_3^2|\leq\frac{B_1^2}{32(1+\alpha)(1+2\alpha)}$$
$$\times\left[4p-\frac{\left[B_1^2\alpha(3-2p)+2|B_2|(1+\alpha-p)+B_1(1+\alpha-2p)\right]^2}{\begin{array}{r}|B_1^4\alpha(2\alpha-1-p\alpha)+\alpha B_1^2B_2(3-2p)+(\alpha+1)B_1B_3-pB_2^2|\\ -B_1^3\alpha(3-2p)-(1+\alpha-p)B_1(2|B_2|+B_1)\end{array}}\right].$$

Proof. For $f\in\mathcal{G}_\alpha(\varphi)$, some calculations shows that

$$\begin{aligned}
a_2&=\frac{B_1c_1}{4},\\
a_3&=\frac{1}{12(1+\alpha)}\left[(\alpha B_1^2-B_1+B_2)c_1^2+2B_1c_2\right],\\
a_4&=\frac{1}{32(\alpha+1)(2\alpha+1)}\Big\{[-3\alpha B_1^2+\alpha(2\alpha-1)B_1^3+B_1(1+\alpha+3\alpha B_2)\\
&\quad+(\alpha+1)(B_3-2B_2)]c_1^3+2[-2(1+\alpha)B_1+3\alpha B_1^2+2(1+\alpha)B_2]c_1c_2\\
&\quad+4(1+\alpha)B_1c_3\Big\}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&a_2a_4 - a_3^2\\
&= \frac{B_1}{128(1+\alpha)(1+2\alpha)}\Bigg[\Big[-3\alpha B_1^2 + \alpha(2\alpha-1)B_1^3 + B_1(1+\alpha) + 3\alpha B_1B_2\\
&\quad + (1+\alpha)(B_3 - 2B_2)\Big]c_1^4 + 2[-2(1+\alpha)B_1 + 3\alpha B_1^2 + 2(1+\alpha)B_2]c_1^2c_2\\
&\quad + 4(1+\alpha)B_1c_1c_3\\
&\quad - \frac{8}{9}\frac{(1+2\alpha)}{(1+\alpha)}\left[\frac{(\alpha B_1^2 - B_1 + B_2)^2}{B_1}c_1^4 + 4(\alpha B_1^2 - B_1 + B_2)c_2c_1^2 + 4B_1c_2^2\right]\Bigg]
\end{aligned}
$$

which yields

$$
\begin{aligned}
&|a_2a_4 - a_3^2|\\
&= T\Big|4(1+\alpha)B_1c_1c_3 + c_1^4\Big[-3\alpha B_1^2 + \alpha(2\alpha-1)B_1^3 + B_1(1+\alpha) + 3\alpha B_1B_2\\
&\quad + (1+\alpha)(B_3 - 2B_2) - p\frac{(\alpha B_1^2 - B_1 + B_2)^2}{B_1}\Big] - 4pB_1c_2^2\\
&\quad + 2c_1^2c_2\left[-2(1+\alpha)B_1 + 3\alpha B_1^2 + 2(1+\alpha)B_2 - 2p(\alpha B_1^2 - B_1 + B_2)\right]\Big| \qquad (3.26)
\end{aligned}
$$

where

$$T = \frac{B_1}{128(1+\alpha)(1+2\alpha)} \quad \text{and} \quad p = \frac{8}{9}\frac{(1+2\alpha)}{(1+\alpha)}.$$

It can be easily verified that for $0 \leq \alpha \leq 1$, $p \in \left[\frac{8}{9}, \frac{4}{3}\right]$. Let

$$
\begin{aligned}
d_1 &= 4(1+\alpha)B_1,\\
d_2 &= 2\left[-2(1+\alpha)B_1 + 3\alpha B_1^2 + 2(1+\alpha)B_2 - 2p(\alpha B_1^2 - B_1 + B_2)\right],\\
d_3 &= -4pB_1,\\
d_4 &= -3\alpha B_1^2 + \alpha(2\alpha-1)B_1^3 + B_1(1+\alpha) + 3\alpha B_1B_2 + (1+\alpha)(B_3 - 2B_2)\\
&\quad - p\frac{(\alpha B_1^2 - B_1 + B_2)^2}{B_1},
\end{aligned}
$$

Then (3.26) becomes

$$|a_2a_4 - a_3^2| = T|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|.$$

Similar as in earlier theorems, it follows that

$$\begin{aligned}
|a_2a_4 - a_3^2| = {} & \frac{T}{4}\Big|c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4-c^2)(d_1+d_2+d_3) \\
& +(4-c^2)x^2(-d_1c^2 + d_3(4-c^2)) + 2d_1c(4-c^2)(1-|x|^2)z\Big| \\
\leq {} & T\Big[c^4\left|B_1^3\alpha(2\alpha - 1 - p\alpha) + \alpha B_1B_2(3-2p) + (\alpha+1)B_3 - p\frac{B_2^2}{B_1}\right| \\
& + \mu c^2(4-c^2)[B_1^2\alpha(3-2p) + 2|B_2|(1+\alpha-p)] + 2c(4-c^2)B_1(1+\alpha) \\
& + \mu^2(4-c^2)B_1(1+\alpha-p)(c-2)\left(c - \frac{2p}{1+\alpha-p}\right)\Big] \\
\equiv {} & F(c,\mu),
\end{aligned}$$

and for fixed $c \in [0,2]$, $\max F(c,\mu) = F(c,1) \equiv G(c)$ with

$$\begin{aligned}
G(c) = T\Bigg[& c^4\Big[\Big|B_1^3\alpha(2\alpha - 1 - p\alpha) + \alpha B_1B_2(3-2p) + (\alpha+1)B_3 - p\frac{B_2^2}{B_1}\Big| \\
& - B_1^2\alpha(3-2p) - (1+\alpha-p)(2|B_2| + B_1)\Big] + 4c^2[B_1^2\alpha(3-2p) \\
& + 2|B_2|(1+\alpha-p) + B_1(1+\alpha-2p)] + 16pB_1\Bigg].
\end{aligned}$$

Let

$$\begin{aligned}
P = {} & \left|B_1^3\alpha(2\alpha - 1 - p\alpha) + \alpha B_1B_2(3-2p) + (\alpha+1)B_3 - p\frac{B_2^2}{B_1}\right| \\
& - B_1^2\alpha(3-2p) - (1+\alpha-p)(2|B_2| + B_1)
\end{aligned}$$

$$Q = 4\left[B_1^2\alpha(3-2p) + 2|B_2|(1+\alpha-p) + B_1(1+\alpha-2p)\right], \tag{3.27}$$

$$R = 16pB_1,$$

By using (3.4), we have

$$|a_2a_4 - a_3^2| \leq T \begin{cases} R, & Q \leq 0,\ P \leq -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \geq 0,\ P \geq -\frac{Q}{8} \text{ or } Q \leq 0,\ P \geq -\frac{Q}{4}; \\ \frac{4PR-Q^2}{4P}, & Q > 0,\ P \leq -\frac{Q}{8} \end{cases}$$

where P, Q, R are given in (3.27). ∎

Remark 3.1 *For $\alpha = 1$, Theorem 3.4 reduces to Theorem 3.2.*

For $0 \leq \alpha < 1$, let $\varphi(z) := (1 + (1-2\alpha)z)/(1-z)$. For this function φ, $B_1 = B_2 = B_3 = 2(1-\alpha)$. In this case, Theorem 3.4 reduces to the following corollary.

Corollary 3.5 [130, Theorem 3.1] *If $f \in \mathcal{G}_\alpha(\varphi)$, then*

$$|a_2a_4 - a_3^2| \leq \frac{4(1-\alpha)^2}{9(1+\alpha)^2}.$$

Remark 3.2 *By letting $\gamma = 0$ and $\tau = 1$ in Theorem 3.3 and $\alpha = 0$ in Theorem 3.4, the two results coincide.*

CHAPTER 4
APPLICATIONS OF DIFFERENTIAL SUBORDINATION FOR FUNCTIONS WITH FIXED SECOND COEFFICIENT

4.1 Introduction and preliminaries

For univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the famous Bieberbach theorem (Theorem 1.2) shows that $|a_2| \leq 2$ and this bound yields the growth and distortion bounds as well as the covering theorem. In view of the influence of the second coefficient in the properties of univalent functions, several authors have investigated functions with fixed second coefficient. For a brief survey of the various developments, mainly on radius problems, from 1920 to this date, see the recent work by Ali *et al.* [13]. The theory of first-order differential subordination was developed by Miller and Mocanu, and a very comprehensive account of the theory and numerous applications can be found in their monograph [61]. Ali *et al.* [10] extended this well-known theory of differential subordination to the functions with preassigned second coefficient. Nagpal and Ravichandran [68] then applied the results in [10] to obtain several extensions of well-known results to the functions with fixed second coefficient. In this chapter, we continue their investigation by deriving several sufficient conditions for starlikeness of functions with fixed second coefficient.

For convenience, let $\mathcal{A}_{n,b}$ denote the class of all functions $f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots$ where $n \in \mathbb{N} = \{1, 2, \dots\}$ and b is a fixed non-negative real number. For fixed $\mu \geq 0$, and $n \in \mathbb{N}$, let $\mathcal{H}_{\mu,n}$ be the class consists of analytic functions p on $\mathbb{D}$ of the form

$$p(z) = 1 + \mu z^n + p_{n+1}z^{n+1} + \cdots \quad (z \in \mathbb{D}).$$

Let Ω be a subset of $\mathbb{C}$ and the class $\Psi_{\mu,n}[\Omega]$ consists of those functions $\psi : \mathbb{C}^2 \to \mathbb{C}$

that are continuous in a domain $D \subset \mathbb{C}^2$ with $(1,0) \in D$, $\psi(1,0) \in \Omega$, and satisfy the admissibility condition: $\psi(i\rho, \sigma) \notin \Omega$ whenever $(i\rho, \sigma) \in D$, $\rho \in \mathbb{R}$, and

$$\sigma \leq -\frac{1}{2}\left(n + \frac{2-\mu}{2+\mu}\right)(1+\rho^2). \tag{4.1}$$

When $\Omega = \{w : \operatorname{Re} w > 0\}$, let $\Psi_{\mu,n} := \Psi_{\mu,n}[\Omega]$. The following theorem is needed to prove our main results.

Theorem 4.1 [10, Theorem 3.4] *Let $p \in \mathcal{H}_{\mu,n}$ with $0 < \mu \leq 2$. Let $\psi \in \Psi_{\mu,n}$ with associated domain D. If $(p(z), zp'(z)) \in D$ and $\operatorname{Re} \psi(p(z), zp'(z)) > 0$, then $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$.*

For $\alpha \neq 1$, let

$$\mathcal{ST}(\alpha) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \right\}.$$

The function $p_\alpha(z) := (1+(1-2\alpha)z)/(1-z)$ maps $\mathbb{D}$ onto $\{w \in \mathbb{C} : \operatorname{Re} w > \alpha\}$ for $\alpha < 1$ and onto $\{w \in \mathbb{C} : \operatorname{Re} w < \alpha\}$ for $\alpha > 1$. Therefore, for $\alpha < 1$, $\mathcal{ST}(\alpha)$ is the class of starlike functions of order α that consist of functions $f \in \mathcal{A}$ for which $\operatorname{Re}(zf'(z)/f(z)) > \alpha$.

Motivated by the works of Lewandowski *et al.* [50], several authors [52, 55, 56, 70, 87, 90, 96, 101, 103, 110, 137] investigated the functions f for which $zf'(z)/f(z) \cdot (\alpha zf''(z)/f'(z) + 1)$ lies in certain region in the right half-plane. For $\alpha \geq 0$ and $\beta < 1$, Ravichandran *et al.* [102] showed that a function f of the form $f(z) = z + a_{n+1}z^{n+1} + \cdots$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right)\right) > \alpha\beta\left(\beta + \frac{n}{2} - 1\right) + \beta - \frac{\alpha n}{2}$$

is starlike of order β. In Theorem 4.4, we will obtain the corresponding result for $f \in \mathcal{A}_{n,b}$.

For function p of the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, Nunokawa *et al.* [88] showed that for certain analytic function w with $w(0) = \alpha$, $\alpha p^2(z)+\beta z p'(z) \prec w(z)$ implies $\operatorname{Re} p(z) > 0$, where $\beta > 0$, $\alpha \geq -\beta/2$. See also [104]. Lemma 4.5 investigates the conditions for the subordination $p^2(z)+\gamma z p'(z) \prec (1 + (1 - 2\delta)z)/(1 - z)$.

For complex numbers β and γ, the differential subordination

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z),$$

where q is analytic and h is univalent with $q(0) = h(0)$, is popularly known as Briot-Bouquet differential subordination. This particular differential subordination has a significant number of important applications in the theory of analytic functions (for details see [61]). The importance of Briot-Bouquet differential subordination inspired many researchers to work in this area and many generalizations and extensions of the Briot-Bouquet differential subordination have recently been obtained. Ali *et al.* [6] obtained several results related to the Briot-Bouquet differential subordination. In Lemma 4.1, the Briot-Bouquet differential subordination is investigated for functions with fixed second coefficient.

4.2 Subordinations for starlikeness

In this section, several sufficient conditions are given for the starlikeness of order β of functions $f \in \mathcal{A}_{n,b}$. We need following lemmas to prove the theorems.

Lemma 4.1 *Let $n \in \mathbb{N}$ be fixed. For $\alpha > 0$, $0 \leq \beta < 1$, $\gamma + \alpha\beta > 0$ and $0 < \mu \leq 2(1 - \beta)$, let*

$$\delta = \begin{cases} \frac{-1}{2}\frac{(1-\beta)}{(\alpha\beta+\gamma)}\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + \beta, & if \quad \gamma + \alpha\beta \geq \alpha(1 - \beta), \\ \frac{-1}{2}\frac{(\alpha\beta+\gamma)}{\alpha^2(1-\beta)}\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + \beta, & if \quad \gamma + \alpha\beta \leq \alpha(1 - \beta). \end{cases}$$

If the function $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$p(z) + \frac{zp'(z)}{\alpha p(z) + \gamma} \prec \frac{1 + (1 - 2\delta)z}{1 - z}, \tag{4.2}$$

then

$$p(z) \prec \frac{1 + (1 - 2\beta)z}{1 - z}.$$

Proof. Let $0 \leq \beta < 1$. Note that

$$\begin{aligned}
\delta &:= \frac{-1}{2} \frac{(1 - \beta)}{(\alpha\beta + \gamma)} \left(n + \frac{2(1 - \beta) - \mu}{2(1 - \beta) + \mu} \right) + \beta \\
&= -(1 - \beta) \left[\frac{1}{2} \frac{1}{(\alpha\beta + \gamma)} \left(n + \frac{2(1 - \beta) - \mu}{2(1 - \beta) + \mu} \right) + 1 \right] + 1 < 1,
\end{aligned}$$

and

$$\begin{aligned}
\delta &:= \frac{-1}{2} \frac{\alpha\beta + \gamma}{\alpha^2(1 - \beta)} \left(n + \frac{2(1 - \beta) - \mu}{2(1 - \beta) + \mu} \right) + \beta \\
&= -(1 - \beta) \left[\frac{1}{2} \frac{\alpha\beta + \gamma}{\alpha^2(1 - \beta)^2} \left(n + \frac{2(1 - \beta) - \mu}{2(1 - \beta) + \mu} \right) + 1 \right] + 1 < 1.
\end{aligned}$$

Then for $p \in \mathcal{H}_{\mu,n}$, (4.2) can be rewritten as

$$\operatorname{Re} \left(p(z) + \frac{zp'(z)}{\alpha p(z) + \gamma} \right) > \delta, \tag{4.3}$$

and we want to show that $\operatorname{Re} p(z) > \beta$. Define the function $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \frac{p(z) - \beta}{1 - \beta} = 1 + \frac{\mu}{1 - \beta} z^n + \frac{a_{n+1}}{1 - \beta} z^{n+1} + \cdots, \quad n \in \mathbb{N}.$$

Then q is analytic on $\mathbb{D}$ and $q \in \mathcal{H}_{\mu',n}$, where $\mu' = \mu/(1-\beta)$. Since $(1-\beta)q(z)+\beta = p(z)$, if we can show that $\operatorname{Re} q(z) > 0$, then it will imply $\operatorname{Re} p(z) > \beta$. Note that

$$p(z) + \frac{zp'(z)}{\alpha p(z) + \gamma} = (1 - \beta)q(z) + \beta + \frac{(1 - \beta)}{\alpha[(1 - \beta)q(z) + \beta] + \gamma} zq'(z).$$

Then by (4.3),

$$\operatorname{Re}\left((1-\beta)q(z)+\frac{(1-\beta)}{\alpha(1-\beta)q(z)+\alpha\beta+\gamma}zq'(z)+\beta\right)>\delta. \tag{4.4}$$

Define $\psi:\mathbb{C}^2\to\mathbb{C}$ by

$$\psi(r,s)=(1-\beta)r+\frac{(1-\beta)}{\alpha(1-\beta)r+\alpha\beta+\gamma}s+\beta-\delta.$$

Then ψ is a continuous function of r and s on $D:=\mathbb{C}\backslash\{-(\alpha\beta+\gamma)/\alpha(1-\beta)\}\times\mathbb{C}$ and $\operatorname{Re}(\psi(1,0))=1-\delta>0$. Note that $(1,0)\in D$ because $(\alpha\beta+\gamma)/\alpha(1-\beta)>0$. Also for $\rho\in\mathbb{R}$, and by (4.1), σ in our case satisfying

$$\sigma\leq-\frac{1}{2}\left(n+\frac{2-\mu'}{2+\mu'}\right)(1+\rho^2), \tag{4.5}$$

where $\mu'=\mu/(1-\beta)$, it follows that

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma)&=\operatorname{Re}\left[(1-\beta)i\rho+\frac{(1-\beta)}{\alpha(1-\beta)i\rho+\alpha\beta+\gamma}\sigma+\beta-\delta\right]\\&=\frac{(1-\beta)(\alpha\beta+\gamma)}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}\sigma+\beta-\delta\\&\leq\frac{(1-\beta)(\alpha\beta+\gamma)}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}\left[-\frac{1}{2}\left(n+\frac{2-\mu'}{2+\mu'}\right)(1+\rho^2)\right]+\beta-\delta\\&=\frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right)\left(\frac{1+\rho^2}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}\right)\\&\quad+\beta-\delta.\end{aligned}$$

Thus, for $\alpha\beta+\gamma\geq\alpha(1-\beta)$,

$$\begin{aligned}\operatorname{Re}\psi(i\rho,\sigma)&\leq\frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right)\left(\frac{1+\rho^2}{(\alpha\beta+\gamma)^2+\alpha^2(1-\beta)^2\rho^2}\right)\\&\quad+\beta-\delta\\&\leq\frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right)\left(\frac{1+\rho^2}{(\alpha\beta+\gamma)^2+(\alpha\beta+\gamma)^2\rho^2}\right)\end{aligned}$$

$$
\begin{aligned}
&+\beta-\delta \\
&= \frac{-1}{2}\frac{(1-\beta)}{(\alpha\beta+\gamma)}\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right)+\beta-\delta \\
&= 0.
\end{aligned}
$$

Similarly, for $\alpha\beta+\gamma \leq \alpha(1-\beta)$,

$$
\begin{aligned}
\operatorname{Re}\psi(i\rho,\sigma) &\leq \frac{-1}{2}(1-\beta)(\alpha\beta+\gamma)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right)\left(\frac{1+\rho^2}{\alpha^2(1-\beta)^2+\alpha^2(1-\beta)^2\rho^2}\right) \\
&+\beta-\delta \\
&= \frac{-1}{2}\frac{(\alpha\beta+\gamma)}{\alpha^2(1-\beta)}\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right)+\beta-\delta \\
&= 0.
\end{aligned}
$$

Hence $\psi \in \Psi_{\mu',n}$, where $\mu' = \mu/(1-\beta)$. Since $q \in \mathcal{H}_{\mu',n}$, by Theorem 4.1, $\operatorname{Re} q(z) > 0$ if $(q(z), zq'(z)) \in D$ and $\operatorname{Re}(\psi(q(z), zq'(z)) > 0$. The former is true by (4.2) and the latter is true by (4.4). So $\operatorname{Re} q(z) > 0$ as desired. ∎

Lemma 4.2 *For $0 \leq \beta < 1$, $\lambda > 0$, and $0 < \mu \leq 2(1-\beta)$, let*

$$
\delta := -\frac{\lambda}{2}(1-\beta)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right)+\beta.
$$

If $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$
p(z)+\lambda zp'(z) \prec \frac{1+(1-2\delta)z}{1-z},
$$

then

$$
p(z) \prec \frac{1+(1-2\beta)z}{1-z}.
$$

Proof. Let $\alpha = 0$ and $\gamma = 1/\lambda$ in Lemma 4.1. ∎

Theorem 4.2 *Let $n \in \mathbb{N}$ be fixed, $0 \leq \beta < 1$ and $0 < \mu = nb \leq 2(1-\beta)$. Let δ_1 be given by*

$$\delta_1 = -\frac{1}{2}(1-\beta)\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + \beta.$$

If $f \in \mathcal{A}_{n,b}$ satisfies the subordination

$$\frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{1+(1-2\delta_1)z}{1-z},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. For a given function $f \in \mathcal{A}_{n,b}$, define the function $p : \mathbb{D} \to \mathbb{C}$ by $p(z) = zf'(z)/f(z)$. A computation shows that $p(z) = 1 + nbz^n + \cdots$, so $p \in \mathcal{H}_{\mu,n}$, where $\mu = nb$. Further calculations yield

$$\frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) = p(z) + zp'(z).$$

Hence by hypothesis and Lemma 4.2, it follows that

$$p(z) = \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

∎

Lemma 4.3 *Let $n \in \mathbb{N}$. For $\epsilon > 0$, $0 \leq \beta < 1$ and $0 < \mu \leq 2(1-\beta)$, let*

$$\delta = \begin{cases} \frac{-\epsilon\beta}{2(1-\beta)}\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + \beta, & if \quad 0 \leq \beta \leq \frac{1}{2}, \\ \frac{-\epsilon}{2\beta}(1-\beta)\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + \beta, & if \quad \frac{1}{2} \leq \beta < 1. \end{cases}$$

If the function $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$p(z) + \epsilon\frac{zp'(z)}{p(z)} \prec \frac{1+(1-2\delta)z}{1-z} \tag{4.6}$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let $\alpha = 1/\epsilon$ and $\gamma = 0$ in Lemma 4.1. ∎

Theorem 4.3 *Let $n \in \mathbb{N}$, $\epsilon > 0$, $0 \leq \beta < 1$ and $0 < \mu = nb \leq 2(1-\beta)$. Let δ_2 be given by*

$$\delta_2 = \begin{cases} \frac{-\epsilon\beta}{2(1-\beta)}\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + \beta, & if \quad 0 \leq \beta \leq \frac{1}{2}, \\ \frac{-\epsilon}{2\beta}(1-\beta)\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + \beta, & if \quad \frac{1}{2} \leq \beta < 1. \end{cases}$$

If $f \in \mathcal{A}_{n,b}$ satisfies the following subordination

$$(1-\epsilon)\frac{zf'(z)}{f(z)} + \epsilon\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+(1-2\delta_2)z}{1-z},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = zf'(z)/f(z)$. Then $p \in \mathcal{H}_{\mu,n}$ and some calculations yield

$$(1-\epsilon)\frac{zf'(z)}{f(z)} + \epsilon\left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z) + \epsilon\frac{zp'(z)}{p(z)}.$$

Hence by hypothesis and Lemma 4.3, it follows that

$$p(z) = \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$ ∎

Lemma 4.4 *Let $n \in \mathbb{N}$. For $\alpha \geq 0$, $0 \leq \beta < 1$, $\gamma > 0$, and $0 < \mu \leq 2(1-\beta)$, let*

$$\delta := -\frac{\gamma}{2}(1-\beta)\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + (1-\alpha)\beta + \alpha\beta^2.$$

If $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$(1-\alpha)p(z) + \alpha p^2(z) + \gamma z p'(z) \prec \frac{1+(1-2\delta)z}{1-z}, \tag{4.7}$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let $0 \le \beta < 1$. Note that

$$\begin{aligned}
\delta &:= -\frac{\gamma}{2}(1-\beta)\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + (1-\alpha)\beta + \alpha\beta^2 \\
&= -\frac{\gamma}{2}(1-\beta)\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) - \alpha\beta(1-\beta) - (1-\beta) + 1 \\
&= -(1-\beta)\left[\frac{\gamma}{2}\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + \alpha\beta + 1\right] + 1 < 1.
\end{aligned}$$

Then for $p \in \mathcal{H}_{\mu,n}$, $\operatorname{Re}((1-\alpha)p(z) + \alpha p^2(z) + \gamma z p'(z)) > \delta$. To show that $\operatorname{Re} p(z) > \beta$, consider the function $q : \mathbb{D} \to \mathbb{C}$ given by

$$q(z) = \frac{p(z)-\beta}{1-\beta}.$$

Then q is analytic on $\mathbb{D}$, $q \in \mathcal{H}_{\mu',n}$ where $\mu' = \mu/(1-\beta)$, and

$$\begin{aligned}
\operatorname{Re}\Big[&(1-\beta)(1-\alpha+2\alpha\beta)q(z) + \alpha(1-\beta)^2 q^2(z) \\
&+\gamma(1-\beta)zq'(z) + (1-\alpha)\beta + \alpha\beta^2\Big] > \delta.
\end{aligned}$$

Define the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\psi(r,s) = (1-\beta)(1-\alpha+2\alpha\beta)r + \alpha(1-\beta)^2 r^2 + \gamma(1-\beta)s + (1-\alpha)\beta + \alpha\beta^2 - \delta.$$

For $\rho \in \mathbb{R}$ and σ satisfying (4.5), it follows that

$$
\begin{aligned}
&\operatorname{Re}\psi(i\rho,\sigma)\\
&= \operatorname{Re}\left[(1-\beta)(1-\alpha+2\alpha\beta)i\rho - \alpha(1-\beta)^2\rho^2 + \gamma(1-\beta)\sigma + (1-\alpha)\beta + \alpha\beta^2 - \delta\right]\\
&= \gamma(1-\beta)\sigma - \alpha(1-\beta)^2\rho^2 + (1-\alpha)\beta + \alpha\beta^2 - \delta\\
&\leq \gamma(1-\beta)\left[-\frac{1}{2}\left(n+\frac{2-\mu'}{2+\mu'}\right)(1+\rho^2)\right] - \alpha(1-\beta)^2\rho^2 + (1-\alpha)\beta + \alpha\beta^2 - \delta\\
&= -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right)(1+\rho^2) - \alpha(1-\beta)^2(\rho^2+1) + \alpha(1-\beta)^2\\
&\quad + (1-\alpha)\beta + \alpha\beta^2 - \delta\\
&= -(1+\rho^2)\left[\frac{\gamma}{2}(1-\beta)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + \alpha(1-\beta)^2\right]\\
&\quad + \alpha(1-\beta)^2 + (1-\alpha)\beta + \alpha\beta^2 - \delta\\
&\leq -\frac{\gamma}{2}(1-\beta)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + (1-\alpha)\beta + \alpha\beta^2 - \delta = 0
\end{aligned}
$$

Hence $\psi \in \Psi_{\mu',n}$. By Theorem 4.1, $\operatorname{Re} q(z) > 0$ or equivalently $\operatorname{Re} p(z) > \beta$. ∎

Remark 4.1 *Lemma 4.2 can also be obtained by letting $\alpha = 0$ in Lemma 4.4.*

Theorem 4.4 *Let $n \in \mathbb{N}$. For $\alpha > 0$, $0 \leq \beta < 1$ and $0 < \mu = nb \leq 2(1-\beta)$, let δ_3 be given by*

$$\delta_3 = -\frac{\alpha}{2}(1-\beta)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + (1-\alpha)\beta + \alpha\beta^2.$$

If $f \in \mathcal{A}_{n,b}$ satisfies the following subordination

$$\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right) \prec \frac{1+(1-2\delta_3)z}{1-z},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. For a given function $f \in \mathcal{A}_{n,b}$, let the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = zf'(z)/f(z)$. Then $p(z) = 1 + nbz^n + \cdots$. So $p \in \mathcal{H}_{\mu,n}$ where $\mu = nb$. Some simple calculations yield

$$\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right) = (1-\alpha)p(z) + \alpha p^2(z) + \alpha zp'(z).$$

Hence by hypothesis and Lemma 4.4, it follows that

$$p(z) = \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}. \qquad \blacksquare$$

If $\mu = 2(1-\beta)$, Theorem 4.4 reduces to the following result.

Corollary 4.1 *[102, Theorem 2.1] If f of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right)\right\} > \alpha\beta\left(\beta + \frac{n}{2} - 1\right) + \left(\beta - \frac{\alpha n}{2}\right) \quad \alpha \geq 0, \beta \leq 1,$$

then $f \in \mathcal{ST}(\beta)$.

Lemma 4.5 *Let $n \in \mathbb{N}$. For $0 \leq \beta < 1$, $\gamma > 0$, and $0 < \mu \leq 2(1-\beta)$, let*

$$\delta := -\frac{\gamma}{2}(1-\beta)\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) + \beta^2.$$

If the function $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$p^2(z) + \gamma zp'(z) \prec \frac{1+(1-2\delta)z}{1-z}, \tag{4.8}$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let $\alpha = 1$ in Lemma 4.4. $\blacksquare$

Lemma 4.6 *Let $n \in \mathbb{N}$. For $0 \le \beta < 1$ and $0 < \mu \le 2(1-\beta)$, let*

$$\delta = \begin{cases} \frac{-\beta}{2(1-\beta)}\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right), & if \quad 0 \le \beta < \frac{1}{2}, \\ \frac{-1}{2\beta}(1-\beta)\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right), & if \quad \frac{1}{2} \le \beta < 1. \end{cases}$$

If the function $p \in \mathcal{H}_{\mu,n}$ satisfies the subordination

$$\frac{zp'(z)}{p(z)} \prec \frac{1+(1-2\delta)z}{1-z},$$

then

$$p(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let $0 \le \beta < 1$. Note that

$$\delta := \frac{-\beta}{2(1-\beta)}\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) < 1,$$

and

$$\delta := \frac{-1}{2\beta}(1-\beta)\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) < 1.$$

Thus for $p \in \mathcal{H}_{\mu,n}$,

$$\operatorname{Re}\frac{zp'(z)}{p(z)} > \delta$$

and we want to show $\operatorname{Re} p(z) > \beta$. Considering the function $q(z) = (p(z)-\beta)/(1-\beta)$ or equivalently $(1-\beta)q(z)+\beta = p(z)$. Then q is analytic on $\mathbb{D}$, $q \in \mathcal{H}_{\mu',n}$ where $\mu' = \mu/(1-\beta)$, and

$$\operatorname{Re}\frac{zp'(z)}{p(z)} = \operatorname{Re}\left(\frac{(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta}\right) > \delta.$$

Define $\psi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\psi(r,s) = \frac{(1-\beta)s}{(1-\beta)r+\beta} - \delta.$$

Then $\psi(r,s)$ is continuous on $(\mathbb{C} - \{-\beta/(1-\beta)\}) \times \mathbb{C}$. For $\rho \in \mathbb{R}$ and σ satisfying (4.5), it follows that

$$\begin{aligned}
\operatorname{Re}\psi(i\rho,\sigma) &= \operatorname{Re}\left(\frac{(1-\beta)}{(1-\beta)i\rho+\beta}\sigma - \delta\right) \\
&= \operatorname{Re}\left(\frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma - \frac{(1-\beta)^2 i\rho}{\beta^2+(1-\beta)^2\rho^2}\sigma - \delta\right) \\
&= \frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\sigma - \delta \\
&\leq \frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2\rho^2}\left[-\frac{1}{2}\left(n+\frac{2-\mu'}{2+\mu'}\right)(1+\rho^2)\right] - \delta \\
&= -\frac{\beta}{2}(1-\beta)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right)\left(\frac{1+\rho^2}{\beta^2+(1-\beta)^2\rho^2}\right) - \delta.
\end{aligned}$$

For $1/2 \leq \beta < 1$,

$$\begin{aligned}
\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\beta}{2}(1-\beta)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right)\frac{1}{\beta^2} - \delta \\
&= \frac{-1}{2\beta}(1-\beta)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) - \delta \\
&= 0.
\end{aligned}$$

For $0 \leq \beta < 1/2$,

$$\begin{aligned}
\operatorname{Re}\psi(i\rho,\sigma) &\leq -\frac{\beta}{2}(1-\beta)\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right)\frac{1}{(1-\beta)^2} - \delta \\
&= \frac{-\beta}{2(1-\beta)}\left(n+\frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right) - \delta \\
&= 0.
\end{aligned}$$

Hence $\psi \in \Psi_{\mu',n}$ and then Theorem 4.1 implies $\operatorname{Re} q(z) > 0$ or equivalently $\operatorname{Re} p(z) > \beta$. ∎

Theorem 4.5 *Let $n \in \mathbb{N}$. For $0 \le \beta < 1$ and $0 < \mu = nb \le 2(1-\beta)$. Let δ_4 be given by*

$$\delta_4 = \begin{cases} \frac{-\beta}{2(1-\beta)}\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right), & if \quad 0 \le \beta < \frac{1}{2}, \\ \frac{-1}{2\beta}(1-\beta)\left(n + \frac{2(1-\beta)-\mu}{2(1-\beta)+\mu}\right), & if \quad \frac{1}{2} \le \beta < 1. \end{cases}$$

If $f \in \mathcal{A}_{n,b}$ satisfies the following subordination

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\delta_4)z}{1-z}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = zf'(z)/f(z)$. Then $p \in \mathcal{H}_{\mu,n}$. Also we have

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)}.$$

Hence by hypothesis and Lemma 4.6, it follows that

$$p(z) = \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}. \qquad \blacksquare$$

The above technique for analytic functions also works for meromorphic functions. Let $\Sigma_{n,b}$ be the class of normalized meromorphic functions, of the form

$$f(z) = \frac{1}{z} + bz^n + a_{n+1}z^{n+1} + \cdots, \quad (b \le 0)$$

that are analytic in the punctured unit disk $\mathbb{D}^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$. For $0 \leq \beta < 1$, the class of meromorphic starlike functions of order β is defined by

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

The following theorems gives sufficient condition for meromorphic functions to be starlike of order β in $\mathbb{D}^*$.

Theorem 4.6 *Let $n \in \mathbb{N}$. For $0 \leq \beta < 1$ and $0 < \mu = -(n+1)b \leq 2(1-\beta)$. Let δ_1 be given as in Theorem 4.2. If $f \in \Sigma_{n,b}$ satisfies the following subordination*

$$-\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \frac{1+(1-2\delta_1)z}{1-z},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let $f \in \Sigma_{n,b}$, and define the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = -zf'(z)/f(z)$ for $z \in \mathbb{D}^*$ and $p(0) = 1$. Then $p(z) = 1-(n+1)bz^{n+1}+\cdots \in \mathcal{H}_{\mu,n}$ with $\mu = -(n+1)b$. Simple computations shows that

$$-\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) = p(z)+zp'(z).$$

Hence the result follows from Lemma 4.2. ∎

Theorem 4.7 *Let $n \in \mathbb{N}$. For $\epsilon > 0$, $0 \leq \beta < 1$ and $0 < \mu = -(n+1)b \leq 2(1-\beta)$. Let δ_2 be given as in Theorem 4.3. If $f \in \Sigma_{n,b}$ satisfies the following subordination*

$$\epsilon\left(1+\frac{zf''(z)}{f'(z)}\right)-(1+\epsilon)\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\delta_2)z}{1-z},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = -zf'(z)/f(z)$ for $z \in \mathbb{D}^*$ and $p(0) = 1$. Then $p \in \mathcal{H}_{\mu,n}$. Simple computations shows that

$$\epsilon\left(1 + \frac{zf''(z)}{f'(z)}\right) - (1+\epsilon)\frac{zf'(z)}{f(z)} = p(z) + \epsilon\frac{zp'(z)}{p(z)}.$$

Hence the result follows from Lemma 4.3. ∎

Theorem 4.8 *Let $n \in \mathbb{N}$, $\alpha \geq 0$, $0 \leq \beta < 1$ and $0 < \mu = -(n+1)b \leq 2(1-\beta)$. Let δ_3 be given as in Theorem 4.4. If $f \in \Sigma_{n,b}$ satisfies*

$$\frac{zf'(z)}{f(z)}\left(2\alpha\frac{zf'(z)}{f(z)} - \alpha\frac{zf''(z)}{f'(z)} - 1\right) \prec \frac{1+(1-2\delta_3)z}{1-z},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let $f \in \Sigma_{n,b}$, and define the function $p : \mathbb{D} \to \mathbb{C}$ given by $p(z) = -zf'(z)/f(z)$ for $z \in \mathbb{D}^*$ and $p(0) = 1$. Then $p \in \mathcal{H}_{\mu,n}$ with $\mu = -(n+1)b$. A simple computation shows that

$$\frac{zf'(z)}{f(z)}\left(2\alpha\frac{zf'(z)}{f(z)} - \alpha\frac{zf''(z)}{f'(z)} - 1\right) = (1-\alpha)p(z) + \alpha p^2(z) + \alpha zp'(z).$$

Hence the result follows from Lemma 4.4. ∎

Theorem 4.9 *Let $n \in \mathbb{N}$, $0 \leq \beta < 1$ and $0 < \mu = -(n+1)b \leq 2(1-\beta)$. Let δ_4 be given as in Theorem 4.5. If $f \in \Sigma_{n,b}$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\delta_4)z}{1-z}$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. Let $p(z) = -zf'(z)/f(z)$ for $z \in \mathbb{D}^*$. It is clear that

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)}.$$

Hence the result follows from Lemma 4.6. ∎

4.3 Subordinations for univalence

Theorem 4.10–4.13 give sufficient conditions for the subordination $f'(z) \prec (1 + (1-2\beta)z)/(1-z)$ to hold. For $\beta = 0$, this latter condition is sufficient for the close-to-convexity and hence univalence of the function f.

Theorem 4.10 *Let $n \in \mathbb{N}$, $0 \leq \beta < 1$ and $0 < \mu = (n+1)b \leq 2(1-\beta)$. Let δ_1 be given as in Theorem 4.2. If $f \in \mathcal{A}_{n,b}$ satisfies following subordination*

$$f'(z) + zf''(z) \prec \frac{1 + (1-2\delta_1)z}{1-z},$$

then

$$f'(z) \prec \frac{1 + (1-2\beta)z}{1-z}.$$

Proof. For $f \in \mathcal{A}_{n,b}$, let the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = f'(z)$. Then $p(z) = 1 + (n+1)bz^n + (n+2)a_{n+2}z^{n+1} + \cdots \in \mathcal{H}_{\mu,n}$, with $\mu = (n+1)b$. Also, we have

$$f'(z) + zf''(z) = p(z) + zp'(z).$$

Hence the result follows from Lemma 4.2. ∎

Theorem 4.11 *Let $n \in \mathbb{N}$, $\epsilon \geq 0$, $0 \leq \beta < 1$ and $0 < \mu = (n+1)b \leq 2(1-\beta)$. Let δ_2 be given as in Theorem 4.3. If $f \in \mathcal{A}_{n,b}$ satisfies following subordination*

$$\epsilon\frac{zf''(z)}{f'(z)} + f'(z) \prec \frac{1 + (1-2\delta_2)z}{1-z},$$

then

$$f'(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. For $p(z) = f'(z)$, we have

$$\epsilon\frac{zf''(z)}{f'(z)} + f'(z) = p(z) + \epsilon\frac{zp'(z)}{p(z)},$$

and the result follows from Lemma 4.3. ∎

Theorem 4.11 reduces to the following result in the case when $\mu = 2(1-\beta)$, $n = 1$ and $\beta = 1/2$.

Corollary 4.2 *[112, Theorem 2, p. 182] For $\alpha \geq 0$, if*

$$\operatorname{Re}\left((1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \frac{1}{2},$$

then $\operatorname{Re} f'(z) > 1/2$.

Theorem 4.12 *Let $n \in \mathbb{N}$, $\alpha \geq 0$, $0 \leq \beta < 1$ and $0 < \mu = (n+1)b \leq 2(1-\beta)$. Let δ_3 be given as in Theorem 4.4. If $f \in \mathcal{A}_{n,b}$ satisfies*

$$f'(z)\left[\alpha\left(\frac{zf''(z)}{f'(z)} + f'(z) - 1\right) + 1\right] \prec \frac{1+(1-2\delta_3)z}{1-z},$$

then

$$f'(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. For $f \in \mathcal{A}_{n,b}$, let the function $p : \mathbb{D} \to \mathbb{C}$ be defined by $p(z) = f'(z)$. Since

$$f'(z)\left(\alpha\left(\frac{zf''(z)}{f'(z)} + f'(z) - 1\right) + 1\right) = (1-\alpha)p(z) + \alpha p^2(z) + \alpha zp'(z),$$

the result follows from Lemma 4.4. ∎

Theorem 4.13 *Let $n \in \mathbb{N}$, $0 \leq \beta < 1$ and $0 < \mu = (n+1)b \leq 2(1-\beta)$. Let δ_4 be given as in Theorem 4.5. If $f \in \mathcal{A}_{n,b}$ satisfies the subordination*

$$\frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\delta_4)z}{1-z}$$

then

$$f'(z) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Proof. For $p(z) = f'(z)$,

$$\frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)},$$

hence the result follows from Lemma 4.6. ∎

If $\mu = 2(1-\beta)$, $n = 1$ and $\beta = (\alpha+1)/2$, Theorem 4.13 reduces to the following result.

Corollary 4.3 *[94, Theorem 1] Let the function $f \in \mathcal{A}$ satisfy the inequality*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1+3\alpha}{2(1+\alpha)}, \quad 0 \leq \alpha < 1,$$

then $\operatorname{Re} f'(z) > (\alpha+1)/2$.

CHAPTER 5

CONVEXITY OF FUNCTIONS SATISFYING CERTAIN DIFFERENTIAL INEQUALITIES AND INTEGRAL OPERATORS

5.1 Introduction and preliminaries

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions f defined in the open unit disk $\mathbb{D}$. In this chapter, instead of considering functions with fixed second coefficient as in Chapter 4, we consider the more general function. For $a \in \mathbb{C}$, and n a positive integer, let

$$\mathcal{H}[a,n] = \left\{ f \in \mathcal{H}(\mathbb{D}) : f(z) = a + \sum_{k=n}^{\infty} a_k z^k \right\},$$

and

$$\mathcal{A}_n = \left\{ f \in \mathcal{H}(\mathbb{D}) : f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \right\}.$$

For $0 \leq \beta < 1$, recall $\mathcal{ST}(\beta)$ the subclass of $\mathcal{A}$ consisting of functions starlike of order β satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in \mathbb{D}.$$

Also for $0 \leq \beta < 1$, $\mathcal{CV}(\beta)$ is the subclass of $\mathcal{A}$ consisting of functions convex of order β satisfying

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, \quad z \in \mathbb{D}.$$

Several authors investigated the sufficient conditions to ensure starlikeness of functions. These include conditions in terms of differential inequalities, see for example [4,5,12,15,32,89,109,135]. Miller and Mocanu [62], Kuroki and Owa [49], and Ali *et al.* [9], determined conditions for starlikeness of functions defined by an integral operator of the form

$$f(z) = \int_0^1 W(r,z)\,dr,$$

or by the double integral operator

$$f(z) = \int_0^1 \int_0^1 W(r, s, z) dr ds.$$

More recently, Chandra *et al.* [28], obtained sufficient conditions for starlikeness of positive order for analytic functions satisfying certain third-order differential inequalities.

In this chapter, conditions that would imply convexity of positive order for functions satisfying certain second-order and third-order differential inequalities are determined. As a consequence, conditions on the kernel of certain integral operators are also obtained to ensure functions defined by these operators are convex.

We are going to use the concepts of subordination to help us to determine the convexity conditions. The following properties of differential subordination are needed.

Lemma 5.1 [38, Theorem 1, p. 192](see also [61, Theorem 3.1b, p. 71]) *Let h be convex in $\mathbb{D}$ with $h(0) = a$, $\gamma \neq 0$ and $\operatorname{Re}\gamma \geq 0$. If $p \in \mathcal{H}_n(a)$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{(\gamma/n)-1} dt.$$

The function q is convex and is the best $(a, n)-$dominant.

Lemma 5.2 [123](see also [61, Theorem 3.1d, p. 76]) *Let h be a starlike function* *with* $h(0) = 0$. *If* $p \in \mathcal{H}_n(a)$ *satisfies*

$$zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z) = a + \frac{1}{n}\int_0^z \frac{h(t)}{t}dt.$$

The function q *is convex and is the best* $(a, n)-$*dominant.*

Lemma 5.3 [3, Theorem 1, p. 13] *Let* n *be a positive integer and* α *real, with* $0 \leq \alpha < n$. *Let* q *be analytic with* $q(0) = 0$, $q'(0) \neq 0$ *and*

$$\operatorname{Re}\frac{zq''(z)}{q'(z)} + 1 > \frac{\alpha}{n}.$$

If $p \in \mathcal{H}_n(0)$ *satisfies*

$$zp'(z) - \alpha p(z) \prec nzq'(z) - \alpha q(z),$$

then $p(z) \prec q(z)$ *and this result is sharp.*

5.2 Convexity of functions satisfying second-order differential inequalities

The following theorem discusses on the function $F(f', f'') = \delta z f''(z) - \alpha(f'(z) - 1)$.

Theorem 5.1 *Let* $f \in \mathcal{A}_n$, $\delta > 0$, $0 \leq \alpha < n\delta$ *and* $0 \leq \beta < 1$. *If*

$$|\delta z f''(z) - \alpha(f'(z) - 1)| < \frac{(1-\beta)(n\delta - \alpha)}{n + 1 - \beta}, \tag{5.1}$$

then $f \in \mathcal{CV}(\beta)$.

Proof. We want to show that $\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta$. For notation simplicity, let

$$Q(z) = 1 + \frac{zf''(z)}{f'(z)}.$$

Then

$$f'(z)[\delta(Q(z)-1)-\alpha] = \delta z f''(z) - \alpha(f'(z)-1) - \alpha.$$

So by the triangle inequality and (5.1),

$$\begin{aligned} |f'(z)||\delta(Q(z)-1)-\alpha| &\le |\delta z f''(z) - \alpha(f'(z)-1| + \alpha \\ &< \frac{(1-\beta)(n\delta-\alpha)}{n+1-\beta} + \alpha. \end{aligned} \tag{5.2}$$

To determine the bound of $|f'(z)|$, note that the inequality (5.1) can be expressed in the subordination form

$$\delta z f''(z) - \alpha\left(f'(z)-1\right) \prec \frac{(1-\beta)(n\delta-\alpha)}{n+1-\beta} z, \qquad z \in \mathbb{D}.$$

By writing

$$P(z) = \delta f'(z) - (\delta+\alpha)\frac{f(z)}{z} = -\alpha + (n\delta-\alpha)a_{n+1}z^n + \cdots \in \mathcal{H}_n(-\alpha),$$

it follows that

$$\begin{aligned} P(z) + zP'(z) &= z\left[\delta f''(z) - (\delta+\alpha)\left(\frac{f'(z)}{z} - \frac{f(z)}{z^2}\right)\right] + \delta f'(z) - (\delta+\alpha)\frac{f(z)}{z} \\ &= \delta z f''(z) - \alpha f'(z) \\ &\prec \frac{(1-\beta)(n\delta-\alpha)}{n+1-\beta} z - \alpha. \end{aligned}$$

Let

$$h(z) = \frac{(1-\beta)(n\delta-\alpha)}{n+1-\beta} z - \alpha.$$

Then h is convex and $h(0) = -\alpha$, and hence Lemma 5.1 with $\gamma = 1$ yields

$$\begin{aligned}
P(z) &= \delta f'(z) - (\delta+\alpha)\frac{f(z)}{z} \\
&\prec \frac{1}{nz^{1/n}} \int_0^z \left[\frac{(1-\beta)(n\delta-\alpha)}{n+1-\beta} t - \alpha\right] t^{(1/n)-1} dt \\
&= \frac{1}{nz^{1/n}} \int_0^z \left[\frac{(1-\beta)(n\delta-\alpha)}{n+1-\beta} t^{(1/n)} - \alpha t^{(1/n)-1}\right] dt \\
&= \frac{(1-\beta)(n\delta-\alpha)}{(n+1)(n+1-\beta)} z - \alpha. \qquad (5.3)
\end{aligned}$$

Now consider

$$p(z) = \delta\left(\frac{f(z)}{z} - 1\right) \quad \text{and} \quad q(z) = \frac{\delta(1-\beta)}{(n+1)(n+1-\beta)} z.$$

Then

$$p \in \mathcal{H}_n(0),\ q(0) = 0,\ q'(0) = \frac{\delta(1-\beta)}{(n+1)(n+1-\beta)} \neq 0,\ \mathrm{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right) = 1 > \frac{\alpha}{\delta n}.$$

Also

$$\begin{aligned}
zp'(z) - \frac{\alpha}{\delta} p(z) &= z\delta\left(\frac{f'(z)}{z} - \frac{f(z)}{z^2}\right) - \frac{\alpha}{\delta}\delta\left(\frac{f(z)}{z} - 1\right) \\
&= \delta f'(z) - (\delta+\alpha)\frac{f(z)}{z} + \alpha,
\end{aligned}$$

and

$$\begin{aligned}
nzq'(z) - \frac{\alpha}{\delta} q(z) &= nz\frac{\delta(1-\beta)}{(n+1)(n+1-\beta)} - \frac{\alpha}{\delta}\frac{\delta(1-\beta)}{(n+1)(n+1-\beta)} z \\
&= \frac{(1-\beta)(n\delta-\alpha)}{(n+1)(n+1-\beta)} z.
\end{aligned}$$

Then the subordination (5.3) can be written as

$$zp'(z) - \frac{\alpha}{\delta}p(z) \prec nzq'(z) - \frac{\alpha}{\delta}q(z).$$

Applying Lemma 5.3 gives $p \prec q$, which implies

$$\frac{f(z)}{z} \prec 1 + \frac{1-\beta}{(n+1)(n+1-\beta)}z.$$

By the definition of subordination, we have

$$\begin{aligned}
\frac{f(z)}{z} &= 1 + \frac{1-\beta}{(n+1)(n+1-\beta)}w(z) \qquad (|w(z)| < 1),\\
\left|\frac{f(z)}{z}\right| &= \left|1 + \frac{1-\beta}{(n+1)(n+1-\beta)}w(z)\right|\\
&> 1 - \frac{1-\beta}{(n+1)(n+1-\beta)}|w(z)|\\
&> 1 - \frac{1-\beta}{(n+1)(n+1-\beta)}.
\end{aligned}$$

From subordination (5.3) and the inequality $-\big|a-b\big| \leq |a| - |b|$,

$$\begin{aligned}
&\left|\delta f'(z) - (\delta+\alpha)\frac{f(z)}{z}\right| < \left|\frac{(1-\beta)(n\delta-\alpha)}{(n+1)(n+1-\beta)}z - \alpha\right|,\\
&|\delta f'(z)| - \big|(\delta+\alpha)\frac{f(z)}{z}\big| > -\left|\frac{(1-\beta)(n\delta-\alpha)}{(n+1)(n+1-\beta)}z - \alpha\right|\\
&\delta|f'(z)| > (\delta+\alpha)\left|\frac{f(z)}{z}\right| - \left|\frac{(1-\beta)(n\delta-\alpha)}{(n+1)(n+1-\beta)}z - \alpha\right|\\
&\qquad > (\delta+\alpha)\left(1 - \frac{(1-\beta)}{(n+1)(n+1-\beta)}\right) - \left(\frac{(1-\beta)(n\delta-\alpha)}{(n+1)(n+1-\beta)} + \alpha\right),\\
&|f'(z)| > \frac{n}{n+1-\beta}.
\end{aligned}$$

Substituting this bound of $|f'(z)|$ in (5.2), we get

$$\frac{n}{n+1-\beta}|\delta(Q(z)-1) - \alpha| < \frac{(1-\beta)(n\delta-\alpha)}{(n+1-\beta)} + \alpha,$$

$$|Q(z) - 1 - \frac{\alpha}{\delta}| < 1 - \beta + \frac{\alpha}{\delta}.$$

Since $-|w| \le \operatorname{Re} w$, it follows that

$$\operatorname{Re}\left(Q(z) - 1 - \frac{\alpha}{\delta}\right) > -\left(1 - \beta + \frac{\alpha}{\delta}\right).$$

Hence $\operatorname{Re} Q(z) > \beta$, i.e.,

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,$$

as desired. ∎

Theorem 5.2 *Let $\delta > 0$, $0 \le \alpha < n\delta$, $0 \le \beta < 1$ and $g \in \mathcal{H}$. If*

$$|g(z)| < \frac{(1-\beta)(n\delta - \alpha)}{n + 1 - \beta},$$

then $f \in \mathcal{A}_n$ given by

$$f(z) = z + \frac{z^{n+1}}{\delta} \int_0^1 \int_0^1 g(rsz) r^{[(n-1)\delta - \alpha]/\delta} s^n \, dr ds$$

is a convex function of order β.

Proof. Let $f \in \mathcal{A}_n$ satisfying the differential equation

$$\delta z f''(z) - \alpha(f'(z) - 1) = z^n g(z). \tag{5.4}$$

From Theorem 5.1, it is clear that the solution f of (5.4) is a convex function of order β. We find the closed form of f. Let $\phi(z) = f'(z) - 1$, then (5.4) becomes

$$\delta z \phi'(z) - \alpha \phi(z) = z^n g(z). \tag{5.5}$$

Using the integrating factor $e^{\int \frac{-\alpha}{\delta z}dz} = z^{\frac{-\alpha}{\delta}}$, (5.5) becomes

$$\begin{aligned}
z^{\frac{-\alpha}{\delta}}\phi'(z) - \frac{\alpha}{\delta}z^{\frac{-\alpha}{\delta}-1}\phi(z) &= \frac{z^{n-1-\frac{\alpha}{\delta}}}{\delta}g(z) \\
\frac{d}{dz}\left[z^{\frac{-\alpha}{\delta}}\phi(z)\right] &= \frac{z^{n-1-\frac{\alpha}{\delta}}}{\delta}g(z) \\
z^{\frac{-\alpha}{\delta}}\phi(z) &= \frac{1}{\delta}\int_0^z \xi^{n-1-\frac{\alpha}{\delta}}g(\xi)d\xi \\
\phi(z) &= \frac{z^{\frac{\alpha}{\delta}}}{\delta}\int_0^z g(\xi)\xi^{n-1-\frac{\alpha}{\delta}}d\xi \\
&= \frac{z^{\frac{\alpha}{\delta}}}{\delta}\int_0^1 g(rz)(rz)^{n-1-\frac{\alpha}{\delta}}zdr \\
&= \frac{z^n}{\delta}\int_0^1 g(rz)r^{n-1-\frac{\alpha}{\delta}}dr.
\end{aligned}$$

Since $\phi(z) = f'(z) - 1$, we have

$$\begin{aligned}
f'(z) - 1 &= \frac{z^n}{\delta}\int_0^1 g(rz)r^{n-1-\frac{\alpha}{\delta}}dr \\
f(z) &= z + \int_0^z \frac{\xi^n}{\delta}\int_0^1 g(r\xi)r^{n-1-\frac{\alpha}{\delta}}drd\xi \\
&= z + \int_0^1 \frac{(sz)^n}{\delta}\int_0^1 g(rsz)r^{n-1-\frac{\alpha}{\delta}}zdrds \\
&= z + \frac{z^{n+1}}{\delta}\int_0^1\int_0^1 g(rsz)r^{[(n-1)\delta-\alpha]/\delta}s^n drds.
\end{aligned}$$

This completes the proof. ∎

The next theorem discusses on the function $F(f, f', f'') = \delta z f''(z) - \alpha(f'(z) - f(z)/z)$.

Theorem 5.3 *Let $f \in \mathcal{A}_n$, $\delta > 0$, and $0 \le \alpha < \delta$, with $0 \le \beta < 1$. If*

$$\left|\delta z f''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{n(1-\beta)\big(\delta(n+1)-\alpha\big)}{(n+1)(n+1-\beta)}, \tag{5.6}$$

then $f \in \mathcal{CV}(\beta)$.

Proof. We again want to show that $\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta$. Let

$$Q(z) = 1 + \frac{zf''(z)}{f'(z)}.$$

Then rewriting (5.6), we have

$$\left|\delta f'(z)(Q(z)-1) - \alpha\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{n(1-\beta)[\delta(n+1)-\alpha]}{(n+1)(n+1-\beta)}$$

and after some computation we have

$$\delta|f'(z)||Q(z)-1| < \frac{n(1-\beta)[\delta(n+1)-\alpha]}{(n+1)(n+1-\beta)} + \alpha\left|f'(z) - \frac{f(z)}{z}\right|. \tag{5.7}$$

So we need to determine the bounds for $|f'(z)|$ and $\left|f'(z) - f(z)/z\right|$.

First, to determine the bound of $\left|f'(z) - f(z)/z\right|$, note that the inequality (5.6) can be expressed in the subordination form

$$\delta zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right) \prec \frac{n(1-\beta)(\delta(n+1)-\alpha)}{(n+1)(n+1-\beta)}z, \qquad z \in \mathbb{D}. \tag{5.8}$$

Writing

$$P(z) = \delta\left(f'(z) - \frac{f(z)}{z}\right) = \delta n a_{n+1} z^n + \cdots \in \mathcal{H}_n(0),$$

it follows that

$$\begin{aligned}
&\left(\frac{\delta-\alpha}{\delta}\right)P(z) + zP'(z) \\
&= z\left[\delta f''(z) - \delta\left(\frac{f'(z)}{z} - \frac{f(z)}{z^2}\right)\right] + \frac{\delta-\alpha}{\delta}\delta\left(f'(z) - \frac{f(z)}{z}\right) \\
&= \delta zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right) \\
&\prec \frac{n(1-\beta)(\delta(n+1)-\alpha)}{(n+1)(n+1-\beta)}z.
\end{aligned}$$

By letting

$$h(z) = \frac{\delta n(1-\beta)(\delta(n+1)-\alpha)}{(\delta-\alpha)(n+1)(n+1-\beta)} z,$$

then h is convex and $h(0) = 0$, and hence by applying Lemma 5.1 with $\gamma = (\delta-\alpha)/\delta$ yields

$$\begin{aligned}
P(z) &= \delta\left(f'(z) - \frac{f(z)}{z}\right) \\
&\prec \frac{(\delta-\alpha)}{\delta}\frac{1}{nz^{(\delta-\alpha)/\delta n}}\int_0^z \left(\frac{\delta n(1-\beta)(\delta(n+1)-\alpha)}{(\delta-\alpha)(n+1)(n+1-\beta)}t\right) t^{(\delta-\alpha)/\delta n-1} dt \\
&= \frac{(1-\beta)(\delta(n+1)-\alpha)}{(n+1)(n+1-\beta)}\frac{1}{z^{(\delta-\alpha)/\delta n}}\int_0^z t^{(\delta-\alpha)/\delta n} dt \\
&= \frac{\delta n(1-\beta)}{(n+1)(n+1-\beta)} z,
\end{aligned}$$

which implies

$$f'(z) - \frac{f(z)}{z} \prec \frac{n(1-\beta)}{(n+1)(n+1-\beta)} z, \tag{5.9}$$

or equivalently

$$\left|f'(z) - \frac{f(z)}{z}\right| < \frac{n(1-\beta)}{(n+1)(n+1-\beta)}. \tag{5.10}$$

Next consider

$$p(z) = \frac{f(z)}{z}, \quad zp'(z) = f'(z) - \frac{f(z)}{z}.$$

Then (5.9) can be rewritten as

$$zp'(z) \prec \frac{n(1-\beta)}{(n+1)(n+1-\beta)} z.$$

Applying Lemma 5.2,

$$p(z) = \frac{f(z)}{z} \prec 1 + \frac{1}{n}\int_0^z \frac{n(1-\beta)}{(n+1)(n+1-\beta)} dt = 1 + \frac{(1-\beta)}{(n+1)(n+1-\beta)} z,$$

and hence (same computation as in the proof of Theorem 5.1)

$$\left|\frac{f(z)}{z}\right| > 1 - \frac{(1-\beta)}{(n+1)(n+1-\beta)}. \tag{5.11}$$

Using $-|a-b| \leq |a| - |b|$, and (5.11), (5.10) becomes

$$\begin{aligned}
|f'(z)| - \left|\frac{f(z)}{z}\right| &> \frac{-n(1-\beta)}{(n+1)(n+1-\beta)} \\
|f'(z)| &> \left(1 - \frac{(1-\beta)}{(n+1)(n+1-\beta)}\right) - \left(\frac{n(1-\beta)}{(n+1)(n+1-\beta)}\right) \\
&= \frac{n}{n+1-\beta}.
\end{aligned} \tag{5.12}$$

Substituting (5.12) and (5.10) in (5.7) yields

$$\begin{aligned}
\frac{\delta n}{n+1-\beta}|Q(z)-1| &< \frac{n(1-\beta)[\delta(n+1)-\alpha]}{(n+1)(n+1-\beta)} + \alpha\left(\frac{n(1-\beta)}{(n+1)(n+1-\beta)}\right), \\
&= \frac{\delta n(1-\beta)}{n+1-\beta}, \\
|Q(z)-1| &< 1-\beta.
\end{aligned}$$

Since $-|w| \leq \operatorname{Re} w$, it follows that

$$\operatorname{Re}(Q(z)-1) > -(1-\beta),$$

and hence $\operatorname{Re} Q(z) > \beta$. This completes the proof. ∎

Theorem 5.4 *Let $\delta > 0$, $0 \leq \alpha < \delta$, $0 \leq \beta < 1$ and $g \in \mathcal{H}$. If*

$$|g(z)| < \frac{n(1-\beta)(\delta(n+1)-\alpha)}{(n+1)(n+1-\beta)},$$

then $f \in \mathcal{A}_n$ defined by

$$f(z) = z + \frac{z^{n+1}}{\delta}\int_0^1\int_0^1 g(rsz)r^{(n\delta-\alpha)/\delta}s^{n-1}drds$$

is a convex function of order β.

Proof. Let $f \in \mathcal{A}_n$ satisfying

$$\delta z f''(z) - \alpha \left(f'(z) - \frac{f(z)}{z} \right) = z^n g(z). \tag{5.13}$$

From Theorem 5.3, it is clear that the solution f of (5.13) is a convex function of order β. Let

$$\phi(z) = \delta \left(f'(z) - \frac{f(z)}{z} \right),$$

then

$$\begin{aligned} z\phi'(z) + \frac{(\delta - \alpha)}{\delta}\phi(z) &= z\delta \left[f''(z) - \left(\frac{f'(z)}{z} - \frac{f(z)}{z^2} \right) \right] + \frac{\delta - \alpha}{\delta}\delta \left(f'(z) - \frac{f(z)}{z} \right) \\ &= \delta z f''(z) - \alpha \left(f'(z) - \frac{f(z)}{z} \right) \\ &= z^n g(z). \end{aligned} \tag{5.14}$$

Using the integrating factor $e^{\int \frac{\delta-\alpha}{\delta z} dz} = z^{\frac{\delta-\alpha}{\delta}}$, (5.14) becomes

$$\begin{aligned} z^{\frac{\delta-\alpha}{\delta}}\phi'(z) + \frac{(\delta - \alpha)}{\delta} z^{\frac{\delta-\alpha}{\delta}-1}\phi(z) &= z^{n-1+\frac{\delta-\alpha}{\delta}} g(z) \\ \frac{d}{dz}\left[z^{\frac{\delta-\alpha}{\delta}}\phi(z) \right] &= z^{n-1+\frac{\delta-\alpha}{\delta}} g(z) \\ z^{\frac{\delta-\alpha}{\delta}}\phi(z) &= \int_0^z \xi^{n-1+\frac{\delta-\alpha}{\delta}} g(\xi) d\xi \\ \phi(z) &= z^{-\frac{\delta-\alpha}{\delta}} \int_0^z g(\xi)\xi^{n-1+\frac{\delta-\alpha}{\delta}} d\xi \\ &= z^{-\frac{\delta-\alpha}{\delta}} \int_0^1 g(rz)(rz)^{n-1+\frac{\delta-\alpha}{\delta}} z dr \\ &= z^n \int_0^1 g(rz) r^{n-1+\frac{\delta-\alpha}{\delta}} dr. \end{aligned}$$

Substituting $\phi(z) = \delta\left(f'(z) - f(z)/z\right)$, we have

$$\begin{aligned}
\delta\left(f'(z) - \frac{f(z)}{z}\right) &= z^n \int_0^1 g(rz) r^{n-1+\frac{\delta-\alpha}{\delta}} dr \\
\delta z \left(\frac{f(z)}{z}\right)' &= z^n \int_0^1 g(rz) r^{n-1+\frac{\delta-\alpha}{\delta}} dr \\
\left(\frac{f(z)}{z}\right)' &= \frac{z^{n-1}}{\delta} \int_0^1 g(rz) r^{n-1+\frac{\delta-\alpha}{\delta}} dr.
\end{aligned}$$

Another integration gives

$$\begin{aligned}
\frac{f(z)}{z} - 1 &= \int_0^z \frac{\xi^{n-1}}{\delta} \int_0^1 g(r\xi) r^{n-1+\frac{\delta-\alpha}{\delta}} dr d\xi \\
&= \frac{1}{\delta} \int_0^1 (zs)^{n-1} \int_0^1 g(rsz) r^{n-1+\frac{\delta-\alpha}{\delta}} z dr ds \\
&= \frac{1}{\delta} \int_0^1 (zs)^{n-1} \int_0^1 g(rsz) r^{n-1+\frac{\delta-\alpha}{\delta}} z dr ds \\
f(z) &= z + \frac{z^{n+1}}{\delta} \int_0^1 \int_0^1 g(rsz) r^{(n\delta-\alpha)/\delta} s^{n-1} dr ds.
\end{aligned}$$

This completes the proof. ∎

5.3 Convexity of functions satisfying third-order differential inequalities

Theorem 5.5 *Let $f \in \mathcal{A}_n$, $0 < \alpha < n\nu$, $\mu > 0$ and $0 \leq \beta < 1$. If*

$$|\gamma z^2 f'''(z) + \delta z f''(z) - \alpha(f'(z) - 1)| < \frac{(1 + n\mu)(1 - \beta)(n\nu - \alpha)}{n + 1 - \beta}, \tag{5.15}$$

where $\nu - \alpha\mu = \delta - \gamma$ and $\nu\mu = \gamma$, then $f \in \mathcal{CV}(\beta)$.

Proof. Let

$$p(z) = \nu z f''(z) - \alpha(f'(z) - 1).$$

Then

$$\begin{aligned}
p(z) + \mu z p'(z) &= \nu z f''(z) - \alpha(f'(z) - 1) + \mu z[\nu f''(z) + \nu z f'''(z) - \alpha f''(z)] \\
&= \mu\nu z^2 f'''(z) + z f''[\nu - \mu\alpha + \mu\nu] - \alpha(f'(z) - 1) \\
&= \gamma z^2 f'''(z) + \delta z f''(z) - \alpha(f'(z) - 1), \qquad (5.16)
\end{aligned}$$

where $\mu\nu = \gamma$, $\nu - \mu\alpha + \mu\nu = \delta$. Hence, by (5.16) and (5.15), we have

$$p(z) + \mu z p'(z) \prec \frac{(1 + n\mu)(1 - \beta)(n\nu - \alpha)}{n + 1 - \beta} z,$$

$$p(z) + \frac{zp'(z)}{\frac{1}{\mu}} \prec \frac{(1 + n\mu)(1 - \beta)(n\nu - \alpha)}{n + 1 - \beta} z.$$

Apply Lemma 5.1 to obtain

$$\begin{aligned}
p(z) &= \nu z f''(z) - \alpha(f'(z) - 1) \\
&\prec \frac{1}{\mu}.\frac{1}{nz^{1/n\mu}} \int_0^z \frac{(1 + n\mu)(1 - \beta)(n\nu - \alpha)}{n + 1 - \beta} t.t^{(1/n\mu)-1} dt \\
&= \frac{1}{\mu n z^{1/n\mu}} \int_0^z \frac{(1 + n\mu)(1 - \beta)(n\nu - \alpha)}{n + 1 - \beta} t^{(1/n\mu)} dt \\
&= \frac{1}{\mu n z^{1/n\mu}} \left[\frac{n\mu(1 + n\mu)(1 - \beta)(n\nu - \alpha)}{(1 + n\mu)(n + 1 - \beta)} z^{(1/n\mu)+1} \right] \\
&= \frac{(1 - \beta)(n\nu - \alpha)}{n + 1 - \beta} z
\end{aligned}$$

which yields

$$|\nu z f''(z) - \alpha(f'(z) - 1)| < \frac{(1 - \beta)(n\nu - \alpha)}{n + 1 - \beta}.$$

Hence, applying Theorem 5.1, $f \in \mathcal{CV}(\beta)$. ∎

Example 5.1 *For the function*

$$f(z) = z + \frac{(1 - \beta)}{(n + 1)(n + 1 - \beta)} z^{n+1}, \quad 0 \le \beta < 1,$$

we have

$$\begin{aligned}
&|\gamma z^2 f'''(z) + \delta z f''(z) - \alpha(f'(z) - 1)| \\
&= \left| \frac{\gamma n(n-1)(1-\beta)z^n}{n+1-\beta} + \frac{\delta n(1-\beta)z^n}{n+1-\beta} - \frac{\alpha(1-\beta)z^n}{n+1-\beta} \right| \\
&= \left| \frac{(1-\beta)(1+n\mu)(n\nu-\alpha)}{n+1-\beta} \right| |z|^n \\
&< \frac{(1-\beta)(1+n\mu)(n\nu-\alpha)}{n+1-\beta}.
\end{aligned}$$

So by Theorem 5.5, $f \in \mathcal{CV}(\beta)$. Indeed we have

$$\begin{aligned}
\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) &= \operatorname{Re}\left(\frac{1 + ((n+1)(1-\beta)/(n+1-\beta))z^n}{1 + ((1-\beta)/(n+1-\beta))z^n}\right) \\
&> \frac{1 - ((n+1)(1-\beta)/(n+1-\beta))}{1 - ((1-\beta)/(n+1-\beta))} = \beta.
\end{aligned}$$

Theorem 5.6 *Let $0 < \alpha < n\nu$, $\mu > 0$, $0 \le \beta < 1$ and $g \in \mathcal{H}$. If*

$$|g(z)| < \frac{(1+\mu n)(1-\beta)(n\nu-\alpha)}{n+1-\beta},$$

then $f \in \mathcal{A}_n$ defined by

$$f(z) = z + \frac{z^{n+1}}{\mu\nu} \int_0^1 \int_0^1 \int_0^1 g(rstz) r^{n-1-\frac{\alpha}{\nu}} s^n t^{n-1+\frac{1}{\mu}} dr ds dt$$

is a convex function of order β.

Proof. Let $f \in \mathcal{A}_n$ satisfying

$$\gamma z^2 f'''(z) + \delta z f''(z) - \alpha\left(f'(z) - 1\right) = z^n g(z) \tag{5.17}$$

where $\nu - \alpha\mu = \delta - \gamma$ and $\nu\mu = \gamma$. From Theorem 5.5, we note that the solution of differential equation (5.17) is convex of order β. Let $p(z) = \nu z f''(z) - \alpha(f'(z) - 1)$.

Then equation (5.17) becomes

$$p(z) + \mu z p'(z) = z^n g(z). \tag{5.18}$$

Using the integrating factor $e^{\int \frac{1}{\mu z} dz} = z^{\frac{1}{\mu}}$, equation (5.18) simplifies to

$$\begin{aligned}
zp'(z) + \frac{1}{\mu} p(z) &= \frac{z^n}{\mu} g(z) \\
z^{\frac{1}{\mu}} p'(z) + \left(\frac{1}{\mu}\right) z^{\frac{1}{\mu}-1} p(z) &= \frac{z^{n-1+\frac{1}{\mu}}}{\mu} g(z) \\
\frac{d}{dz}\left[z^{\frac{1}{\mu}} p(z)\right] &= \frac{z^{n-1+\frac{1}{\mu}}}{\mu} g(z) \\
z^{\frac{1}{\mu}} p(z) &= \frac{1}{\mu} \int_0^z \xi^{n-1+\frac{1}{\mu}} g(\xi) d\xi \\
p(z) &= \frac{z^{-\frac{1}{\mu}}}{\mu} \int_0^z g(\xi) \xi^{n-1+\frac{1}{\mu}} d\xi \\
&= \frac{z^{-\frac{1}{\mu}}}{\mu} \int_0^1 g(tz)(tz)^{n+\frac{1}{\mu}-1} z dt \\
&= \frac{z^n}{\mu} \int_0^1 g(tz) t^{n+\frac{1}{\mu}-1} dt \\
&= z^n \phi(z),
\end{aligned}$$

where

$$\phi(z) = \frac{1}{\mu} \int_0^1 g(tz) t^{n+\frac{1}{\mu}-1} dt. \tag{5.19}$$

Note that the function f in Theorem 5.2 satisfies $\delta z f''(z) - \alpha(f'(z) - 1) = z^n g(z)$. Then by replacing the appropriate parameters, the equation

$$\nu z f''(z) - \alpha(f'(z) - 1) = z^n \phi(z)$$

has a solution

$$\begin{aligned}
f(z) &= z + \frac{z^{n+1}}{\nu} \int_0^1 \int_0^1 \phi(rsz) r^{n-\frac{\alpha}{\nu}-1} s^n drds \\
&= z + \frac{z^{n+1}}{\nu} \int_0^1 \int_0^1 \left[\frac{1}{\mu} \int_0^1 g(trsz) t^{n+\frac{1}{\mu}-1} dt \right] r^{n-\frac{\alpha}{\nu}-1} s^n drds \\
&= z + \frac{z^{n+1}}{\mu\nu} \int_0^1 \int_0^1 \int_0^1 g(rstz) r^{n-\frac{\alpha}{\nu}-1} s^n t^{n+\frac{1}{\mu}-1} drdsdt.
\end{aligned}$$

This completes the proof. ∎

Theorem 5.7 *Let $f \in \mathcal{A}_n$, $0 < \mu < 1$, $0 \leq \alpha < (1-\mu)\nu$ and $0 \leq \beta < 1$. If*

$$\left| \gamma z^2 f'''(z) + \delta z f''(z) - \alpha \left(f'(z) - \frac{f(z)}{z} \right) \right| < \frac{n[(n+1)(\nu-\gamma)-\alpha](1-\beta)(1+n\mu)}{(1-\mu)(n+1)(n+1-\beta)}, \tag{5.20}$$

where $\nu - \frac{\alpha\mu}{1-\mu} = \delta - \gamma$ and $\nu\mu = \gamma$, then $f \in \mathcal{CV}(\beta)$.

Proof. Let $p(z) = \nu z f''(z) - \frac{\alpha}{1-\mu} \left(f'(z) - \frac{f(z)}{z} \right)$. Then

$$\begin{aligned}
p(z) + \mu z p'(z) &= \nu z f''(z) - \frac{\alpha}{1-\mu} \left(f'(z) - \frac{f(z)}{z} \right) + \mu\nu z^2 f'''(z) + \mu\nu z f''(z) \\
&\quad - \frac{\mu\alpha}{1-\mu} z f''(z) + \frac{\mu\alpha}{1-\mu} \left(f'(z) - \frac{f(z)}{z} \right) \\
&= \mu\nu z^2 f'''(z) + \left(\mu\nu - \frac{\alpha\mu}{1-\mu} + \nu \right) z f''(z) - \alpha \left(f'(z) - \frac{f(z)}{z} \right) \\
&= \gamma z^2 f'''(z) + \delta z f''(z) - \alpha \left(f'(z) - \frac{f(z)}{z} \right),
\end{aligned} \tag{5.21}$$

where $\mu\nu = \gamma$ and $\mu\nu - \frac{\alpha\mu}{1-\mu} + \nu = \delta$.

By (5.21) and (5.20),

$$p(z) + \mu z p'(z) \prec \frac{n(1-\beta)(1+n\mu)[(n+1)(\nu-\gamma)-\alpha]}{(1-\mu)(n+1)(n+1-\beta)} z.$$

By applying Lemma 5.1, we obtain

$$\begin{aligned}
p(z) &\prec \frac{1}{\mu}\frac{1}{nz^{\frac{1}{n\mu}}}\int_0^z \frac{(1+n\mu)(1-\beta)}{(n+1-\beta)(n+1)(1-\mu)} n[(n+1)(\nu-\gamma)-\alpha]tt^{\frac{1}{n\mu}-1}dt \\
&= \frac{1}{\mu n z^{\frac{1}{n\mu}}}\int_0^z \frac{(1+n\mu)(1-\beta)}{(n+1-\beta)(n+1)(1-\mu)} n[(n+1)(\nu-\gamma)-\alpha]t^{\frac{1}{n\mu}}dt \\
&= \frac{n(1-\beta)[(n+1)(\nu-\gamma)-\alpha]}{(1-\mu)(n+1)(n+1-\beta)}z, \\
&= \frac{n(1-\beta)}{(n+1)(n+1-\beta)}\left[\frac{(n+1)(\nu-\nu\mu)}{(1-\mu)} - \frac{\alpha}{(1-\mu)}\right]z \\
&= \frac{n(1-\beta)}{(n+1)(n+1-\beta)}\left[\nu(n+1) - \frac{\alpha}{(1-\mu)}\right]z.
\end{aligned}$$

Since $p(z) = \nu z f''(z) - \frac{\alpha}{1-\mu}\left(f'(z) - \frac{f(z)}{z}\right)$, the above subordination implies

$$\left|\nu z f''(z) - \frac{\alpha}{1-\mu}\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{n(1-\beta)\left(\nu(n+1) - \frac{\alpha}{1-\mu}\right)}{(n+1)(n+1-\beta)}.$$

Hence, by Theorem 5.3, $f \in \mathcal{CV}(\beta)$. ∎

Theorem 5.8 *Let $0 < \mu < 1$, $0 \le \alpha < (1-\mu)\nu$, $0 \le \beta < 1$ and $g \in \mathcal{H}$. If*

$$|g(z)| < \frac{n(1-\beta)(1+n\mu)[(n+1)(\nu-\gamma)-\alpha]}{(1-\mu)(n+1)(n+1-\beta)},$$

where $\nu\mu = \gamma$, then

$$f(z) = z + \frac{z^{n+1}}{\gamma}\int_0^1\int_0^1\int_0^1 g(rstz)r^{\frac{1}{\nu}[n\nu - \frac{\alpha}{1-\mu}]}s^n t^{n-1+\frac{1}{\mu}}drdsdt$$

is a convex function of order β.

Proof. Suppose $f \in \mathcal{A}_n$ satisfies

$$\gamma z^2 f'''(z) + \delta z f''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right) = z^n g(z), \tag{5.22}$$

where $\gamma + \nu - \frac{\alpha\mu}{1-\mu} = \delta$. Then by Theorem 5.7, f is a convex function of order β.

Let

$$p(z) = \nu z f''(z) - \frac{\alpha}{1-\mu}\left(f'(z) - \frac{f(z)}{z}\right).$$

Now equation (5.22) can be written as

$$p(z) + \mu z p'(z) = z^n g(z). \tag{5.23}$$

Similarly as in the proof of Theorem 5.6, equation (5.23) has the solution

$$p(z) = z^n \phi(z),$$

that is,

$$\nu z f''(z) - \frac{\alpha}{1-\mu}\left(f'(z) - \frac{f(z)}{z}\right) = z^n \phi(z),$$

where

$$\phi(z) = \frac{1}{\mu}\int_0^1 g(tz) t^{n+\frac{1}{\mu}-1} dt.$$

On the other hand, note that the function f in Theorem 5.4 satisfies

$$\delta z f''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right) = z^n g(z).$$

Hence by Theorem 5.4, the equation

$$\nu z f''(z) - \frac{\alpha}{1-\mu}\left(f'(z) - \frac{f(z)}{z}\right) = z^n \phi(z)$$

has the solution

$$\begin{aligned} f(z) &= z + \frac{z^{n+1}}{\nu}\int_0^1\int_0^1 \phi(rsz) r^{\frac{1}{\nu}[n\nu - \frac{\alpha}{1-\mu}]} s^{n-1} dr ds \\ &= z + \frac{z^{n+1}}{\nu}\int_0^1\int_0^1 \left[\frac{1}{\mu}\int_0^1 g(trsz) t^{n+\frac{1}{\mu}-1} dt\right] r^{\frac{1}{\nu}[n\nu - \frac{\alpha}{1-\mu}]} s^{n-1} dr ds \end{aligned}$$

$$= z + \frac{z^{n+1}}{\gamma} \int_0^1 \int_0^1 \int_0^1 g(rstz) r^{\frac{1}{\nu}[n\nu - \frac{\alpha}{1-\mu}]} s^{n-1} t^{n-1+\frac{1}{\mu}} drdsdt.$$

This completes the proof. ∎

CHAPTER 6

CLOSE-TO-CONVEXITY AND STARLIKENESS OF ANALYTIC FUNCTIONS

6.1 Introduction and preliminaries

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. For a fixed $p \in \mathbb{N} := \{1, 2, \dots\}$, let $\mathcal{A}_p$ be the class of all analytic functions of the form $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \dots$, that are p-valent (multivalent) in the open unit disk, with $\mathcal{A} := \mathcal{A}_1$. Let $\mathcal{A}_{p,n}$ be the class of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ of the form $f(z) = z^p + a_{n+p}z^{n+p} + a_{n+p+1}z^{n+p+1} + \dots$ with $\mathcal{A}_p := \mathcal{A}_{p,1}$ and $\mathcal{A} := \mathcal{A}_{1,1}$.

For studies related to multivalent functions, see [57,83–86]. Also recall $\mathcal{CCV}(\alpha)$, the subclass of $\mathcal{A}$ consisting of functions which are close-to-convex of order α in $\mathbb{D}$ $(0 \le \alpha < 1)$ defined by

$$\mathcal{CCV}(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > \alpha, \quad g \in \mathcal{CV} \right\}.$$

Singh and Singh [113] obtained several interesting conditions for functions $f \in \mathcal{A}$ satisfying inequalities involving $f'(z)$ and $zf''(z)$ to be univalent or starlike in $\mathbb{D}$. Owa *et al.* [94] generalized the results of Singh and Singh [113] and also obtained several sufficient conditions for close-to-convexity, starlikeness and convexity of functions $f \in \mathcal{A}$. In fact, they proved the following theorems.

Theorem 6.1 [94, Theorems 1-3] *Let $0 \le \alpha < 1$ and $\beta, \gamma \ge 0$. If $f \in \mathcal{A}$, then*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1+3\alpha}{2(1+\alpha)} \Longrightarrow \operatorname{Re}\left(f'(z)\right) > \frac{1+\alpha}{2},$$

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3+2\alpha}{(2+\alpha)} \Longrightarrow \left|f'(z) - 1\right| < 1 + \alpha,$$

$$\left|f'(z) - 1\right|^{\beta} |zf''(z)|^{\gamma} < \frac{(1-\alpha)^{\beta+\gamma}}{2^{\beta+2\gamma}} \Longrightarrow \operatorname{Re}\left(f'(z)\right) > \frac{1+\alpha}{2}.$$

Theorem 6.2 [94, Theorem 4] *Let $1 < \lambda < 3$. If $f \in \mathcal{A}$, then*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < \begin{cases} \frac{5\lambda-1}{2(\lambda+1)}, & 1<\lambda\leq 2; \\ \\ \frac{\lambda+1}{2(\lambda-1)}, & 2<\lambda<3, \end{cases} \quad \Longrightarrow \frac{zf'(z)}{f(z)} \prec \frac{\lambda(1-z)}{\lambda-z}.$$

In this chapter, the above results are extended to functions $f \in \mathcal{A}_{p,n}$.

6.2 Close-to-convexity and Starlikeness

For the proof of our main results, we need the following lemma.

Lemma 6.1 [61, Lemma 2.2a] *Let $z_0 \in \mathbb{D}$ and $r_0 = |z_0|$. Let $f(z) = a_n z^n + a_{n+1}z^{n+1} + \cdots$ be continuous on $\overline{\mathbb{D}}_{r_0}$ and analytic on $\mathbb{D}_{r_0} \cup \{z_0\}$ with $f(z) \not\equiv 0$ and $n \geq 1$. If*

$$|f(z_0)| = \max\{|f(z)| : z \in \overline{\mathbb{D}}_{r_0}\},$$

then there exists an $m \geq n$ such that

1. $\dfrac{z_0 f'(z_0)}{f(z_0)} = m$, *and*
2. $\operatorname{Re}\dfrac{z_0 f''(z_0)}{f'(z_0)} + 1 \geq m$.

Recall that for $f \in \mathcal{A}$, the condition $\operatorname{Re} f'(z) > 0$ implies the close-to-convexity and univalence of f. Similarly, for $f \in \mathcal{A}_p$, the inequality $\operatorname{Re}(f'(z)/z^{p-1}) > 0$ implies p-valency of f, see [127,128]. A function $f \in \mathcal{A}_p$ is close-to-convex if there is a p-valent convex function ϕ such that $\operatorname{Re}(f'(z)/\phi(z)) > 0$. In particular, they are all close-to-convex with respect to $\phi(z) = z^p$.

Theorem 6.3 *Let $0 \leq \alpha < 1$. If the function $f \in \mathcal{A}_{p,n}$ satisfies the inequality*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{(2p-n)+\alpha(2p+n)}{2(\alpha+1)}, \tag{6.1}$$

then

$$\operatorname{Re}\left(\frac{f'(z)}{pz^{p-1}}\right) > \frac{1+\alpha}{2}.$$

Proof. Let the function w be defined by

$$\frac{f'(z)}{pz^{p-1}} = \frac{1+\alpha w(z)}{1+w(z)}. \tag{6.2}$$

Then w can be written as

$$w(z) = \frac{1}{\alpha-1}\left[\frac{(n+p)}{p}a_{n+p}z^n - \frac{(n+p)^2}{p^2(1-\alpha)}a_{n+p}^2 z^{2n} + \cdots\right],$$

hence it is analytic in $\mathbb{D}$ with $w(0) = 0$. From (6.2), logarithmic differentiation yield

$$1 + \frac{zf''(z)}{f'(z)} = p + \frac{\alpha zw'(z)}{1+\alpha w(z)} - \frac{zw'(z)}{1+w(z)}. \tag{6.3}$$

Suppose there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.$$

Applying Lemma 6.1, there exists $m \geq n$ such that

$$z_0 w'(z_0) = mw(z_0), \quad (w(z_0) = e^{i\theta}; \theta \in \mathbb{R}). \tag{6.4}$$

Thus, by using (6.3) and (6.4), it follows that

$$\begin{aligned}
\operatorname{Re}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= p + \operatorname{Re}\left(\frac{\alpha m w(z_0)}{1+\alpha w(z_0)}\right) - \operatorname{Re}\left(\frac{mw(z_0)}{1+w(z_0)}\right) \\
&= p + \operatorname{Re}\left(\frac{\alpha m e^{i\theta}}{1+\alpha e^{i\theta}}\right) - \operatorname{Re}\left(\frac{me^{i\theta}}{1+e^{i\theta}}\right).
\end{aligned}$$

Note that

$$\frac{\alpha m e^{i\theta}}{1+\alpha e^{i\theta}} = \frac{\alpha m e^{i\theta}(1+\alpha e^{-i\theta})}{(1+\alpha e^{i\theta})(1+\alpha e^{-i\theta})} = \frac{\alpha^2 m + \alpha m\cos\theta + i\alpha m\sin\theta}{1+\alpha^2+2\alpha\cos\theta},$$

so

$$\operatorname{Re}\left(\frac{\alpha m e^{i\theta}}{1+\alpha e^{i\theta}}\right) = \frac{\alpha^2 m + \alpha m\cos\theta}{1+\alpha^2+2\alpha\cos\theta}.$$

Similarly,

$$\operatorname{Re}\left(\frac{m e^{i\theta}}{1+e^{i\theta}}\right) = \frac{m}{2}.$$

Hence

$$\begin{aligned}
\operatorname{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) &= p + \frac{\alpha m(\alpha+\cos\theta)}{1+\alpha^2+2\alpha\cos\theta} - \frac{m}{2} \\
&= p - \frac{m(\alpha+1)(1-\alpha)}{2(1+\alpha^2+2\alpha\cos\theta)} \\
&\le p - \frac{m(1-\alpha)}{2(1+\alpha)} \\
&\le p - \frac{n(1-\alpha)}{2(1+\alpha)} \\
&= \frac{(2p-n)+\alpha(2p+n)}{2(\alpha+1)},
\end{aligned}$$

which contradicts the hypothesis (6.1). It follows that $|w(z)| < 1$, for all $z \in \mathbb{D}$, that is,

$$\left|\frac{1-\frac{f'(z)}{pz^{p-1}}}{\frac{f'(z)}{pz^{p-1}}-\alpha}\right| < 1,$$

or equivalently

$$\operatorname{Re}\left(\frac{f'(z)}{pz^{p-1}}\right) > \frac{1+\alpha}{2}. \qquad \blacksquare$$

A function $f \in \mathcal{A}_p$ is starlike if

$$\operatorname{Re}\frac{1}{p}\frac{zf'(z)}{f(z)} > 0.$$

Owa [91] shows that a function $f \in \mathcal{A}_p$ satisfying $\operatorname{Re}(1 + zf''(z)/f'(z)) < p + 1/2$ implies f is p-valently starlike (that is, $f \in \mathcal{A}_p$ is starlike). Our next theorem investigates the close-to-convexity of this type of functions. For related results, see [54, 93, 136].

Theorem 6.4 *For $0 \leq \alpha < 1$, if the function $f \in \mathcal{A}_{p,n}$ satisfies the inequality*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{(p+n)\alpha + (2p+n)}{(\alpha+2)}, \tag{6.5}$$

then

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right| < 1 + \alpha.$$

Proof. Consider the function w defined by

$$\frac{f'(z)}{pz^{p-1}} = (1+\alpha)w(z) + 1. \tag{6.6}$$

It can be checked similarly as in Theorem 6.3 that w is analytic in $\mathbb{D}$ with $w(0) = 0$. From (6.6), some computation yields

$$1 + \frac{zf''(z)}{f'(z)} = p + \frac{(1+\alpha)zw'(z)}{(1+\alpha)w(z) + 1}. \tag{6.7}$$

Suppose there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.$$

By applying Lemma 6.1, there exists $m \geq n$ such that

$$z_0 w'(z_0) = m w(z_0), \quad (w(z_0) = e^{i\theta}; \theta \in \mathbb{R}). \tag{6.8}$$

Thus, by using (6.7) and (6.8), it follows that

$$\begin{aligned}\operatorname{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) &= p+\operatorname{Re}\left(\frac{(1+\alpha)mw(z_0)}{(1+\alpha)w(z_0)+1}\right)\\ &= p+\operatorname{Re}\left(\frac{(1+\alpha)me^{i\theta}}{(1+\alpha)e^{i\theta}+1}\right).\end{aligned}$$

Note that

$$\frac{(1+\alpha)me^{i\theta}}{(1+\alpha)e^{i\theta}+1}=\frac{(1+\alpha)^2 m+(1+\alpha)m\cos\theta+i(1+\alpha)m\sin\theta}{1+(1+\alpha)^2+2\cos\theta(1+\alpha)},$$

so

$$\operatorname{Re}\left(\frac{(1+\alpha)me^{i\theta}}{(1+\alpha)e^{i\theta}+1}\right)=\frac{m(1+\alpha)(1+\alpha+\cos\theta)}{1+(1+\alpha)^2+2\cos\theta(1+\alpha)}.$$

Hence

$$\begin{aligned}\operatorname{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) &= p+\frac{m(1+\alpha)(1+\alpha+\cos\theta)}{1+(1+\alpha)^2+2\cos\theta(1+\alpha)}\\ &\geq p+\frac{m(1+\alpha)}{(\alpha+2)}\geq p+\frac{n(1+\alpha)}{(\alpha+2)}\\ &= \frac{(p+n)\alpha+(2p+n)}{(\alpha+2)},\end{aligned}$$

which contradicts the hypothesis (6.5). It follows that $|w(z)|<1$, for all $z\in\mathbb{D}$, that is,

$$\left|\frac{f'(z)}{pz^{p-1}}-1\right|<1+\alpha.$$

This completes the proof. ∎

Owa [92] has also showed that a function $f\in\mathcal{A}$ satisfying $|f'(z)/g'(z)-1|^{\beta}|zf''(z)/g'(z)-zf'(z)g''(z)/(g'(z))^2|^{\gamma}<(1+\alpha)^{\beta+\alpha}$, for $0\leq\alpha<1$, $\beta\geq 0$, $\gamma\geq 0$ and g a convex function, is close-to-convex. Also, see [53]. Our next theorem investigates the close-to-convexity of similar class of functions.

Theorem 6.5 *Let $0 \leq \alpha < 1$ and $\beta, \gamma \geq 0$. If $f \in \mathcal{A}_{p,n}$, then*

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right|^{\beta} \left|\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}}\right|^{\gamma} < \frac{(pn)^{\gamma}(1-\alpha)^{\beta+\gamma}}{2^{\beta+2\gamma}} \tag{6.9}$$

implies

$$\mathrm{Re}\left(\frac{f'(z)}{pz^{p-1}}\right) > \frac{1+\alpha}{2},$$

and

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right|^{\beta} \left|\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}}\right|^{\gamma} < (pn)^{\gamma}|1-\alpha|^{\beta+\gamma} \tag{6.10}$$

implies

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right| < 1 - \alpha.$$

Proof. For the analytic function w defined by

$$\frac{f'(z)}{pz^{p-1}} = \frac{1+\alpha w(z)}{1+w(z)}, \tag{6.11}$$

we can rewrite (6.11) as

$$\frac{f'(z)}{pz^{p-1}} - 1 = \frac{(\alpha-1)w(z)}{1+w(z)},$$

which yields

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right|^{\beta} = \frac{|w(z)|^{\beta}|1-\alpha|^{\beta}}{|1+w(z)|^{\beta}}. \tag{6.12}$$

By using logarithmic differentiation on (6.11) and some computation,

$$\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}} = \frac{p(\alpha-1)zw'(z)}{(1+w(z))^2}$$

or equivalently

$$\left|\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}}\right|^{\gamma} = \frac{p^{\gamma}|zw'(z)|^{\gamma}|1-\alpha|^{\gamma}}{|1+w(z)|^{2\gamma}}. \tag{6.13}$$

From (6.12) and (6.13), it follows that

$$\left|\frac{f'(z)}{pz^{p-1}}-1\right|^{\beta}\left|\frac{f''(z)}{z^{p-2}}-(p-1)\frac{f'(z)}{z^{p-1}}\right|^{\gamma}=\frac{p^{\gamma}|w(z)|^{\beta}(1-\alpha)^{\beta+\gamma}|zw'(z)|^{\gamma}}{|1+w(z)|^{\beta+2\gamma}}.$$

Suppose there exists a point $z_0\in\mathbb{D}$ such that

$$|w(z_0)|=1 \text{ and } |w(z)|<1 \text{ when } |z|<|z_0|.$$

Then (6.4) and Lemma 6.1 yield

$$\begin{aligned}\left|\frac{f'(z_0)}{pz_0^{p-1}}-1\right|^{\beta}\left|\frac{f''(z_0)}{z_0^{p-2}}-(p-1)\frac{f'(z_0)}{z_0^{p-1}}\right|^{\gamma}&=\frac{p^{\gamma}(1-\alpha)^{\beta+\gamma}|w(z_0)|^{\beta}|mw(z_0)|^{\gamma}}{|1+e^{i\theta}|^{\beta+2\gamma}}\\&=\frac{p^{\gamma}m^{\gamma}(1-\alpha)^{\beta+\gamma}}{(2+2\cos\theta)^{(\beta+2\gamma)/2}}\\&\geq\frac{p^{\gamma}n^{\gamma}(1-\alpha)^{\beta+\gamma}}{2^{\beta+2\gamma}},\end{aligned}$$

which contradicts the hypothesis (6.9). Hence $|w(z)|<1$, for all $z\in\mathbb{D}$, that is

$$\left|\frac{1-\frac{f'(z)}{pz^{p-1}}}{\frac{f'(z)}{pz^{p-1}}-\alpha}\right|<1,$$

or equivalently

$$\operatorname{Re}\left(\frac{f'(z)}{pz^{p-1}}\right)>\frac{1+\alpha}{2}.$$

For the second implication, now consider the function w defined by

$$\frac{f'(z)}{pz^{p-1}}=1+(1-\alpha)w(z). \tag{6.14}$$

Then

$$\left|\frac{f'(z)}{pz^{p-1}}-1\right|^{\beta}=|1-\alpha|^{\beta}|w(z)|^{\beta}. \tag{6.15}$$

Also with some computation on (6.14), we get

$$\left|\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}}\right|^{\gamma} = p^{\gamma}|zw'(z)|^{\gamma}|1-\alpha|^{\gamma}. \tag{6.16}$$

From (6.15) and (6.16), it is clear that

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right|^{\beta}\left|\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}}\right|^{\gamma} = p^{\gamma}|w(z)|^{\beta}|1-\alpha|^{\beta+\gamma}|zw'(z)|^{\gamma}.$$

Suppose there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.$$

Then by applying Lemma 6.1 and using (6.4), it follows that

$$\begin{aligned}\left|\frac{f'(z_0)}{pz_0^{p-1}} - 1\right|^{\beta}\left|\frac{f''(z_0)}{z_0^{p-2}} - (p-1)\frac{f'(z_0)}{z_0^{p-1}}\right|^{\gamma} &= p^{\gamma}|w(z_0)|^{\beta}|1-\alpha|^{\beta+\gamma}|z_0w'(z_0)|^{\gamma} \\ &= p^{\gamma}m^{\gamma}|1-\alpha|^{\beta+\gamma} \\ &\geq (pn)^{\gamma}|1-\alpha|^{\beta+\gamma},\end{aligned}$$

but now this contradicts the hypothesis (6.10). Hence $|w(z)| < 1$ and this implies

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right| < 1-\alpha, \quad \text{or} \quad \operatorname{Re}\left(\frac{f'(z)}{pz^{p-1}}\right) > \alpha.$$

Therefore the proof is complete. ∎

In the next theorem, we will use the concept of subordination. Again, recall that for f and g analytic on $\mathbb{D}$, we say f is *subordinate* to g, written $f \prec g$, if there is an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ such that $f = g \circ w$.

Recall that a function $f \in \mathcal{A}_p$ is starlike if

$$\operatorname{Re} \frac{1}{p} \frac{zf'(z)}{f(z)} > 0.$$

Note that for $f \in \mathcal{A}_p$ the subordination

$$\frac{1}{p} \frac{zf'(z)}{f(z)} \prec \frac{\lambda(1-z)}{\lambda - z}$$

implies $f \in \mathcal{A}_p$ is starlike as

$$\operatorname{Re} \frac{1}{p} \frac{zf'(z)}{f(z)} = \operatorname{Re} \frac{\lambda(1-w(z))}{\lambda - w(z)} > 0.$$

Theorem 6.6 *Let λ_1 and λ_2 be given by*

$$\begin{aligned} \lambda_1 &= \frac{n+2}{4p+n-2p}, \\ \lambda_2 &= \frac{n+2}{2-n}, \end{aligned}$$

and $1 \leq \lambda_1 < \lambda < \lambda_2 \leq 3$. If the function $f \in \mathcal{A}_{p,n}$ satisfies the inequality

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) < \begin{cases} \frac{(4p+n)\lambda - n}{2(\lambda+1)}, & \lambda_1 < \lambda \leq \frac{p+n}{p}; \\ \frac{n(\lambda+1)}{2(\lambda-1)}, & \frac{p+n}{p} < \lambda < \lambda_2, \end{cases} \tag{6.17}$$

then

$$\frac{1}{p} \frac{zf'(z)}{f(z)} \prec \frac{\lambda(1-z)}{\lambda - z}. \tag{6.18}$$

The result is sharp for the function f given by

$$f(z) = z^p \, (\lambda - z)^{p(\lambda - 1)}. \tag{6.19}$$

Proof. Let w be

$$\frac{1}{p}\frac{zf'(z)}{f(z)} = \frac{\lambda(1-w(z))}{\lambda - w(z)}. \tag{6.20}$$

By doing the logarithmic differentiation on (6.20), we get

$$1+\frac{zf''(z)}{f'(z)} = \frac{p\lambda(1-w(z))}{\lambda - z} - \frac{zw'(z)}{1-w(z)} + \frac{zw'(z)}{\lambda - w(z)}.$$

Assume that there exists a point $z_0 \in \mathbb{D}$ such that $|w(z_0)| = 1$ and $|w(z)| < 1$ when $|z| < |z_0|$. By applying Lemma 6.1, it follows that

$$\operatorname{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) = \operatorname{Re}\left(\frac{p\lambda(1-e^{i\theta})}{\lambda - e^{i\theta}}\right) - \operatorname{Re}\left(\frac{me^{i\theta}}{1-e^{i\theta}}\right) + \operatorname{Re}\left(\frac{me^{i\theta}}{\lambda - e^{i\theta}}\right)$$

Since

$$\frac{1-e^{i\theta}}{\lambda - e^{i\theta}} = \frac{\lambda+1-(1+\lambda)\cos\theta + i(1-\lambda)\sin\theta}{\lambda^2+1-2\lambda\cos\theta},$$

$$\frac{e^{i\theta}}{1-e^{i\theta}} = \frac{\cos\theta - 1 + i\sin\theta}{2-2\cos\theta},$$

and

$$\frac{e^{i\theta}}{\lambda - e^{i\theta}} = \frac{\lambda\cos\theta - 1 + i\lambda\sin\theta}{\lambda^2+1-2\lambda\cos\theta},$$

it follows that

$$\begin{aligned}
\operatorname{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) &= \frac{p\lambda(\lambda+1)(1-\cos\theta)}{\lambda^2+1-2\lambda\cos\theta} + \frac{m}{2} + \frac{m(\lambda\cos\theta - 1)}{\lambda^2+1-2\lambda\cos\theta} \\
&= \frac{2p\lambda(\lambda+1)(1-\cos\theta) + m(\lambda^2-1)}{2(\lambda^2+1-2\lambda\cos\theta)} \\
&= \frac{\lambda+1}{2}p + \frac{(\lambda^2-1)[(p+m)-p\lambda]}{2(\lambda^2+1-2\lambda\cos\theta)} \\
&\geq \frac{\lambda+1}{2}p + \frac{(\lambda^2-1)[(p+n)-p\lambda]}{2(\lambda^2+1-2\lambda\cos\theta)}.
\end{aligned}$$

For $\lambda_1 < \lambda \leq (p+n)/p$, some computations yield

$$\mathrm{Re}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) \geq \frac{(4p+n)\lambda - n}{2(\lambda+1)}, \tag{6.21}$$

and for $(p+n)/p < \lambda < \lambda_2$, we get

$$\mathrm{Re}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) \geq \frac{n(\lambda+1)}{2(\lambda-1)}. \tag{6.22}$$

Since (6.21) and (6.22) obviously contradict hypothesis (6.17), it follows that $|w(z)| < 1$. Hence by the definition of subordination, (6.20) becomes (6.18).

Finally, for (6.18) to be sharp, consider

$$\frac{1}{p}\frac{zf'(z)}{f(z)} = \frac{\lambda(1-z)}{\lambda - z}. \tag{6.23}$$

By integrating both sides of the equality and after some arrangement, we get $f(z) = z^p\,(\lambda - z)^{p(\lambda-1)}$. This completes the proof. ∎

Remark 6.1 *The subordination* (6.18) *can be written in equivalent form as*

$$\left|\frac{\lambda(zf'(z)/pf(z) - 1)}{zf'(z)/pf(z) - \lambda}\right| < 1,$$

or by further computation, as

$$\left|\frac{1}{p}\frac{zf'(z)}{f(z)} - \frac{\lambda}{\lambda+1}\right| < \frac{\lambda}{\lambda+1}.$$

The last inequality shows that f is p-valent starlike in $\mathbb{D}$.

Remark 6.2 *When $p = 1$ and $n = 1$, Theorems 6.3–6.6 reduce to Theorems 6.1 and 6.2.*

CONCLUSION

This thesis investigates complex-valued analytic (and meromorphic) functions on simply connected proper domains. By the Riemann Mapping Theorem, these functions f can be assumed to take the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined on the unit disk $\mathbb{D}$. The collection of these functions is denoted by $\mathcal{A}$.

The subclass $\mathcal{S}$ of $\mathcal{A}$ consists of univalent functions. When $f \in \mathcal{S}$, then f^{-1} need not belong to $\mathcal{S}$. Bi-univalent functions are those functions $f \in \mathcal{S}$ satisfying both f and f^{-1} are in $\mathcal{S}$. This thesis finds initial coefficient estimates for the Ma Minda class of bi-univalent convex and starlike functions. Under this general framework, it gets more computationally involved, but on the other hand, several earlier known results are obtained as simple consequences. The results obtained in this thesis rest on using known coefficient bounds of functions with positive real part. However the expressions to be optimized involved non-linear combinations of these coefficients, of which no known estimates are available. Further work on getting good estimates of these non-linear expressions would lead to better estimates for the initial coefficients of bi-univalent functions.

This thesis also determines the bound for the second Hankel determinant $H_2(2) = a_2a_4 - a_3^2$ of analytic Ma Minda starlike and convex functions. Similar problems were also treated for related classes defined by subordination. Since the classes introduced by subordination naturally include several well known classes of univalent functions, earlier known results for these classes are simple consequences of the theorems obtained in this thesis. A good continuation to the work done here would be to investigate the Hankel determinant for transforms of analytic functions.

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, the sharp estimate $|a_2| \leq 2$ yields the growth, distortion as well as the Koebe disk for the class $\mathcal{S}$. In view of the influence of the second coefficient on the geometric properties of univalent functions, this thesis examines the class of functions with fixed second coefficient. By extending the theory of differential subordination, certain known properties for the class $\mathcal{S}$ are shown to extend to the class of functions with fixed second coefficient. In particular, several sufficient conditions related to starlikeness, meromorphic starlikeness and univalence of normalized analytic functions are derived.

Integral operators involving double and triple integrals from the class $\mathcal{A}$ to $\mathcal{A}$ are discussed in this thesis. For the classes of functions treated here, the important technique of duality is not applicable. Thus the technique used in this thesis involves differential subordination and exploited the properties of best dominant solutions to these differential equations. Convexity conditions for analytic functions defined in the open unit disk satisfying certain second-order and third-order differential inequalities are obtained. As a consequence, sufficient conditions are determined that ensure that the double and triple integral operators map functions in $\mathcal{A}$ into convex functions of positive order.

The thesis concludes by finding sufficient conditions for close-to-convexity and starlikeness of a subclass of multivalent functions. Relevant connections with previously known results are also made.

REFERENCES

[1] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, Ann. of Math. (2) **17** (1915), no. 1, 12–22.

[2] H. S. Al-Amiri and M. O. Reade, On a linear combination of some expressions in the theory of the univalent functions, Monatsh. Math. **80** (1975), no. 4, 257–264.

[3] H. Al-Amiri and P. T. Mocanu, Some simple criteria of starlikeness and convexity for meromorphic functions, Mathematica **37(60)** (1995), no. 1-2, 11–20.

[4] R. M. Ali, On a subclass of starlike functions, Rocky Mountain J. Math. **24** (1994), no. 2, 447–451.

[5] R. M. Ali and V. Singh, Convexity and starlikeness of functions defined by a class of integral operators, Complex Variables Theory Appl. **26** (1995), no. 4, 299–309.

[6] R. M. Ali, V. Ravichandran and N. Seenivasagan, On Bernardi's integral operator and the Briot-Bouquet differential subordination, J. Math. Anal. Appl. **324** (2006), no. 1, 663–668.

[7] R. M. Ali, V. Ravichandran and N. Seenivasagan, Coefficient bounds for p-valent functions, Appl. Math. Comput. **187** (2007), no. 1, 35–46.

[8] R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, The Fekete-Szegő coefficient functional for transforms of analytic functions, Bull. Iranian Math. Soc. **35** (2009), no. 2, 119–142, 276.

[9] R. M. Ali, K. G. Subramanian, S.K. Lee, and A. Swaminathan, A third-order differential equation and starlikeness of a double integral operator, Abstr. Appl. Anal. **2011**, Art. ID 901235, 10 pp.

[10] R. M. Ali, S. Nagpal and V. Ravichandran, Second-order differential subordination for analytic functions with fixed initial coefficient, Bull. Malays. Math. Sci. Soc. (2) **34** (2011), no. 3, 611–629.

[11] R. M. Ali, and V. Ravichandran, Uniformly convex and uniformly starlike functions, Math. Newsletter, **21** (2011), no. 1, 16–30.

[12] R. M. Ali, A. O. Badghaish, V. Ravichandran and A. Swaminathan, Starlikeness of integral transforms and duality, J. Math. Anal. Appl. **385** (2012), no. 2, 808–822.

[13] R. M. Ali, N. E. Cho, N. Jain and V. Ravichandran, Radii of starlikeness and convexity of functions defined by subordination with fixed second coefficients, Filomat **26** (2012), no. 3, 553–561.

[14] R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Appl. Math. Lett. **25** (2012), no. 3, 344–351.

[15] R. M. Ali, M. M. Nargesi and V. Ravichandran, Convexity of integral transforms and duality, Complex Var. Elliptic Equ. **58** (2013), no. 11, 1569-1590.

[16] M. Arif, K. I. Noor and M. Raza, Hankel determinant problem of a subclass of analytic functions, J. Inequal. Appl. **2012**, 2012:22, 7 pp.

[17] D. Bansal, Fekete-Szegö problem for a new class of analytic functions, Int. J. Math. Math. Sci. **2011**, Art. ID 143095, 5 pp.

[18] D. Bansal, Upper bound of second Hankel determinant for a new class of analytic functions, Appl. Math. Lett. **26** (2013), no. 1, 103–107.

[19] L. Bieberbach, Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, Semesterberichte Preuss. Akad. Wiss. **38** (1916), 940–955.

[20] L. de Branges, A proof of the Bieberbach conjecture, Acta Math. **154** (1985), no. 1-2, 137–152.

[21] D. A. Brannan, J. Clunie and W. E. Kirwan, Coefficient estimates for a class of star-like functions, Canad. J. Math. **22** (1970), 476–485.

[22] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, Studia Univ. Babeş-Bolyai Math. **31** (1986), no. 2, 70–77.

[23] D. Bshouty, W. Hengartner and G. Schober, Estimates for the Koebe constant and the second coefficient for some classes of univalent functions, Canad. J. Math. **32** (1980), no. 6, 1311–1324.

[24] S. Bulut, Coefficient estimates for initial Taylor-Maclaurin coefficients for a subclass of analytic and bi-univalent functions defined by Al-Oboudi differential operator, The Scientific World Journal, vol. 2013, Article ID 171039, 6 pages, 2013. doi:10.1155/2013/171039.

[25] S. Bulut, Coefficient estimates for a class of analytic and bi-univalent functions, Novi Sad J. Math., **43** (2013), no. 2, 59-65.

[26] M. Çağlar, H. Orhan and N. Yagmur, Coefficient bounds for new subclasses of bi-univalent functions, Filomat **27** (2013), no. 7, 1165-1171, DOI 10.2298/FIL1307165C.

[27] D. G. Cantor, Power series with integral coefficients, Bull. Amer. Math. Soc. **69** (1963), 362–366.

[28] R. Chandrashekar, R. M. Ali, K. G. Subramanian and A. Swaminathan, Starlikeness of functions defined by third-order differential inequalities and integral operators, Abstr. Appl. Anal., accepted.

[29] P. L. Duren, Univalent functions, Grundlehren der Mathematischen Wissenschaften, 259, Springer, New York, 1983.

[30] M. M. ElHosh, On the second Hankel determinant of univalent functions, Bull. Malaysian Math. Soc. (2) **9** (1986), no. 1, 23–25.

[31] M. M. ElHosh, On the second Hankel determinant of close-to-convex functions, Bull. Malaysian Math. Soc. (2) **9** (1986), no. 2, 67–68.

[32] R. Fournier and P.T. Mocanu, Differential inequalities and starlikeness, Complex Variables Theory Appl. **48** (2003), 283–292.

[33] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett. **24** (2011), no. 9, 1569–1573.

[34] P. R. Garabedian and M. A. Schiffer, A proof of the Bieberbach conjecture for the fourth coefficient, J. Rational Mech. Anal. **4** (1955), 427–465.

[35] A. W. Goodman, An invitation to the study of univalent and multivalent functions, Internat. J. Math. Math. Sci. **2** (1979), no. 2, 163–186.

[36] A. W. Goodman, Univalent functions. Vol. I, II, Mariner, Tampa, FL, 1983.

[37] U. Grenander and G. Szegö, Toeplitz forms and their applications, California Monographs in Mathematical Sciences Univ. California Press, Berkeley, 1958.

[38] D. J. Hallenbeck and S. Ruscheweyh, Subordination by convex functions, Proc. Amer. Math. Soc. **52** (1975), 191–195.

[39] S. G. Hamidi, S. A. Halim and J. M. Jahangiri, Coefficent estimates for bi-univalent strongly starlike and Bazilevic functions, Int. J. Math. Research, **5** (2013), no. 1, 87–96.

[40] T. Hayami and S. Owa, Hankel determinant for p-valently starlike and convex functions of order α, Gen. Math. **17** (2009), no. 4, 29–44.

[41] T. Hayami and S. Owa, Generalized Hankel determinant for certain classes, Int. J. Math. Anal. (Ruse) **4** (2010), no. 49-52, 2573–2585.

[42] T. Hayami and S. Owa, Applications of Hankel determinant for p-valently starlike and convex functions of order α, Far East J. Appl. Math. **46** (2010), no. 1, 1–23.

[43] W. K. Hayman, On the second Hankel determinant of mean univalent functions, Proc. London Math. Soc. (3) **18** (1968), 77–94.

[44] A. Janteng, S. A. Halim and M. Darus, Hankel determinant for starlike and convex functions, Int. J. Math. Anal. (Ruse) **1** (2007), no. 13-16, 619–625.

[45] W. Kaplan, Close-to-convex schlicht functions, Michigan Math. J. **1** (1952), 169–185 (1953).

[46] A. W. Kędzierawski, Some remarks on bi-univalent functions, Ann. Univ. Mariae Curie-Skłodowska Sect. A **39** (1985), 77–81 (1988).

[47] A. Kędzierawski and J. Waniurski, Bi-univalent polynomials of small degree, Complex Variables Theory Appl. **10** (1988), no. 2-3, 97–100.

[48] S. S. Kumar, V. Kumar and V. Ravichandran, Estimates for the initial coefficients of bi-univalent functions, Tamsui Oxf. J. Inf. Math. Sci., accepted.

[49] K. Kuroki and S. Owa, Double integral operators concerning starlike of order β, Int. J. Differential Equations, **2009**, Article ID 737129, 13 pages.

[50] Z. Lewandowski, S. Miller and E. Złotkiewicz, Generating functions for some classes of univalent functions, Proc. Amer. Math. Soc. **56** (1976), 111–117.

[51] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. **18** (1967), 63–68.

[52] J.-L. Li and S. Owa, Sufficient conditions for starlikeness, Indian J. Pure Appl. Math. **33** (2002), no. 3, 313–318.

[53] L. J. Lin and H. M. Srivastava, Some starlikeness conditions for analytic functions, Nihonkai Math. J. **7** (1996), no. 2, 101–112.

[54] J. L. Liu, A remark on certain multivalent functions, Soochow J. Math. **21** (1995), no. 2, 179–181.

[55] M.-S. Liu, Y.-C. Zhu and H. M. Srivastava, Properties and characteristics of certain subclasses of starlike functions of order β, Math. Comput. Modelling **48** (2008), no. 3-4, 402–419.

[56] M.-S. Liu, Y.-Y. Liu and Z.-W. Liu, Properties and characteristics of certain subclass of analytic functions with positive coefficients, Tamsui Oxford J. Inform. Math. Sci. **27** (2011), no. 1, 39–60.

[57] A. E. Livingston, p-valent close-to-convex functions, Trans. Amer. Math. Soc. **115** (1965), 161–179.

[58] K. Löwner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I, Math. Ann. **89** (1923), no. 1-2, 103–121.

[59] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157–169, Internat. Press, Cambridge, MA.

[60] N. Magesh, T. Rosy and S. Varma, Coefficient estimate problem for a new subclass of biunivalent functions, J. Complex Anal. **2013**, Art. ID 474231, 3 pp.

[61] S. S. Miller and P. T. Mocanu, Differential Subordinations, Dekker, New York, 2000.

[62] S. S. Miller and P. T. Mocanu, Double integral starlike operators, Integral Transforms Spec. Funct. **19** (2008), no. 7-8, 591–597.

[63] A. K. Mishra and P. Gochhayat, Second Hankel determinant for a class of analytic functions defined by fractional derivative, Int. J. Math. Math. Sci. **2008**, Art. ID 153280, 10 pp.

[64] A. K. Mishra and P. Gochhayat, Fekete-Szegö problem for a class defined by an integral operator, Kodai Math. J. **33** (2010), no. 2, 310–328.

[65] N. Mohamed, D. Mohamad and S. Cik Soh, Second Hankel determinant for certain generalized classes of analytic functions, Int. J. Math. Anal. (Ruse) **6** (2012), no. 17-20, 807–812.

[66] G. Murugusundaramoorthy and N. Magesh, Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant, Bull. Math. Anal. Appl. **1** (2009), no. 3, 85–89.

[67] G. Murugusundaramoorthy, N. Magesh and V. Prameela, Coefficient bounds for certain subclasses of bi-univalent function, Abstr. Appl. Anal. **2013**, Art. ID 573017, 3 pp.

[68] S. Nagpal and V. Ravichandran, Applications of theory of differential subordination for functions with fixed initial coefficient to univalent functions, Ann. Polon. Math. **105** (2012), 225–238.

[69] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Rational Mech. Anal. **32** (1969), 100–112.

[70] J. Nishiwaki and S. Owa, Coefficient inequalities for certain analytic functions, Int. J. Math. Math. Sci. **29** (2002), no. 5, 285–290.

[71] J. W. Noonan and D. K. Thomas, On the Hankel determinants of areally mean p-valent functions, Proc. London Math. Soc. (3) **25** (1972), 503–524.

[72] J. W. Noonan, Coefficient differences and Hankel determinants of areally mean p-valent functions, Proc. Amer. Math. Soc. **46** (1974), 29–37.

[73] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of areally mean p-valent functions, Trans. Amer. Math. Soc. **223** (1976), 337–346.

[74] K. I. Noor, On analytic functions related with functions of bounded boundary rotation, Comment. Math. Univ. St. Paul. **30** (1981), no. 2, 113–118.

[75] K. I. Noor, On meromorphic functions of bounded boundary rotation, Caribbean J. Math. **1** (1982), no. 3, 95–103.

[76] K. I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, Rev. Roumaine Math. Pures Appl. **28** (1983), no. 8, 731–739.

[77] K. I. Noor and S. A. Al-Bany, On Bazilevic functions, Internat. J. Math. Math. Sci. **10** (1987), no. 1, 79–88.

[78] K. I. Noor, Higher order close-to-convex functions, Math. Japon. **37** (1992), no. 1, 1–8.

[79] K. I. Noor, On the Hankel determinant problem for strongly close-to-convex functions, J. Natur. Geom. **11** (1997), no. 1, 29–34.

[80] K. I. Noor and I. M. A. Al-Naggar, On the Hankel determinant problem, J. Natur. Geom. **14** (1998), no. 2, 133–140.

[81] K. I. Noor, On certain analytic functions related with strongly close-to-convex functions, Appl. Math. Comput. **197** (2008), no. 1, 149–157.

[82] K. Noshiro, On the theory of schlicht functions, J. Fac. Sci., Hokkaido Univ. **2** (1934-35), 129–155.

[83] M. Nunokawa, On the theory of multivalent functions, Tsukuba J. Math. **11** (1987), no. 2, 273–286.

[84] M. Nunokawa and S. Owa, On certain subclass of analytic functions, Indian J. Pure Appl. Math. **19** (1988), no. 1, 51–54.

[85] M. Nunokawa, On the multivalent functions, Indian J. Pure Appl. Math. **20** (1989), no. 6, 577–582.

[86] M. Nunokawa, S. Owa and H. Saitoh, Notes on multivalent functions, Indian J. Pure Appl. Math. **26** (1995), no. 8, 797–805.

[87] M. Nunokawa, S. Owa, S. K. Lee, M. Obradovic, M. K. Aouf, H. Saitoh, A. Ikeda and N. Koike, Sufficient conditions for starlikeness, Chinese J. Math. **24** (1996), no. 3, 265–271.

[88] M. Nunokawa, S. Owa, N. Takahashi and H. Saitoh, Sufficient conditions for Carathéodory functions, Indian J. Pure Appl. Math. **33** (2002), no. 9, 1385–1390.

[89] M. Obradovic, Simple sufficient conditions for univalence, Mat. Vesnik. **49** (1997) 241-244.

[90] M. Obradović and S. B. Joshi, On certain classes of strongly starlike functions, Taiwanese J. Math. **2** (1998), no. 3, 297–302.

[91] S. Owa, On Nunokawa's conjecture for multivalent functions, Bull. Austral. Math. Soc. **41** (1990), no. 2, 301–305.

[92] S. Owa, On sufficient conditions for close-to-convex functions of order alpha, Bull. Inst. Math. Acad. Sinica **20** (1992), no. 1, 43–51.

[93] S. Owa, M. Nunokawa and H. M. Srivastava, A certain class of multivalent functions, Appl. Math. Lett. **10** (1997), no. 2, 7–10.

[94] S. Owa, M. Nunokawa, H. Saitoh and H. M. Srivastava, Close-to-convexity, starlikeness, and convexity of certain analytic functions, Appl. Math. Lett. **15** (2002), no. 1, 63–69.

[95] M. Ozawa, An elementary proof of the Bieberbach conjecture for the sixth coefficient, Kōdai Math. Sem. Rep. **21** (1969), 129–132.

[96] K. S. Padmanabhan, On sufficient conditions for starlikeness, Indian J. Pure Appl. Math. **32** (2001), no. 4, 543–550.

[97] R. N. Pederson, A proof of the Bieberbach conjecture for the sixth coefficient, Arch. Rational Mech. Anal. **31** (1968/1969), 331–351.

[98] R. Pederson and M. Schiffer, A proof of the Bieberbach conjecture for the fifth coefficient, Arch. Rational Mech. Anal. **45** (1972), 161–193.

[99] Ch. Pommerenke, On the coefficients and Hankel determinants of univalent functions, J. London Math. Soc. **41** (1966), 111–122.

[100] Ch. Pommerenke, On the Hankel determinants of univalent functions, Mathematika **14** (1967), 108–112.

[101] C. Ramesha, S. Kumar and K. S. Padmanabhan, A sufficient condition for starlikeness, Chinese J. Math. **23** (1995), no. 2, 167–171.

[102] V. Ravichandran, C. Selvaraj and R. Rajalaksmi, Sufficient conditions for starlike functions of order α, J. Inequal. Pure Appl. Math. **3** (2002), no. 5, Art. 81, 6 pp.

[103] V. Ravichandran, Certain applications of first order differential subordination, Far East J. Math. Sci. **12** (2004), no. 1, 41–51.

[104] V. Ravichandran and M. Jayamala, On sufficient conditions for Caratheodory functions, Far East J. Math. Sci. **12** (2004), no. 2, 191–201.

[105] M. I. S. Robertson, On the theory of univalent functions, Ann. of Math. (2) **37** (1936), no. 2, 374–408.

[106] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. **118** (1993), no. 1, 189–196.

[107] B. Şeker, On certain new subclass of close-to-convex functions, Appl. Math. Comput. **218** (2011), no. 3, 1041–1045.

[108] T. N. Shanmugam, C. Ramachandran and V. Ravichandran, Fekete-Szegő problem for subclasses of starlike functions with respect to symmetric points, Bull. Korean Math. Soc. **43** (2006), no. 3, 589–598.

[109] H. Shiraishi, K. Kuroki and S. Owa, Some classes of order α for second-order differential inequalities, Electron. J. Math. Anal. Appl. **1** (2013), no. 2, 149–155.

[110] S. Singh and S. Gupta, A differential subordination and starlikeness of analytic functions, Appl. Math. Lett. **19** (2006), no. 7, 618–627.

[111] S. Singh, S. Gupta and S. Singh, On a problem of univalence of functions satisfying a differential inequality, Math. Inequal. Appl. **10** (2007), no. 1, 95–98.

[112] V. Singh, S. Singh and S. Gupta, A problem in the theory of univalent functions, Integral Transforms Spec. Funct. **16** (2005), no. 2, 179–186.

[113] R. Singh and S. Singh, Some sufficient conditions for univalence and starlikeness, Colloq. Math. **47** (1982), no. 2, 309–314 (1983).

[114] H. V. Smith, Bi-univalent polynomials, Simon Stevin **50** (1976/77), no. 2, 115–122.

[115] H. V. Smith, Some results/open questions in the theory of bi-univalent functions, J. Inst. Math. Comput. Sci. Math. Ser. **7** (1994), no. 3, 185–195.

[116] J. Sokół and J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, Zeszyty Nauk. Politech. Rzeszowskiej Mat. No. 19 (1996), 101–105.

[117] H. M. Srivastava, Some inequalities and other results associated with certain subclasses of univalent and bi-univalent analytic functions, in *Nonlinear Analysis*, Springer Series on Optimization and Its Applications, Vol. 68, pp. 607-630, Springer, Berlin, New York and Heidelberg, 2012.

[118] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. **23** (2010), no. 10, 1188–1192.

[119] H. M. Srivastava, S. Bulut, M. Çağlar and N. Yagmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat **27** (2013), no. 5, 831-842, DOI 10.2298/FIL1305831S.

[120] H. M. Srivastava, G. Murugusundaramoorthy and N. Magesh, On certain subclasses of bi-univalent functions associated with Hohlov operator, Global Journal of Mathematical Analysis, **1** (2013), no. 2 67–73.

[121] D. Styer and D. J. Wright, Results on bi-univalent functions, Proc. Amer. Math. Soc. **82** (1981), no. 2, 243–248.

[122] T. J. Suffridge, A coefficient problem for a class of univalent functions, Michigan Math. J. **16** (1969), 33–42.

[123] T. J. Suffridge, Some remarks on convex maps of the unit disk, Duke Math. J. **37** (1970), 775–777.

[124] A. Swaminathan, Sufficient conditions for hypergeometric functions to be in a certain class of analytic functions, Comput. Math. Appl. **59** (2010), no. 4, 1578–1583.

[125] D. L. Tan, Coefficient estimates for bi-univalent functions, Chinese Ann. Math. Ser. A **5** (1984), no. 5, 559–568.

[126] H. Tang, G.-T. Deng and S.-H. Li, Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions, J. Inequal. Appl. **2013**, 2013:317, 10 pp.

[127] T. Umezawa, On the theory of univalent functions, Tôhoku Math. J. (2) **7** (1955), 212–228.

[128] T. Umezawa, Multivalently close-to-convex functions, Proc. Amer. Math. Soc. **8** (1957), 869–874.

[129] R. Vein and P. Dale, Determinants and their applications in mathematical physics, Applied Mathematical Sciences, 134, Springer, New York, 1999.

[130] S. Verma, S. Gupta, and S. Singh, Bounds of Hankel determinant for a class of univalent functions, Int. J. Math. Math. Sci. **2012** (2012), Article ID 147842, 6 pages.

[131] S. E. Warschawski, On the higher derivatives at the boundary in conformal mapping, Trans. Amer. Math. Soc. **38** (1935), no. 2, 310–340.

[132] R. Wilson, Determinantal criteria for meromorphic functions, Proc. London Math. Soc. (3) **4** (1954), 357–374.

[133] Q.-H. Xu, H.-G. Xiao, H.M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, App. Math. Comput. **218** (2012), no. 23, 11461–11465.

[134] Q.-H. Xu, H.-G. Xiao, H.M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett. **25** (2012), no. 6, 990–994.

[135] N. Yagmur and H. Orhan, Starlikeness and convexity of generalized Struve functions, Abstr. Appl. Anal. **2013**, Art.ID 954513, 6 pp.

[136] R. Yamakawa, A proof of Nunokawa's conjecture for starlikeness of p-valently analytic functions, in Current topics in analytic function theory, 387–392, World Sci. Publ., River Edge, NJ.

[137] D. Yang, S. Owa and K. Ochiai, Sufficient conditions for Carathéodory functions, Comput. Math. Appl. **51** (2006), no. 3-4, 467–474.

PUBLICATIONS

1. R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Appl. Math. Lett., **25** (2012), Pages 344-351. (ISI)

2. S. K. Lee, V. Ravichandran and S. Supramaniam, Bounds for the second Hankel determinant of certain univalent functions, J. Inequal. Appl. **2013**, 2013:281, 17 pp. (ISI)

3. S. K. Lee, V. Ravichandran and S. Supramaniam, Applications of differential subordination for functions with fixed second coefficient to geometric function theory, Tamsui Oxf. J. Inf. Math. Sci., **29** (2013), no. 2, 267-284. (SCOPUS)

4. S. K. Lee, V. Ravichandran and S. Supramaniam, Initial coefficients of bi-univalent functions, Abstr. Appl. Anal. Volume 2014 (2014), Article ID 640856, 6 pages. (ISI)

5. S. K. Lee, V. Ravichandran and S. Supramaniam, Close-to-convexity and starlikeness of analytic functions, Tamkang Journal of Mathematics, accepted. (SCOPUS)

6. R. M. Ali, R. Chandrashekar, S. K. Lee and S. Supramaniam, Convexity of functions defined by differential inequalities and integral operators, submitted.