# DIFFERENTIAL SUBORDINATION AND SUPERORDINATION FOR ANALYTIC AND MEROMORPHIC FUNCTIONS DEFINED BY LINEAR OPERATORS

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**by**

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**SUBORDINASI DAN SUPERORDINASI PEMBEZA UNTUK FUNGSI ANALISIS DAN FUNGSI MEROMORFI YANG TERTAKRIF OLEH PENGOPERASI LINEAR**

## ABSTRAK


Suatu fungsi $f$ yang tertakrif pada cakera unit terbuka $U$ dalam satah kompleks $\mathcal{C}$ disebut univalen jika fungsi tersebut memetakan titik berlainan dalam $U$ ke titik berlainan dalam $C$. Suatu fungsi $f$ disebut subordinat terhadap suatu fungsi univalen $g$ jika $f(0) = g(0)$ dan $f(U) \subset g(U)$. Fungsi ternormalkan $f$ disebut bak-bintang Janowski jika $zf'(z)/f(z)$ adalah subordinat terhadap $(1 + Az)/(1 + Bz)$, $(-1 \leq B \leq A \leq 1)$. Dengan menggunakan teori subordinasi pembeza peringkat pertama, kami memperoleh beberapa syarat cukup untuk fungsi $f$ menjadi bak-bintang Janowski. Syarat-syarat cukup ini diperoleh dengan mengkaji implikasi

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + Dz}{1 + Ez} \Rightarrow p(z) \prec \frac{1 + Az}{1 + Bz}$$

dimana simbol $\prec$ menandai subordinasi antara fungsi analisis dan implikasi-implikasi lain yang serupa yang melibatkan $1 + \beta zp'(z)$, $1 + \beta\frac{zp'(z)}{p(z)}$ dan $1 + \beta\frac{zp'(z)}{p^2(z)}$. Keputusan-keputusan ini digunakan kemudian untuk memperoleh syarat-syarat cukup bagi fungsi analisis menjadi bak-bintang Janowski.

Andaikan $\Omega$ sebagai set dalam $\mathcal{C}$, fungsi $q_1$ adalah univalen dan $q_2$ adalah analisis dalam $\mathcal{U}$. Juga andaikan $\psi : \mathcal{C}^3 \times U \to \mathcal{C}$. Miller dan Mocanu [*Differential Subordinations*, Dekker, New York, 2000.] telah mengkaji teori subordinasi pembeza peringkat pertama dan kedua. Baru-baru ini, Miller dan Mocanu (Subordinants of differential superordinations, Complex Var. Theory Appl. **48**(10) (2003), 815–826.) mengkaji konsep kedualan superordinasi pembeza dan berjaya mendapat beberapa keputusan 'tersepit'. Dengan menggunakan teori subordinasi pembeza, kami menentukan kelas fungsi sedemikian

$$\psi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}\right) \prec h$$

mengimplikasikan $zf'(z)/f(z) \prec q(z)$, dimana simbol $\{f, z\}$ menandai terbitan Schwarz fungsi $f$. Kami juga memperoleh beberapa keputusan serupa yang melibatkan nisbah antara fungsi yang tertakrif melalui operator linear Dziok-Srivastava dan transformasi

pendarab bagi fungsi analisis. Kami juga memperoleh bagi superordinasi yang sepadan beberapa keputusan tersepit. Tambahan pula, kami mengkaji masalah yang serupa bagi fungsi meromorfi yang tertakrif melalui pengoperasi linear Liu-Srivastava dan transformasi pendarab.

Bagi fungsi analisis $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ yang tetap dan tertakrif pada cakera unit terbuka dan $\gamma < 1$, andaikan $T_g(\gamma)$ sebagai kelas semua fungsi analisis $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ yang memenuhi syarat $\sum_{n=2}^{\infty} |a_n g_n| \leq 1 - \gamma$. Bagi fungsi $f \in T_g(\gamma)$ dan fungsi cembung $h$, kami menunjukkan bahawa

$$\frac{g_2}{2g_2 + 1 - \gamma}(f * h) \prec h,$$

dan menggunakan keputusan ini untuk memperoleh batas bawah bagi $\Re f(z)$. Keputusan ini meranggkumi beberapa keputusan awal sebagai kes khas.

Andaikan $\varphi(z)$ fungsi analisis dengan bahagian nyata positif pada cakera unit $U$, dengan $\varphi(0) = 1$ and $\varphi'(0) > 0$, yang memetakan $U$ secara keseluruh kesuatu rantau bak-bintang terhadap $1$ dan simetri terhadap paksi nyata. Ma dan Minda (A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Int. Press, Cambridge, MA.) memperkenalkan kelas $\mathcal{S}^*(\varphi)$ yang terdiri daripada semua fungsi analisis ternormalkan $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ sedemikian $zf'(z)/f(z) \prec \varphi(z)$. Andaikan $\mathcal{A}_p$ menandakan kelas semua fungsi analisis berbentuk $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in U,\ p \in \mathcal{N} := \{1, 2, 3 \ldots\})$. Andaikan $S_p^*(\varphi)$ sebagai subkelas $\mathcal{A}_p$ yang ditakrifkan sebagai

$$S_p^*(\varphi) = \left\{ f \in \mathcal{A}_p : \frac{zf'(z)}{pf(z)} \prec \varphi(z) \right\}.$$

Bagi fungsi $f \in S_p^*(\varphi)$, batas atas tepat bagi fungsian pekali $|a_{p+2} - \mu a_{p+1}^2|$ dan $|a_{p+3}|$ telah diperoleh; batas-batas ini menghasilkan batas atas tepat bagi pekali kedua, ketiga dan keempat. Seterusnya kami mengkaji masalah pekali yang serupa bagi fungsi-fungsi dalam subkelas yang tertakrif dengan ungkapan $1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right)$, $1 + \frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}} - 1\right)$, $\frac{1+\alpha(1-p)}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\frac{z^2 f''(z)}{f(z)}$, $\frac{1-\alpha}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)$ dan $\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha}$. Keputusan-keputusan ini digunakan kemudian untuk memperoleh ketaksamaan bak Fekete-Szegö bagi beberapa kelas fungsi yang tertakrif melalui konvolusi.

**DIFFERENTIAL SUBORDINATION AND SUPERORDINATION FOR ANALYTIC AND MEROMORPHIC FUNCTIONS DEFINED BY LINEAR OPERATORS**

## ABSTRACT

A function $f$ defined on the open unit disk $U$ of the complex plane $\mathcal{C}$ is univalent if it maps different points of $U$ to different points of $\mathcal{C}$. The function $f$ is subordinate to an univalent function $g$ if $f(0) = g(0)$ and $f(U) \subset g(U)$. A normalized function $f$ is Janowski starlike if $zf'(z)/f(z)$ is subordinated to $(1 + Az)/(1 + Bz)$, $(-1 \leq B \leq A \leq 1)$. By making use of the theory of first order differential subordination, we obtain several sufficient conditions for a function $f$ to be Janowski starlike. These sufficient conditions are obtained by investigating the implication

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + Dz}{1 + Ez} \Rightarrow p(z) \prec \frac{1 + Az}{1 + Bz}$$

where $\prec$ denotes subordination between analytic functions and other similar implications involving $1 + \beta zp'(z)$, $1 + \beta \frac{zp'(z)}{p(z)}$ and $1 + \beta \frac{zp'(z)}{p^2(z)}$. These results are then applied to obtain sufficient conditions for analytic functions to be Janowski starlike.

Let $\Omega$ be any set in $\mathcal{C}$ and the functions $q_1$ be univalent and $q_2$ be analytic in $U$. Let $\psi : \mathcal{C}^3 \times U \to \mathcal{C}$. Miller and Mocanu [*Differential Subordinations*, Dekker, New York, 2000.] have investigated the theory of first and second order differential subordination. Recently Miller and Mocanu (Subordinants of differential superordinations, Complex Var. Theory Appl. **48**(10) (2003), 815–826.) investigated the dual concept of differential superordination to obtain several sandwich results. By using the theory of differential subordination, we determine the class of functions so that

$$\psi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}\right) \prec h$$

implies $zf'(z)/f(z) \prec q(z)$, where $\{f, z\}$ denotes the Schwarzian derivative of the function $f$. We also obtain similar results involving the ratios of functions defined through the Dziok-Srivastava linear operator and the multiplier transformation of analytic functions. We also obtain the corresponding superordination and sandwich type results. Further, we

investigate similar problems for meromorphic functions defined through the Liu-Srivastava linear operator and the multiplier transformation.

For a fixed analytic function $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ defined on the open unit disk and $\gamma < 1$, let $T_g(\gamma)$ denote the class of all analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying $\sum_{n=2}^{\infty} |a_n g_n| \leq 1 - \gamma$. For a function $f \in T_g(\gamma)$ and a convex function $h$, we show that

$$\frac{g_2}{2g_2 + 1 - \gamma}(f * h) \prec h$$

and use this to obtain the lower bound for $\Re f(z)$. These results includes several earlier results as special cases.

Let $\varphi(z)$ be an analytic function with positive real part in the unit disk $U$ with $\varphi(0) = 1$ and $\varphi'(0) > 0$, maps $U$ onto a region starlike with respect to $1$ and symmetric with respect to real axis. Ma and Minda (A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Int. Press, Cambridge, MA.) introduced the class $\mathcal{S}^*(\varphi)$ consisting of all normalized analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying $zf'(z)/f(z) \prec \varphi(z)$. Let $\mathcal{A}_p$ denote the class of all analytic functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in U,\ p \in \mathcal{N} := \{1, 2, 3 \ldots\})$. Let $S_p^*(\varphi)$ be a subclass of $\mathcal{A}_p$ defined by

$$S_p^*(\varphi) = \left\{ f \in \mathcal{A}_p : \frac{zf'(z)}{pf(z)} \prec \varphi(z) \right\}.$$

For the function $f \in S_p^*(\varphi)$, the sharp upper bounds for the coefficient functionals $|a_{p+2} - \mu a_{p+1}^2|$ and $|a_{p+3}|$ are obtained; these bounds yield the sharp upper bounds for the second, third and fourth coefficients. Further we investigate a similar coefficient problem for functions in the subclasses defined by the expressions $1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right)$, $1 + \frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}} - 1\right)$, $\frac{1+\alpha(1-p)}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\frac{z^2 f''(z)}{f(z)}$, $\frac{1-\alpha}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)$ and $\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha}$. These are then applied to obtain Fekete-Szegö-like inequalities for several classes of functions defined by convolution.

## SYMBOLS

| Symbol | Description |
|---|---|
| $\mathcal{A}_p$ | Class of all $p$-valent analytic functions of the form $f(z) = z^p + \sum_{k=1+p}^{\infty} a_k z^k \quad (z \in U)$ |
| $\mathcal{A} := \mathcal{A}_1$ | Class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U)$ |
| $(a)_n$ | Pochhammer symbol or shifted factorial |
| $\arg$ | Argument |
| $\mathcal{C}$ | Complex plane |
| $C$ | Class of normalized convex functions in $U$ |
| $C(\alpha)$ | Class of normalized convex functions of order $\alpha$ in $U$ |
| $f * g$ | Convolution or Hadamard product of functions $f$ and $g$ |
| $\{f, z\}$ | Schwarzian derivative of $f$ |
| ${}_1F_1(a, b, c; z)$ | Confluent hypergeometric functions |
| ${}_2F_1(a, b, c; z)$ | Gaussian hypergeometric functions |
| ${}_lF_m\left(\begin{array}{ll}\alpha_1, \ldots, \alpha_l; & \\ \beta_1, \ldots, \beta_m; & z\end{array}\right)$ | Generalized hypergeometric functions |
| $\mathcal{H}(U)$ | Class of analytic functions in $U$ |
| $\mathcal{H}[a, n]$ | Class of analytic functions in $U$ of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (z \in U)$ |
| $\mathcal{H}_0 := \mathcal{H}[0, 1]$ | Class of analytic functions in $U$ of the form $f(z) = a_1 z + a_2 z^2 + \cdots \quad (z \in U)$ |
| $\mathcal{H} := \mathcal{H}[1, 1]$ | Class of analytic functions in $U$ of the form $f(z) = 1 + a_1 z + a_2 z^2 + \cdots \quad (z \in U)$ |
| $H_p^{l,m}$ | Dziok-Srivastava / Liu-Srivastava linear operator |
| $\prec$ | Subordinate to |
| $\Im$ | Imaginary part of a complex number |
| $I_p(n, \lambda),\ (\lambda + p > 0, n \in \mathcal{N})$ | Multiplier transformation from $\mathcal{A}_p \to \mathcal{A}_p$ |
| $K$ | Class of close to convex functions in $\mathcal{A}$ |

| | |
|---|---|
| $k(z)$ | Koebe function |
| $\mathcal{N}$ | Set of all positive integers |
| $\mathcal{R}$ | Set of all real numbers |
| $R[A,B]$ | $\{f \in \mathcal{A} : f'(z) \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1)\}$ |
| $R[\alpha]$ | $\{f \in \mathcal{A} : \lvert f'(z)-1\rvert < 1-\alpha \quad (z \in U,\ 0 \le \alpha < 1)\}$ |
| $\Re$ | Real part of a complex number |
| $\mathcal{S}$ | Class of all normalized univalent functions of the form |
| | $f(z) = z + a_2 z^2 + \cdots \quad z \in U$ |
| $S^*$ | Class of normalized starlike functions in $U$ |
| $S^*_\alpha$ | Class of normalized starlike functions of order $\alpha$ in $U$ |
| $\mathcal{S}^*[A,B]$ | $\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1)\}$ |
| $T_g(\gamma)$ | $\{f(z) \in \mathcal{A} : \sum_{n=2}^{\infty} \lvert a_n g_n\rvert \le 1-\gamma,$ |
| | $g(z) = z + \sum_{n=2}^{\infty} g_n z^n, g_n \ge g_2 > 0, n \ge 2, \gamma < 1\}$ |
| $\Sigma_p$ | Class of all $p$-valent functions of the form |
| | $f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (z \in U^*)$ |
| $\Sigma := \Sigma_1$ | Class of all functions of the form |
| | $f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \quad (z \in U^*)$ |
| $U$ | Open unit disk $\{z \in \mathcal{C} : \lvert z\rvert < 1\}$ |
| $U^*$ | Punctured unit disk $U \setminus \{0\}$ |
| $U_r$ | Open disk of radius $r$, $\{z \in \mathcal{C} : \lvert z\rvert < r\}$ |
| $\partial U$ | Boundary of unit disk $U$, $\{z \in \mathcal{C} : \lvert z\rvert = 1\}$ |
| $\Psi_n[\Omega,q],\ \Phi_H[\Omega,q],\ \Phi_I[\Omega,q]$ | |
| $\Theta_H[\Omega,q],\ \Theta_I[\Omega,q],\ \Theta_I[\Omega,M]$ | Classes of admissible functions |
| $\mathcal{Z}$ | Set of all integers |

# CHAPTER 1

# INTRODUCTION

## 1.1. UNIVALENT FUNCTIONS

Let $\mathcal{C}$ be the complex plane and $U := \{z \in \mathcal{C} : |z| < 1\}$ be the open unit disk in $\mathcal{C}$. Let $\mathcal{H}(U)$ be the class of functions analytic in $U$. Let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$ and let $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$. Let $\mathcal{A}$ denote the class of all analytic functions defined in $U$ and normalized by $f(0) = 0, f'(0) = 1$. A function $f \in \mathcal{A}$ has the Taylor series of the form

(1.1.1)
$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U).$$

More generally, let $\mathcal{A}_p$ denote the class of all analytic functions of the form

(1.1.2)
$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in U,\ p \in \mathcal{N} := \{1, 2, 3 \ldots\}).$$

A function $f \in \mathcal{H}(U)$ is *univalent* if it is one to one in $U$. The function $f \in \mathcal{H}(U)$ is *locally univalent* at $z_0 \in U$ if it is univalent in some neighborhood of $z_0$. The function $f(z)$ is $p$-*valent* (or *multivalent* of order $p$) if for each $w_0$ with infinity included, the equation $f(z) = w_0$ has at most $p$ roots in $U$, where the roots are counted with their multiplicities, and for some $w_1$ the equation $f(z) = w_1$ has exactly $p$ roots in $U$ [**31**]. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Thus $\mathcal{S}$ is the class of all normalized univalent functions in $U$.

In 1916, Bieberbach [**12**] studied the second coefficient $a_2$ of a function $f \in \mathcal{S}$ of the form (1.1.1). He has shown that $|a_2| \leq 2$, with equality if and only if $f$ is a rotation of the Koebe function $k(z) = z/(1-z)^2$ and he mentioned "$|a_n| \leq n$ is generally valid". This statement is known as the Bieberbach conjecture. The Koebe function

(1.1.3)
$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \sum_{n=1}^{\infty} n z^n \quad (z \in U)$$

which maps $U$ onto the complex plane except for a slit along the half-line $(-\infty, -1/4]$ is the "largest" function in $\mathcal{S}$. The function $e^{-i\beta}k(e^{i\beta}z)$, $(\beta \in \mathcal{R})$ also belongs to $\mathcal{S}$ and is referred to as a rotation of the Koebe function. These functions play a very important role in the study of the class $\mathcal{S}$ and are the only extremal functions for various extremal problems in $\mathcal{S}$.

In 1923, Löwner [**53**] proved the Bieberbach conjecture for n $=$ 3. Schaeffer and Spencer [**96**], Jenkins [**37**], Garabedian and Schiffer [**29**], Charzyński and Schiffer [**16, 17**], Pederson [**76, 77**], Ozawa [**66**] and Pederson and Schiffer [**78**] have investigated the Bieberbach conjecture for certain values of $n$. Finally, in 1985, de Branges proved the Bieberbach conjecture for all coefficients $n$ with the help of the hypergeometric functions.

Since the Bieberbach conjecture was difficult to settle, several authors have considered classes defined by geometric conditions. Notable among them are the classes of convex functions, starlike functions and close-to-convex functions. A set $\mathcal{D}$ in the complex plane is called *convex* if for every pair of points $w_1$ and $w_2$ lies in the interior of $\mathcal{D}$, the line segment joining $w_1$ and $w_2$ lies in the interior of $\mathcal{D}$. If a function $f \in \mathcal{A}$ maps $U$ onto a convex domain, then $f(z)$ is called a convex function. Let $C$ denotes the class of all convex functions in $\mathcal{A}$. An analytic description of the class $C$ is given by $C := \{f \in \mathcal{A} : \Re\left(1 + zf''(z)/f'(z)\right) > 0\}$ [**24, 30, 31, 32, 80**]. Let $w_0$ be an interior point of $\mathcal{D}$. A set $\mathcal{D}$ in the complex plane is called *starlike* with respect to $w_0$ if the line segment joining $w_0$ to every other point $w \in \mathcal{D}$ lies in the interior of $\mathcal{D}$. If a function $f \in \mathcal{A}$ maps $U$ onto a domain starlike, then $f(z)$ is called a starlike function. The class of starlike functions with respect to origin is denoted by $\mathcal{S}^*$. Analytically, $\mathcal{S}^* := \{f \in \mathcal{A} : \Re\left(zf'(z)/f(z)\right) > 0\}$ [**24, 30, 31, 32, 80**].

A function $f \in \mathcal{A}$ is said to be *close-to-convex* if there is a convex function $g(z)$ such that $\Re\left(f'(z)/g'(z)\right) > 0$ for all $z \in U$. The class of all close-to-convex functions in $\mathcal{A}$ is denoted by $K$.

A function in any one of these classes is characterized by either of the quantities $1 + zf''(z)/f'(z)$, $zf'(z)/f(z)$ or $f'(z)/g'(z)$ lying in a given region in the right half plane; the region is often convex and symmetric with respect to the real axis [**54**]. Let

$f$ and $F$ be members of $\mathcal{H}(U)$. The function $f(z)$ is said to be *subordinate* to $F(z)$, or $F(z)$ is *superordinate* to $f(z)$, if there exists a function $w(z)$, analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = F(w(z))$. In such a case, we write $f(z) \prec F(z)$. If $F$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let $\varphi$ be an analytic function with positive real part in the unit disk $U$, $\varphi(0) = 1$ and $\varphi'(0) > 0$, and map $U$ onto a region starlike with respect to $1$ and symmetric with respect to real axis. Ma and Minda [**54**] introduced the classes $\mathcal{S}^*(\varphi)$ and $C(\varphi)$ by

(1.1.4) $$\mathcal{S}^*(\varphi) \;=\; \left\{ f \in \mathcal{A} : \; \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$

(1.1.5) $$C(\varphi) \;=\; \left\{ f \in \mathcal{A} : \; 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

The classes $\mathcal{S}^*(\varphi)$ and $C(\varphi)$ include the subclasses of starlike and convex functions as special cases. When

$$\varphi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B \leq A \leq 1),$$

the classes $\mathcal{S}^*(\varphi)$ and $C(\varphi)$ reduce to the class $\mathcal{S}^*[A, B]$ of *Janowski starlike functions* and the class $C[A, B]$ of *Janowski convex functions* respectively [**35, 79**]. Thus

$$\mathcal{S}^*[A, B] =: \mathcal{S}^* \left( \frac{1+Az}{1+Bz} \right) \text{ and } \quad C[A, B] =: C \left( \frac{1+Az}{1+Bz} \right).$$

Also

$$\mathcal{S}^* = \mathcal{S}^*[1, -1] = \mathcal{S}^* \left( \frac{1+z}{1-z} \right) \text{ and } \quad C = C[1, -1] = C \left( \frac{1+z}{1-z} \right)$$

are the familiar classes of starlike and convex functions respectively.

For $0 \leq \alpha < 1$, the class $\mathcal{S}^*[1-2\alpha, -1]$ is the class $\mathcal{S}^*_\alpha$ of starlike functions of order $\alpha$. An equivalent analytic description of $\mathcal{S}^*_\alpha$ is given by

$$\mathcal{S}^*_\alpha := \left\{ f \in \mathcal{A} : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad (0 \leq \alpha < 1) \right\}.$$

For $0 \leq \alpha < 1$,

$$\mathcal{S}^*(\alpha) := \mathcal{S}^*[1-\alpha, 0] = \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha \quad (z \in U, \; 0 \leq \alpha < 1) \right\}.$$

For $0 < \alpha \le 1$, Parvatham [**74**] introduced and studied the class $S^*[\alpha]$ where

$$\begin{aligned} \mathcal{S}^*[\alpha] &:= \mathcal{S}^*[\alpha, -\alpha] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+\alpha z}{1-\alpha z} \right\} \\ &= \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \alpha \left| \frac{zf'(z)}{f(z)} + 1 \right| \quad (z \in U,\ 0 < \alpha \le 1) \right\}. \end{aligned} \tag{1.1.6}$$

For $0 \le \alpha < 1$, $C(\alpha) := C[1-2\alpha, -1]$ is the class of convex functions of order $\alpha$. Equivalently

$$\begin{aligned} C(\alpha) &= \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \right\} \\ &= \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (0 \le \alpha < 1) \right\}. \end{aligned}$$

The transform

$$\int_0^z \frac{f(t)}{t} dt$$

is called the Alexander transform of $f(z)$. It is clear that $f \in C(\alpha)$ if and only if $zf' \in \mathcal{S}^*_\alpha$ or equivalently $f \in \mathcal{S}^*_\alpha$ if and only if the Alexander transform of $f(z)$ is in $C(\alpha)$.

For real $\alpha$, let

$$\mathcal{M}(\alpha, f; z) \equiv (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

The class of $\alpha$-convex functions is defined by

$$\mathcal{M}_\alpha = \{f \in \mathcal{A} : \Re\mathcal{M}(\alpha, f; z) > 0\}.$$

This class $\mathcal{M}_\alpha$ is a subclass of $\mathcal{S}$, and was introduced and studied by Miller et al. [**56**]. It has the additional properties that $\mathcal{M}_\alpha \subset \mathcal{M}_\beta \subset \mathcal{M}_0 = \mathcal{S}^*$ for $0 \le \alpha/\beta \le 1$, and $\mathcal{M}_\alpha \subset \mathcal{M}_1 \subset C$ for $\alpha \ge 1$.

More information on univalent functions can be found in the text books [**24, 30, 31, 32, 33, 58, 80**].

## 1.2. HYPERGEOMETRIC FUNCTIONS

The use of the hypergeometric functions in the celebrated de Branges proof of the Bieberbach conjecture prompted renewed interest in the investigation of special functions.

Prior to this proof, there had been only a few articles in the literature dealing with the relationships between these special functions and univalent function theory.

Let $a$ and $c$ be any complex numbers with $c \neq 0, -1, \cdots$, and consider the function defined by

$$(1.2.1)\quad \Phi(a,c;z) = \ {}_1F_1(a,c;z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2\,!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{z^3}{3\,!} + \cdots .$$

This function, called a *confluent (or Kummer) hypergeometric function* is analytic in $\mathcal{C}$ and satisfies Kummer's differential equation

$$zw''(z) + (c-z)w'(z) - aw(z) = 0.$$

The Pochhammer symbol $(a)_n$ is defined by

$$(1.2.2)\qquad (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n=0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathcal{N}) \end{cases}$$

where $\Gamma(a),\ (a \in \mathcal{N})$ denotes the Gamma function. Then (1.2.1) can be written in the form

$$(1.2.3)\qquad \Phi(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k\,!} = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)}\frac{z^k}{k\,!}.$$

Let $a$, $b$ and $c$ be any complex numbers with $c \neq 0, -1, \cdots$, and consider the function defined by

$$(1.2.4)\qquad F(a,b,c;z) = \ {}_2F_1(a,b,c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2\,!} + \cdots .$$

This function, called a *(Gaussian) hypergeometric function* is analytic in $U$ and satisfies the hypergeometric differential equation

$$z(1-z)w''(z) + [c-(a+b+1)z]w'(z) - abw(z) = 0.$$

Using the notation (1.2.2) in (1.2.4), $F$ can be written as

$$(1.2.5)\qquad F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k\,!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k\,!}.$$

More generally, for $\alpha_j \in \mathcal{C} \quad (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathcal{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots m)$, the *generalized hypergeometric function*

$$ {}_lF_m(z) := {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$$

is defined by the infinite series

$$ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m+1; l, m \in \mathcal{N}_0 := \mathcal{N} \cup \{0\})$$

where $(a)_n$ is the Pochhammer symbol defined by (1.2.2). The absence of parameters is emphasized by a dash. For example,

$$ {}_0F_1(-; b; z) = \sum_{k=0}^{\infty} \frac{z^k}{(b)_k k\ !},$$

is the Bessel's function. Also

$$ {}_0F_0(-; -; z) = \sum_{k=0}^{\infty} \frac{z^k}{k\ !} = \exp(z)$$

and

$$ {}_1F_0(a; -; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k\ !} = \frac{1}{(1-z)^a}.$$

Similarly,

$$ {}_2F_1(a, b; b; z) = \frac{1}{(1-z)^a}, \qquad {}_2F_1(1, 1; 1; z) = \frac{1}{1-z},$$

$$ {}_2F_1(1, 1; 2; z) = \frac{-\ln(1-z)}{z}, \text{ and } {}_2F_1(1, 2; 1; z) = \frac{1}{(1-z)^2}.$$

For two functions $f(z)$ given by (1.1.2) and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$, the *Hadamard product* (or convolution) of $f$ and $g$ is defined by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.2.6}$$

Corresponding to the function

$$h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z^p \ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the *Dziok-Srivastava operator* [**25**] (see also [**107**])

$$H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{A}_p \to \mathcal{A}_p$$

is defined by the Hadamard product

$$
\begin{aligned}
H_p^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z) &:= h_p(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)*f(z) \\
&= z^p+\sum_{n=p+1}^{\infty}\frac{(\alpha_1)_{n-p}\ldots(\alpha_l)_{n-p}}{(\beta_1)_{n-p}\ldots(\beta_m)_{n-p}}\frac{a_nz^n}{(n-p)!}. \qquad (1.2.7)
\end{aligned}
$$

For brevity,

$$H_p^{l,m}[\alpha_1]f(z):=H_p^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z).$$

The linear (convolution) operator $H_p^{l,m}[\alpha_1]f(z)$ includes, as its special cases, many earlier linear (convolution) operators investigated in geometric function theory. Some of these special cases are described below.

The linear operator $\mathcal{F}(\alpha,\beta,\gamma)$ defined by

$$\mathcal{F}(\alpha,\beta,\gamma)=H_1^{2,1}(\alpha,\beta;\gamma)f(z)$$

is Hohlov linear operator [**34**]. The linear operator $\mathcal{L}(\alpha,\gamma)$ defined by

$$\mathcal{L}(\alpha,\gamma)=H_1^{2,1}(\alpha,1;\gamma)f(z)=\mathcal{F}(\alpha,1,\gamma)$$

is the Carlson and Shaffer linear operator [**15**]. The differential operator $\mathcal{D}^\lambda:\mathcal{A}\to\mathcal{A}$ defined by the Hadamard product:

$$\mathcal{D}^\lambda f(z):=\frac{z}{(1-z)^{\lambda+1}}*f(z)=H_1^{2,1}(\lambda+1,1;1)f(z),\quad (\lambda\geq 1, f\in\mathcal{A})$$

is the Ruscheweyh derivative operator [**92**]. This operator can also be defined by,

$$\mathcal{D}^n f(z):=\frac{z(z^{n-1}f(z))^{(n)}}{n\,!},\quad (n\in\mathcal{N}_0, f(z)\in\mathcal{A}).$$

In 1969, Bernardi [**10**] considered the linear integral operator $F:\mathcal{A}\to\mathcal{A}$ defined by

$$F(z):=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt. \qquad (1.2.8)$$

When $c=1$, this operator was investigated by Libera [**47**] and Livingston [**48**]. Therefore the operator in (1.2.8) is called the generalized Bernardi-Libera-Livingston linear operator. Clearly

$$F(z)=H_1^{2,1}(c+1,1;c+2)f(z),\quad (c>-1, f\in\mathcal{A}).$$

It is well-known [**10**] that the classes of starlike, convex and close-to-convex functions are closed under the Bernardi-Libera-Livingston integral operator.

DEFINITION 1.2.1. [**67, 71**] The *fractional integral of order* $\lambda$ is defined by

$$D_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda < 0)$$

where $f(z)$ is an analytic function in a simply connected region of the complex $z$-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

DEFINITION 1.2.2. [**67, 71**] The *fractional derivative of order* $\lambda$ is defined by

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \leq \lambda < 1)$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed as in definition (1.2.1) above.

DEFINITION 1.2.3. [**67, 71**] Under the hypothesis of Definition 1.2.2, the *fractional derivative of order* $n+\lambda$ is defined, by

$$D_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \leq \lambda < 1, \ n \in \mathcal{N}_0).$$

In 1987, Srivastava and Owa [**105**] studied a fractional derivative operator $\Omega^\lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$\Omega^\lambda f(z) := \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z).$$

The fractional derivative operator is a special case of the Dziok-Srivastava linear operator since

$$\begin{aligned} \Omega^\lambda f(z) &= H_1^{2,1}(2,1;2-\lambda) f(z) \\ &= \mathcal{L}(2, 2-\lambda) f(z), \quad (\lambda \notin \mathcal{N} \setminus \{1\}, f \in \mathcal{A}). \end{aligned}$$

## 1.3. MULTIPLIER TRANSFORMATIONS

The Sălăgean [**95**] derivative operator $\mathcal{D}^m f(z)$ order $m$ $(m \in \mathcal{N})$ is defined by

$$\mathcal{D}^m f(z) := f(z) * \left( z + \sum_{n=2}^{\infty} n^m z^n \right) = z + \sum_{n=2}^{\infty} n^m a_n z^n \quad (m \in \mathcal{N}, f \in \mathcal{A}).$$

Clearly $\mathcal{D}^0 f(z) = f(z)$, $\mathcal{D}^1 f(z) = z f'(z)$ and in general

$$\mathcal{D}^m f(z) = z(\mathcal{D}^{m-1} f(z))' \quad (m \in \mathcal{N}, f \in \mathcal{A}).$$

In 1990, Komatu [**42**] introduced a certain integral operator $\mathcal{I}_a^\lambda$ $(a > 0, \lambda \geq 0)$ defined by

$$\begin{aligned} \mathcal{I}_a^\lambda f(z) &:= \frac{a^\lambda}{\Gamma(\lambda)} \int_0^1 t^{a-2} \left(\log \frac{1}{t}\right)^{\lambda-1} f(zt)dt, \\ &= z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1}\right)^\lambda a_n z^n \quad (z \in U,\ a > 0,\ \lambda \geq 0,\ f \in \mathcal{A}). \end{aligned} \tag{1.3.1}$$

When $a = 2$, the integral operator $\mathcal{I}_a^\lambda f(z)$ is essentially the multiplier transformation studied by Flett [**27**]. Subsequently Junk *et al.* [**38**] studied the following one-parameter families of integral operators:

$$P^\alpha f(z) := \frac{2^\alpha}{z\Gamma(\alpha)} \int_0^1 \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)dt \quad (\alpha > 0),$$

$$Q_\alpha^\beta f(z) := \binom{\alpha+\beta}{\alpha} \frac{\beta}{z^\alpha} \int_0^1 t^{\alpha-1} \left(1 - \frac{t}{z}\right)^{\beta-1} f(t)dt \quad (\beta > 0, \alpha > -1),$$

and

$$F(z) := \frac{\alpha+1}{z^\alpha} \int_0^z t^{\alpha-1} f(t)dt \quad (\alpha > -1),$$

where $\Gamma(\alpha)$ is Gamma function, $\alpha \in \mathcal{N}$. The operator $P^\alpha, Q_\alpha^\beta$ and $F(z)$ were considered by Bernardi [**10, 11**]. Further, for a real number $\alpha > -1$, the operator $F(z)$ was used by several authors [**70, 74, 103, 104**].

For $f \in \mathcal{A}$ given by (1.1.1), Jung *et al.* [**38**] obtained

$$P^\alpha f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\alpha a_n z^n \quad (\alpha > 0), \tag{1.3.2}$$

$$Q_\alpha^\beta f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+\beta+n)} a_n z^n \quad (\beta > 0, \alpha > -1), \tag{1.3.3}$$

and

$$F(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha+1}{\alpha+n}\right) a_n z^n \quad (\alpha > -1). \tag{1.3.4}$$

By virtue of (1.3.1), (1.3.2), (1.3.3) and (1.3.4), we see that

$$\mathcal{I}_2^\lambda f(z) = P^\lambda f(z) \quad (\lambda > 0)$$

and

$$F(z) = Q_\alpha^1 f(z) \quad (\alpha > -1).$$

Ali and Singh [**4**] and Fournier and Ruscheweyh [**28**] have studied integral operators $V_\lambda$ of functions $f \in \mathcal{A}$ by

(1.3.5) $$V_\lambda f(z) := \int_0^1 \lambda(t) \frac{f(zt)}{t} dt \quad (\lambda(t) \in \Phi),$$

where

$$\Phi := \left\{ \lambda(t) : \lambda(t) \geq 0 \quad (0 \leq t \leq 1) \quad \text{and} \quad \int_0^1 \lambda(t) dt = 1 \right\}.$$

Recently, Li and Srivastava [**46**] have studied an integral operator $V_\lambda^\alpha$ of functions $f \in \mathcal{A}$ defined by

(1.3.6) $$V_\lambda^\alpha f(z) := \int_0^1 \lambda_\alpha(t) \frac{f(zt)}{t} dt,$$

where the real valued functions $\lambda_\alpha$ and $\lambda_{\alpha-1}$ satisfy the following conditions:

(i) For a suitable parameter $\alpha$,

$$\lambda_{\alpha-1} \in \Phi, \lambda_\alpha \in \Phi \quad \text{and} \quad \lambda_\alpha(1) = 0;$$

(ii) There exists a constant $c$ $(-1 < c \leq 2)$ such that

$$c\lambda_\alpha(t) - t\lambda'_{\alpha-1}(t) = (c+1)\lambda_{\alpha-1} \quad (0 < t < 1; -1 < c \leq 2).$$

Further Li and Srivastava [**46**] found a relation between $V_\lambda^\alpha f(z)$, $P^\alpha f(z)$ and $Q_\alpha^\beta f(z)$ by setting particular values of $\lambda_\alpha(t)$. By setting

$$\lambda_\alpha(t) = \binom{\alpha+\beta}{\alpha} \alpha (1-t)^{\alpha-1} t^\beta \quad (\alpha > 0, \beta > -1)$$

in (1.3.6), they obtained $V_\lambda^\alpha f(z) = Q_\beta^\alpha f(z)$; similarly by setting

$$\lambda_\alpha(t) = \frac{2^\alpha}{\Gamma(\alpha)} t \left( \log \frac{1}{t} \right)^{\alpha-1} \quad (\alpha > 0),$$

in (1.3.6), they obtained $V_\lambda^\alpha f(z) = P^\alpha f(z)$.

Motivated by these operators , Cho and Him [**20**, Definition, p. 400] introduced a more general linear operator called the multiplier transformation. For any integer $n$, the multiplier transformation $I_\lambda^n : \mathcal{A} \to \mathcal{A}$ is defined by

(1.3.7) $$I_\lambda^n f(z) := z + \sum_{k=2}^{\infty} \left( \frac{k+\lambda}{1+\lambda} \right)^n a_k z^k \quad (\lambda \geq 0,\ n \in \mathcal{Z}).$$

For $\lambda = 1$, the operator $I_\lambda^n$ was studied by Uralegaddi and Somanatha [**111**]. The operator $I_\lambda^n$ is closely related to the Komatu integral operators [**42**] and the differential and integral operators defined by Sălăgean [**95**].

Motivated by the multiplier transformation on $\mathcal{A}$, we define the operator $I_p(n,\lambda)$ on $\mathcal{A}_p$ by the following infinite series

$$I_p(n,\lambda)f(z) := z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k \quad (\lambda \geq -p,\ n \in \mathcal{Z}). \tag{1.3.8}$$

The operator $I_1(m,0)$ is the Sălăgean derivative operator $\mathcal{D}^m$ [**95**]. The operator $I_1(n,\lambda)$ was studied recently by Cho and Srivastava [**19**] and Cho and Kim [**20**]. The operator $I_1(n,1)$ was studied by Uralegaddi and Somanatha [**111**]. The operator $I_1(-1,c)$ is the generalized Bernardi-Libera-Livingston linear operator [**10, 11**].

## 1.4. SUBORDINATION AND SUPERORDINATION

Let $\psi(r,s,t;z) : \mathcal{C}^3 \times U \to \mathcal{C}$ and let $h(z)$ be univalent in $U$. If $p(z)$ is analytic in $U$ and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.4.1}$$

then $p(z)$ is called a *solution* of the differential subordination. The univalent function $q(z)$ is called a *dominant of the solution* of the differential subordination or more simply *dominant*, if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.4.1). A dominant $q_1(z)$ satisfying $q_1(z) \prec q(z)$ for all dominants $q(z)$ of (1.4.1) is said to be the *best dominant* of (1.4.1). The best dominant is unique up to a rotation of $U$. If $p(z) \in \mathcal{H}[a,n]$, then $p(z)$ will be called an $(a,n)$-*solution*, $q(z)$ an $(a,n)$-*dominant*, and $q_1(z)$ the *best* $(a,n)$-*dominant*. Let $\Omega \subset \mathcal{C}$ and let (1.4.1) be replaced by

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega, \text{ for all } z \in U. \tag{1.4.2}$$

Even though this is a "differential inclusion" and $\psi(p(z), zp'(z), z^2p''(z); z)$ may not be analytic in $U$, the condition in (1.4.2) will also be referred as a *second order differential subordination*, and the same definition of solution, dominant and best dominant as given above can be extended to this generalization. See [**58**] for more information on differential subordination.

Let $\psi(r,s,t;z) : \mathcal{C}^3 \times U \to \mathcal{C}$ and let $h(z)$ be analytic in $U$. If $p(z)$ and

$$\psi(p(z), zp'(z), z^2p''(z); z)$$

are univalent in $U$ and satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \tag{1.4.3}$$

then $p(z)$ is called a *solution* of the differential superordination. An analytic function $q(z)$ is called a *subordinant of the solution* of the differential superordination or more simply *subordinant*, if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.4.3). A univalent subordinant $q_1(z)$ satisfying $q(z) \prec q_1(z)$ for all subordinants $q(z)$ of (1.4.3) is said to be the *best subordinant* of (1.4.3). The best subordinant is unique up to a rotation of $U$. Let $\Omega \subset \mathcal{C}$ and let (1.4.3) be replaced by

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) | z \in U\}. \tag{1.4.4}$$

Even though this more general situation is a "*differential containment*", the condition in (1.4.4) will also be referred as a *second order differential superordination* and the definition of solution, subordinant and best subordinant can be extended to this generalization. See [**59**] for more information on the differential superordination.

Denote by $\mathcal{Q}$ the set of all functions $q(z)$ that are analytic and injective on $\overline{U} \setminus E(q)$ where

$$E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.4.1. [**58**, Definition 2.3a, p. 27] Let $\Omega$ be a set in $\mathcal{C}$, $q \in \mathcal{Q}$ and $n$ be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition

$$\psi(r,s,t;z) \notin \Omega \tag{1.4.5}$$

whenever $r = q(\zeta),\ s = k\zeta q'(\zeta)$, and

$$\Re\left\{\frac{t}{s}+1\right\} \geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

If $\psi : \mathcal{C}^2 \times U \to \mathcal{C}$, then the admissible condition (1.4.5) reduces to

(1.4.6) $$\psi(q(\zeta), k\zeta q'(\zeta); z) \notin \Omega,$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq n$.

In particular when $q(z) = M\frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then $q(U) = U_M := \{w : |w| < M\}$, $q(0) = a$, $E(q) = \emptyset$ and $q \in \mathcal{Q}$. In this case, we set $\Psi_n[\Omega, M, a] := \Psi_n[\Omega, q]$, and in the special case when the set $\Omega = U_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 1.4.2. [**59**, Definition 3, p. 817] Let $\Omega$ be a set in $\mathcal{C}$, $q(z) \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition

(1.4.7) $$\psi(r, s, t; \zeta) \in \Omega$$

whenever $r = q(z), s = \frac{zq'(z)}{m}$, and

$$\Re\left\{\frac{t}{s}+1\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$. When $n = 1$, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

If $\psi : \mathcal{C}^2 \times \overline{U} \to \mathcal{C}$ If $\psi : \mathcal{C}^2 \times \overline{U} \to \mathcal{C}$, then the admissible condition (1.4.7) reduces to

(1.4.8) $$\psi(q(z), zq'(z)/m; \zeta) \notin \Omega,$$

$z \in U$, $\zeta \in \partial U$ and $m \geq n$.

## 1.5. SCOPE AND MOTIVATION OF THIS WORK

In the present work, certain properties of analytic functions and meromorphic functions are investigated. In particular, certain sufficient conditions for Janowski starlikeness are obtained for various classes of analytic functions. Certain general differential subordination and superordination results are also obtained. These are then used to obtain differential sandwich results. Also non-linear coefficient problems involving the first three coefficients of a $p$-valent function are discussed.

In 1969, Bernardi [**10**] introduced and studied a linear integral operator

$$F(z) := \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1),$$

now called the *Bernardi integral operator*. He proved that the classes of starlike, convex and close-to-convex functions are closed under the Bernardi integral operator. In the year 2000, Parvatham [**74**] extended Bernardi's results for functions in $\mathcal{S}^*[\alpha]$ defined in (1.1.6). We have extended the results of Parvatham [**74**] by considering a more general subordinate function. In the year 1999, Silverman [**99**] introduced and studied classes of functions obtained by the quotient of expressions defining the convex and starlike functions. Later Obradovič and Tuneski [**64**] and Tuneski [**109**] improved the results of Silverman [**99**]. Further many researchers [**61, 62, 63, 85, 88**] have studied these classes. These results are extended by considering a more general subordinate function. Sufficient conditions for *Janowski starlikeness* for functions in several subclasses of analytic functions are also obtained. These results are presented in Chapter 2.

By using differential subordination, Miller and Mocanu [**58**] found some sufficient conditions relating the Schwarzian derivative to the starlikeness or convexity of $f \in \mathcal{A}$. Aouf et al. [**7**] and Kim and Srivastava [**41**] derived several inequalities associated with some families of integral and convolution operators that are defined for the class of normalized analytic functions in the open unit disk $U$. Recently Aghalary et al. [**2**] obtained some inequalities for analytic functions in the open unit disk that are associated with the Dziok-Srivastava linear operator and the multiplier transformation. Similar results for meromorphic functions defined through a linear operator are considered by Liu and

Owa [**49**]. These results motivate our main results in Chapters 3 and 4. Chapter 3 deals with the applications of differential subordination and superordination to obtain sufficient conditions on the *Schwarzian derivative* of normalized analytic functions. Subordination and superordination results for analytic functions associated with the *Dziok-Srivastava linear operator* and *multiplier transformation* are obtained. Additionally sandwich results are obtained. In Chapter 4, we consider the Liu-Srivastava linear operator for the case of meromorphic functions [**51, 52**] and also the multiplier transformation for meromorphic functions. Differential subordination and superordination results are obtained for meromorphic functions in the punctured unit disk that are associated with the *Liu-Srivastava linear operator* and the *multiplier transformation*. These results are obtained by investigating appropriate class of admissible functions. Certain related sandwich-type results are also obtained.

In 1975, Silverman [**98**] studied the class of analytic functions whose Taylor coefficients are negative. Several other authors (see for example, Al-Amiri [**3**], Attiya [**9**], Srivastava and Attiya [**102**], Owa and Srivastava [**72**], as well as Owa and Nishiwaki [**68**]) have studied several classes of analytic functions with negative coefficients. These results are shown to be special cases of our main results in Chapter 5. For a fixed analytic function $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ defined on the open unit disk and $\gamma < 1$, let $T_g(\gamma)$ denote the class of all analytic functions $f \in \mathcal{A}$ of the form (1.1.1) satisfying $\sum_{n=2}^{\infty} |a_n g_n| \leq 1 - \gamma$. For functions in $T_g(\gamma)$, a subordination result is derived involving the convolution with a normalized convex function.

In 1992, Ma and Minda [**54**] obtained sharp distortion, growth, rotation and covering theorems for the classes $C(\varphi)$ and $\mathcal{S}^*(\varphi)$. In addition, they obtained some sharp results for coefficient problems, particularly, the sharp bound on the coefficient functional $|a_3 - \mu a_2^2|$, $-\infty < \mu < \infty$, which implies sharp upper bounds for the second and third coefficients. They also studied some convolution properties. Also, several authors [**1, 23, 39, 40, 43, 82, 89**] have studied the coefficient problems for various classes of univalent functions. In Chapter 6, sharp upper bounds for the coefficient functionals $|a_{p+2} - \mu a_{p+1}^2|$ and $|a_{p+3}|$ are derived for certain $p$-valent analytic functions. These are then applied to obtain Fekete-Szegö like inequalities for several classes of functions defined by convolution.

## CHAPTER 2

# SUFFICIENT CONDITIONS FOR JANOWSKI STARLIKENESS

## 2.1. INTRODUCTION

For the class $\mathcal{S}^*[\alpha] = \{f \in \mathcal{A} : zf'(z)/f(z) \prec (1+\alpha z)/(1-\alpha z) \quad (0 < \alpha \leq 1)\}$, Parvatham proved the following:

THEOREM 2.1.1. [**74**, Theorem 1, p. 438] *Let $c \geq 0$, $0 < \alpha \leq 1$ and $\delta$ be given by*

$$\delta := \alpha \left[ \frac{2+\alpha+c(1-\alpha)}{1+2\alpha+c(1-\alpha)} \right].$$

*If $f \in \mathcal{S}^*[\delta]$, then the function $F(z)$ given by the Bernardi's integral as defined in (1.2.8) belong to $\mathcal{S}^*[\alpha]$.*

It is well-known [**10**] that the classes of starlike, convex and close-to-convex functions are closed under the Bernardi's integral operator. Since $\delta \geq \alpha$, Theorem 2.1.1 extends the result of Bernardi [**10**].

Parvatham also considered a similar problem for the class $R[\alpha]$ of functions $f \in \mathcal{A}$ satisfying

$$|f'(z) - 1| < \alpha\,|f'(z) + 1| \quad (z \in U,\ 0 < \alpha \leq 1),$$

or equivalently

$$f'(z) \prec \frac{1+\alpha z}{1-\alpha z} \quad (z \in U,\ 0 < \alpha \leq 1),$$

and proved the following:

THEOREM 2.1.2. [**74**, Theorem 2, p. 440] *Let $c \geq 0$, $0 < \alpha \leq 1$ and $\delta$ be given by*

$$\delta := \alpha \left[ \frac{2-\alpha+c(1-\alpha)}{1+c(1-\alpha)} \right].$$

*If $f \in R[\delta]$, then the function $F(z)$ given by the Bernardi's integral (1.2.8) is in $R[\alpha]$.*

The class $R[\alpha]$ can be extended to the general class $R[A, B]$ consisting of all analytic functions $f(z) \in \mathcal{A}$ satisfying

$$f'(z) \prec \frac{1 + Az}{1 + Bz}, \quad ( -1 \le B < A \le 1),$$

or the equivalent inequality,

$$|f'(z) - 1| < |A - Bf'(z)| \quad (z \in U, \ -1 \le B < A \le 1).$$

For $0 \le \alpha < 1$, the class $R[1 - 2\alpha, -1]$ consists of functions $f \in \mathcal{A}$ for which

$$\Re f'(z) > \alpha \quad (z \in U, \ 0 < \alpha \le 1),$$

and $R[1 - \alpha, 0] =: R_\alpha$ is the class of functions $f \in \mathcal{A}$ satisfying the condition

$$|f'(z) - 1| < 1 - \alpha \quad (z \in U, \ 0 \le \alpha < 1).$$

When $0 < \alpha \le 1$, the class $R[\alpha, -\alpha]$ is the class $R[\alpha]$ considered by Parvatham [**74**].

Silverman [**99**], Obradovič and Tuneski [**64**] and many others (see [**61, 62, 63, 85, 88**]) have studied properties of functions defined in terms of the quotient

$$\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}.$$

In fact, Silverman [**99**] have obtained the order of starlikeness for functions in the class $G_b$ defined by

$$G_b := \left\{ f \in \mathcal{A} : \left| \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1 \right| < b, \ 0 < b \le 1, \ z \in U \right\}.$$

Obradovič and Tuneski [**64**] improved the result of Silverman [**99**] by showing

$$G_b \subset \mathcal{S}^*[0, -b] \subset \mathcal{S}^*(2/(1 + \sqrt{1 + 8b})).$$

Later Tuneski [**109**] obtained conditions for the inclusion $G_b \subset \mathcal{S}^*[A, B]$ to hold. If we let $zf'(z)/f(z) =: p(z)$, then $G_b \subset \mathcal{S}^*[A, B]$ becomes

(2.1.1) $$1 + \frac{zp'(z)}{p(z)^2} \prec 1 + bz \Rightarrow p(z) \prec \frac{1 + Az}{1 + Bz}.$$

Let $f \in \mathcal{A}$ and $0 \le \alpha < 1$. Frasin and Darus [**26**] have shown that

$$\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \prec \frac{(1 - \alpha)z}{2 - \alpha} \Rightarrow \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha.$$

By writing $z^2 f'(z)/(f(z))^2$ as $p(z)$, we see that the above implication is a special case of

$$1 + \beta \frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}.$$

Another special case of the above implication was considered by Ponnusamy and Rajasekaran [**81**].

Obradovič *et. al.* [**60**] have shown that if $p(z)$ is analytic in $U$, $p(0) = 1$ and

$$1 + zp'(z) \prec 1 + z, \quad \text{then } p(z) \prec 1 + z.$$

Using this, they have obtained a criterion for a normalized analytic function to be univalent.

In this chapter, we extend Theorems 2.1.1 and 2.1.2 to hold true for the more general classes $\mathcal{S}^*[A, B]$ and $R[A, B]$ respectively. In fact a more general result for functions $p(z)$ with $p(0) = 1$ satisfying

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1+Dz}{1+Ez} \text{ implies } p(z) \prec \frac{1+Az}{1+Bz}$$

is obtained and by applying this result, we investigate the Bernardi's integral operator on the classes $S^*[A, B]$ and $R[D, E]$. Similar results are obtained by considering the expressions $1 + \beta zp'(z)$, $1 + \beta \frac{zp'(z)}{p^2(z)}$ and $1 + \beta \frac{zp'(z)}{p(z)}$. These results are then applied to obtain sufficient conditions for analytic functions to be Janowski starlike.

## 2.2. A BRIOT-BOUQUET DIFFERENTIAL SUBORDINATION

THEOREM 2.2.1. *Let* $-1 \leq B < A \leq 1$ *and* $-1 \leq E \leq 0 < D \leq 1$. *For* $\beta \geq 0$ *and* $\beta + \gamma > 0$, *let* $G := A\beta + B\gamma$, $H := (\beta + \gamma)(D - E)$, $I := (A\beta + B\gamma)(D - E) + (BD - AE)(\beta + \gamma) - kE(A - B)$, $J := (A\beta + B\gamma)(BD - AE)$, *and* $L := \beta + \gamma + k$. *In addition, for all* $k \geq 1$, *let*

$$(L^2 + G^2)[(H + J)I - 4H|J|] + 4LGHJ \geq LG[(H - J)^2 + I^2]. \tag{2.2.1}$$

*Further assume that*

$$\frac{[\beta(1 + A) + \gamma(1 + B) + 1](A - B)}{[\beta(1 + A) + \gamma(1 + B)][D(1 + B) - E(1 + A)] - E(A - B)} \geq 1. \tag{2.2.2}$$

*Let $p(z)$ be analytic in $U$ with $p(0) = 1$. If*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + Dz}{1 + Ez},$$

*then*

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

PROOF. Define $P(z)$ by

$$P(z) := p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \tag{2.2.3}$$

and $w(z)$ by

$$w(z) := \frac{p(z) - 1}{A - Bp(z)},$$

or equivalently by

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{2.2.4}$$

Then $w(z)$ is meromorphic in $U$ and $w(0) = 0$. We need to show that $|w(z)| < 1$ in $U$. By a computation from (2.2.4), it follows that

$$p'(z) = \frac{(A - B)zw'(z)}{(1 + Bw(z))^2}$$

and using this in (2.2.3),

$$P(z) = \frac{1 + Aw(z)}{1 + Bw(z)} + \frac{(A - B)zw'(z)}{(1 + Bw(z))[\beta(1 + Aw(z)) + \gamma(1 + Bw(z))]}.$$

Therefore

$$\frac{P(z) - 1}{D - EP(z)} = \frac{(A - B)[(\beta + \gamma)w(z) + (A\beta + B\gamma)w^2(z) + zw'(z)]}{\begin{array}{c}[(D - E) + (BD - AE)w(z)][\beta + \gamma \\ +(A\beta + B\gamma)w(z)] - E(A - B)zw'(z)\end{array}}.$$

Assume that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then by [**93**, Lemma 1.3, p. 28], there exists $k \geq 1$ such that $z_0 w'(z_0) = kw(z_0)$. Let $w(z_0) = e^{i\theta}$. For this $z_0$, we have

$$\left| \frac{P(z_0) - 1}{D - EP(z_0)} \right| = \left| \frac{(A - B)[L + Gw(z_0)]}{H + Iw(z_0) + Jw(z_0)^2} \right| = (A - B)[\varphi(\cos\theta)]^{\frac{1}{2}}$$

where

$$\begin{aligned}
\varphi(\cos\theta) &:= \frac{|L+Ge^{i\theta}|^2}{|He^{-i\theta}+Je^{i\theta}+I|^2} \\
&= \frac{L^2+G^2+2LG\cos\theta}{[I+(H+J)\cos\theta]^2+[(J-H)\sin\theta]^2} \\
&= \frac{L^2+G^2+2LG\cos\theta}{I^2+(H+J)^2\cos^2\theta+2I(H+J)\cos\theta+(J-H)^2\sin^2\theta} \\
&= \frac{L^2+G^2+2LG\cos\theta}{H^2+J^2+I^2+2HJ\cos 2\theta+2I(H+J)\cos\theta}.
\end{aligned}$$

Define the function

$$\varphi(t) := \frac{L^2+G^2+2LGt}{4HJt^2+2I(H+J)t+(H-J)^2+I^2}. \tag{2.2.5}$$

We shall show that $\varphi(t)$ is a decreasing function. A simple computation using (2.2.5) yields

$$\varphi'(t) = -\frac{4LGHJt^2+4HJ(L^2+G^2)t+I(L^2+G^2)(H+J)-LG[I^2+(H-J)^2]}{[4HJt^2+2I(H+J)t+(H-J)^2+I^2]^2}.$$

The function $\varphi(t)$ is a decreasing function if $\varphi'(t)<0$ or equivalently if

$$4LGHJt^2+4HJ(L^2+G^2)t+I(L^2+G^2)(H+J)-LG[I^2+(H-J)^2] \geq 0.$$

In view of the fact that

$$\min\{at^2+bt+c : -1\leq t\leq 1\} = \begin{cases} \frac{4ac-b^2}{4a}, & \text{if } a>0 \text{ and } |b|<2a \\ a-|b|+c, & \text{otherwise,} \end{cases}$$

the condition (2.2.1) shows that $\varphi(t)$ is a decreasing function of $t=\cos\theta$. Thus

$$\varphi(t) \geq \varphi(1) = \left[\frac{L+G}{I+J+H}\right]^2.$$

Consider the function

$$\begin{aligned}
\psi(k) &:= \frac{L+G}{I+J+H} \\
&= \frac{(1+A)\beta+(1+B)\gamma+k}{[(1+B)D-(1+A)E][(1+A)\beta+(1+B)\gamma]-kE(A-B)}.
\end{aligned}$$

Since

$$\psi'(k) = \frac{[(1+A)\beta+(1+B)\gamma](1+B)(D-E)}{[[(1+B)D-(1+A)E][(1+A)\beta+(1+B)\gamma]-kE(A-B)]^2},$$

clearly $\psi'(k) > 0$ and hence $\psi(k)$ is an increasing function of $k$. Since $k \geq 1$, we have $\psi(k) \geq \psi(1)$ and therefore

$$\left|\frac{P(z_0)-1}{D-EP(z_0)}\right| \geq \frac{[\beta(1+A)+\gamma(1+B)+1](A-B)}{[\beta(1+A)+\gamma(1+B)][D(1+B)-E(1+A)]-E(A-B)},$$

which by (2.2.2) is greater than or equal to 1. This contradicts that $P(z) \prec (1+Dz)/(1+Ez)$ and completes the proof. $\square$

## 2.3. APPLICATION TO THE BERNARDI'S INTEGRAL OPERATOR

THEOREM 2.3.1. *Let the conditions of Theorem 2.2.1 hold with* $\beta = 1$ *and* $\gamma = c > -1$. *If* $f \in \mathcal{S}^*[D,E]$, *then the function* $F(z)$ *given by the Bernardi's integral (1.2.8) is in* $\mathcal{S}^*[A,B]$.

PROOF. Differentiation of the Bernardi's integral (1.2.8) yields

$$(c+1)f(z) = zF'(z) + cF(z).$$

Logarithmic differentiation now yields

$$\frac{zf'(z)}{f(z)} = p(z) + \frac{zp'(z)}{p(z)+c},$$

with $p(z) = zF'(z)/F(z)$. The result now follows from Theorem 2.2.1. $\square$

Observe that when $J = 0$, the condition (2.2.1) reduces to the equivalent form

$$(LI - GH)(LH - GI) \geq 0. \tag{2.3.1}$$

REMARK 2.3.1. If $A = \alpha$, $B = -\alpha$, $D = \delta$ and $E = -\delta$ $(0 < \alpha, \delta \leq 1)$, then $G = \alpha(1-c)$, $H = 2\delta(1+c)$, $I = 2\alpha\delta(1+k-c)$, $J = 0$ and $L = 1+c+k$. In this case, $LI - GH = 2\alpha\delta k(2+k) > 0$. In addition, $LH - GI \geq 0$ becomes $(1+c)(1+c+k) \geq \alpha^2(1-c)(1-c+k)$. Clearly this condition holds when $c \geq 0$. In the case $-1 < c < 0$, since

$$\frac{(1+c)(2+c)}{(1-c)(2-c)} \leq \frac{(1+c)(1+c+k)}{(1-c)(1-c+k)},$$

condition (2.3.1) holds provided $\alpha^2 \leq (1+c)(2+c)/(1-c)(2-c)$. Thus Theorem 2.3.1 not only reduces to Theorem 2.1.1 for $c \geq 0$, but also extends it for the case $-1 < c < 0$.

Corollary 2.3.1. *Let* $-1 < c < 0$, $0 < \alpha \le \sqrt{(1+c)(2+c)/(1-c)(2-c)}$, *and* $\delta$ *be as in Theorem 2.4.2. If* $f \in \mathcal{S}^*[\delta]$, *then the function* $F(z)$ *given by the Bernardi's integral (1.2.8) belongs to* $\mathcal{S}^*[\alpha]$.

Remark 2.3.2. Let $A = 1-\alpha$, $B = 0$, $D = 1-\delta$ and $E = 0$ $(0 \le \alpha, \delta < 1)$. Then $G = 1-\alpha$, $H = (1-\delta)(1+c)$, $I = (1-\alpha)(1-\delta)$, $J = 0$ and $L = 1+c+k$. Since $J = 0$, the condition (2.3.1) reduces to

$$(1+c)(1+c+k) - (1-\alpha)^2 \ge 0. \tag{2.3.2}$$

Since $(1+c)(1+c+k)-(1-\alpha)^2 \ge (1+c)(2+c)-(1-\alpha)^2$, the inequality (2.3.2) holds provided $\alpha \ge 1-\sqrt{(1+c)(2+c)}$. This condition holds for $c \ge \left[\sqrt{4(\alpha-1)^2+1}-3\right]/2$. This yields the following result for the class $\mathcal{S}^*(\delta)$.

Corollary 2.3.2. *Let* $\delta := \alpha - (1-\alpha)/(2+c-\alpha)$, $f(z) \in \mathcal{S}^*(\delta)$ *and* $F(z)$ *be given by the Bernardi's integral (1.2.8). If* $\alpha_0 \le \alpha < 1$, *then* $F(z) \in \mathcal{S}^*(\alpha)$ *for all* $c > -1$. *Here* $\alpha_0 := (3+c-\sqrt{(3+c)^2-4})/2$.

Theorem 2.3.2. *Under the conditions stated in Theorem 2.2.1 with* $\beta = 0$ *and* $\gamma = c+1$, *if* $f \in R[D,E]$, *then the function* $F(z)$ *given by the Bernardi's integral (1.2.8) is in* $R[A,B]$.

Proof. Since

$$(c+1)f(z) = zF'(z) + cF(z),$$

it follows that

$$f'(z) = \frac{zF''(z)}{c+1} + F'(z). \tag{2.3.3}$$

The result now follows from Theorem 2.2.1 with $p(z) = F'(z)$, $\beta = 0$ and $\gamma = c+1$. ☐

Remark 2.3.3. For $A = \alpha$, $B = -\alpha$, $D = \delta$ and $E = -\delta$ $(0 < \alpha, \delta \le 1)$, then $G = -\alpha(1+c)$, $H = 2\delta(1+c)$, $I = 2\alpha\delta(k-1-c)$, $J = 0$ and $L = 1+c+k$. The condition (2.3.1) becomes

$$4\alpha\delta^2k^2(1+c)[(1+c)(1-\alpha^2)+k(1+\alpha^2)] \ge 0$$

which holds for any $c > -1$. This shows that Theorem 2.3.2 reduces to Theorem 2.1.2 and that the assertion even holds in the case $-1 < c < 0$.

REMARK 2.3.4. For $A = \delta$, $B = 0$, $D = \alpha$ and $E = 0$ $(0 < \alpha, \delta \leq 1)$, then $G = I = J = 0$, $H = \alpha(1 + c)$, and $L = 1 + c + k$. In this case the condition (2.3.1) holds for any $c > -1$. Thus Theorem 2.3.2 extends the earlier result of Anbudurai [**6**, Theorem 2.1, p. 20] even in the case $-1 < c < 0$.

REMARK 2.3.5. For $A = 1 - \alpha$, $B = 0$, $D = 1 - \delta$ and $E = 0$ $(0 \leq \alpha, \delta < 1)$, then $G = 0$, $H = (1 - \delta)(1 + c)$, $I = 0$, $J = 0$ and $L = 1 + c + k$. Theorem 2.3.2 yields the following:

COROLLARY 2.3.3. *Let* $c > -1$, $1/(2 + c) \leq \alpha < 1$ *and* $\delta := \alpha - (1 - \alpha)/(1 + c)$. *If* $f(z) \in R_\delta$, *then* $F(z) \in R_\alpha$.

## 2.4. ANOTHER DIFFERENTIAL SUBORDINATION

LEMMA 2.4.1. *Let* $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$ *and* $\beta \neq 0$. *Assume that*

$$(A - B)|\beta| \geq (D - E)(1 + B^2) + |2B(D - E) - E\beta(A - B)|\,. \tag{2.4.1}$$

*If* $p(z)$ *is analytic in* $U$ *with* $p(0) = 1$ *and*

$$1 + \beta z p'(z) \prec \frac{1 + Dz}{1 + Ez},$$

*then*

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

PROOF. Define the function $P(z)$ by

$$P(z) := 1 + \beta z p'(z) \tag{2.4.2}$$

and the function $w(z)$ by (2.2.4). Then $w(z)$ is meromorphic in $U$ and $w(0) = 0$. Using (2.2.4) in (2.4.2), we get

$$P(z) = \frac{(1 + Bw(z))^2 + (A - B)\beta z w'(z)}{(1 + Bw(z))^2},$$

and therefore

$$\frac{P(z)-1}{D-EP(z)} = \frac{(A-B)\beta z w'(z)}{(D-E)(1+Bw(z))^2 - E(A-B)\beta z w'(z)}.$$

If $|w(z)| \not< 1$ in $|z|<1$, then there is a point $z_0$ in $U$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1$$

and by [**93**, Lemma 1.3, p. 28], there exists $k \geq 1$ such that $z_0 w'(z_0) = kw(z_0)$. Let $w(z_0) = e^{i\theta}$. For this $z_0$, we have

$$\begin{aligned}\left|\frac{P(z_0)-1}{D-EP(z_0)}\right| &= \frac{(A-B)k|\beta|}{[I^2+(H-J)^2+4HJt^2+4I(H+J)t]^{\frac{1}{2}}} \\ &\geq \frac{(A-B)k|\beta|}{\max\limits_{-1\leq t\leq 1}\{[I^2+(H-J)^2+4HJt^2+4I(H+J)t]^{\frac{1}{2}}\}}\end{aligned}$$

where $I := 2B(D-E) - k\beta E(A-B)$, $J := (D-E)B^2$, $H := (D-E)$ and $t := \cos\theta$. A computation shows that

$$\left|\frac{P(z_0)-1}{D-EP(z_0)}\right| \geq \frac{(A-B)|\beta|k}{H+|I|+J}.$$

Yet another calculation shows that the function

$$\psi(k) := \frac{(A-B)|\beta|k}{H+|I|+J}$$

is an increasing function of $k$. Since $k \geq 1$, $\psi(k) \geq \psi(1)$ and therefore

$$\left|\frac{P(z_0)-1}{D-EP(z_0)}\right| \geq \frac{(A-B)|\beta|}{(D-E)(1+B^2)+|2B(D-E)-E\beta(A-B)|},$$

which by (2.4.1) is greater than or equal to 1. This contradicts $P(z) \prec (1+Dz)/(1+Ez)$ and completes the proof. □

COROLLARY 2.4.1. [**60**, Lemma 1, p. 1035] *If $p(z)$ is analytic in $U$ and $zp'(z) \prec z$, then $p(z) \prec 1+z$.*

PROOF. The result follows from Lemma 2.4.1 by taking $\beta = 1$, $E = 0 = B$ and $D = 1 = A$. □

COROLLARY 2.4.2. [**60**, Theorem 1, p. 1036] *If $f \in \mathcal{A}$ satisfies $|(z/f(z))''| \leq 1$ with $f(z)/z \neq 0$, then $f(z)$ is univalent in $U$.*

PROOF. Define the function $p(z)$ by

$$p(z) = \frac{z^2 f'(z)}{f(z)^2} = \frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' \quad (f \in \mathcal{A},\ f(z)/z \neq 0).$$

Then we get

$$p'(z) = -z\left(\frac{z}{f(z)}\right)''.$$

Since $|(z/f(z))''| \leq 1$, it follows that

$$-z^2\left(\frac{z}{f(z)}\right)'' \prec z \text{ or } zp'(z) \prec z.$$

From Corollary 2.4.1, we have

$$p(z) \prec 1 + z \text{ or } \frac{z^2 f'(z)}{f(z)^2} \prec 1 + z.$$

Thus we have $|z^2 f'(z)/f(z)^2 - 1| < 1$ and hence $f(z)$ is univalent in $U$ by [**65**, Theorem 2, p. 394] . □

By taking $p(z) = zf'(z)/f(z)$ in Lemma 2.4.1, the following result is obtained.

THEOREM 2.4.1. *Let the conditions of Lemma 2.4.1 holds. If $f \in \mathcal{A}$ satisfies*

$$1 + \beta \frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{1 + Dz}{1 + Ez},$$

*then $f \in \mathcal{S}^*[A, B]$.*

COROLLARY 2.4.3. *Let $0 < \alpha \leq 1$ and $\delta = \alpha/(1 + \alpha)^2$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| < \delta\left|2 + \frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right|,$$

*then $f(z) \in \mathcal{S}^*[\alpha]$.*

PROOF. The result follows from Theorem 2.4.1 by taking $\beta = 1$, $A = \alpha = -B$ and $D = -E = \delta$ $(0 < \alpha, \delta \leq 1)$. □

COROLLARY 2.4.4. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| < \frac{1 - \alpha}{3} \quad (0 \leq \alpha < 1), \tag{2.4.3}$$

*then $f(z) \in \mathcal{S}^*_\alpha$.*

PROOF. The result follows from Theorem 2.4.1 by taking $\beta = 1$ $A = 1-2\alpha, B = -1$, $D = (1-\alpha)/3$ and $E = 0$ $(0 \leq \alpha < 1)$. $\square$

By replacing $p(z)$ by $1/p(z)$, $\beta = -1$, $A$ replaced by $-B$ and $B$ replaced by $-A$ in Lemma 2.4.1, the following result is obtained.

LEMMA 2.4.2. *Let* $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$. *Assume that*

$$(A-B) \geq (D-E)(1+A^2) + |E(A-B) - 2A(D-E)|. \tag{2.4.4}$$

*If* $p(z)$ *is analytic in* $U$ *with* $p(0) = 1$ *and*

$$1 + \frac{zp'(z)}{p^2(z)} \prec \frac{1+Dz}{1+Ez},$$

*then*

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

THEOREM 2.4.2. *Let* $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$. *Assume that (2.4.4) holds. If* $f \in \mathcal{A}$ *satisfies*

$$\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec \frac{1+Dz}{1+Ez},$$

*then* $f \in \mathcal{S}^*[A, B]$.

PROOF. Let the function

$$p(z) = \frac{zf'(z)}{f(z)}. \tag{2.4.5}$$

By a computation from (2.4.5), it follows that

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

Thus

$$1 + \frac{zp'(z)}{p^2(z)} = \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}.$$

The result now follows from Lemma 2.4.2. $\square$

COROLLARY 2.4.5. [**109**, Corollary 2.6, p. 203] *Let* $-1 \leq B \leq A \leq 1$, *then* $G_b \subset \mathcal{S}^*[A, B]$ *where* $b = (A-B)/(1+|A|)^2$.

PROOF. The result follows from Theorem 2.4.2 by taking $E = 0$ and $D = b \quad (0 < b \leq 1)$. $\square$

COROLLARY 2.4.6. [**64**, Theorem 1, p. 61] *Let* $f \in G_b$ $(0 < b \leq 1)$, *then*

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1 + bz}.$$

PROOF. The result follows from Theorem 2.4.2 by taking $A = 0 = E$ and $D = -B = b \quad (0 < b \leq 1)$. $\square$

When $A = 0 = E$ and $D = -B = 1$, Theorem 2.4.2 reduces to:

COROLLARY 2.4.7. [**99**, Corollary 1, p. 76] *If* $f \in G_1$, *then* $f \in \mathcal{S}^*(1/2)$.

EXAMPLE 2.4.1. If $f \in G_{\frac{1-\alpha}{(2-\alpha)^2}}$, $(0 \leq \alpha < 1)$, then $f \in \mathcal{S}^*(\alpha)$.

The result follows by taking $A = 1 - \alpha$, $B = 0$ in Corollary 2.4.5.

EXAMPLE 2.4.2. If $f \in \mathcal{A}$ satisfies

(2.4.6)
$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \beta \left|1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)}\right| \quad (\beta = \frac{\alpha}{1 + 3\alpha + \alpha^2},\ 0 < \alpha \leq 1),$$

then $f \in \mathcal{S}^*[\alpha]$.

The result follows from Theorem 2.4.2 by taking $A = -B = \alpha$ and $D = -E = \delta \quad (0 < \alpha \leq 1)$.

EXAMPLE 2.4.3. If $f \in \mathcal{A}$ satisfies (2.4.6) with $\beta = (1-\alpha)/[1+(1-2\alpha)^2+|5\alpha-3|]$ $(0 \leq \alpha < 1)$, then $f \in \mathcal{S}^*_\alpha$.

The result follows from Theorem 2.4.2 by taking $A = 1 - 2\alpha,\ B = 0$ and $D = -E = \delta \quad (0 \leq \alpha < 1)$.

LEMMA 2.4.3. *Let* $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$, $AB \geq 0$ *and* $\beta \neq 0$. *Assume that*

(2.4.7) $$|\beta|(A - B) \geq (D - E)(1 + AB) + |(D - E)(A + B) - E\beta(A - B)|.$$

*Let $p(z)$ be analytic in $U$ with $p(0) = 1$ and*

$$1 + \beta \frac{zp'(z)}{p(z)} \prec \frac{1 + Dz}{1 + Ez},$$

*then*

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

PROOF. The proof is similar to the proof of Lemma 2.4.1, and is therefore omitted.

□

REMARK 2.4.1. When $E \leq 0,\ AB \leq 0$, Lemma 2.4.3 is valid provided the following conditions holds:

$$(1 - A\beta)^2 \left\{ 2E\beta(A + B)(D - E) - (A - B)[(D - E)^2 + (E\beta)^2] \right\} \geq 4\beta^2(A - B)AB$$

instead of (2.4.7).

By taking $\beta = -1,\ A = \lambda = E$ and $D = B = 0 \quad (|\lambda| \leq 1)$ in Lemma 2.4.3, we have the following result:

COROLLARY 2.4.8. [**81**, Theorem 1(iii), p. 195] *If $p(z)$ is analytic in $U$, and*

$$\frac{zp'(z)}{p(z)} \prec \frac{\lambda z}{1 + \lambda z} \quad (|\lambda| \leq 1),$$

*then*

$$p(z) \prec 1 + z.$$

By taking $\beta = 1, B = 0,\ D = A/(1 + A)$ and $E = 0$ in Lemma 2.4.3, we have the following result:

COROLLARY 2.4.9. *Let $0 < A \leq 1$. Let $p(z)$ be analytic in $U$ with $p(0) = 1$. If $|zp'(z)/p(z)| < A/(1 + A)$, then $p(z) \prec 1 + Az$.*

COROLLARY 2.4.10. *If $f \in \mathcal{A}$ satisfies*

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < \frac{1 - \alpha}{2 - \alpha} \quad (0 \leq \alpha < 1),$$

*then $f \in \mathcal{S}^*(\alpha)$.*

PROOF. Let the function

$$p(z) = \frac{zf'(z)}{f(z)}. \tag{2.4.8}$$

By a computation from (2.4.8), it follows that

$$\frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}.$$

Hence the required result follows by taking $A = 1 - \alpha$ in Corollary 2.4.9. □

THEOREM 2.4.3. *Let the conditions of Lemma 2.4.3 holds. If $f \in \mathcal{A}$ satisfies*

$$1 + \beta\left(\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)}\right) \prec \frac{1+Dz}{1+Ez}, \tag{2.4.9}$$

*then*

$$\frac{z^2 f'(z)}{f^2(z)} \prec \frac{1+Az}{1+Bz}.$$

PROOF. Let the function $p(z)$ be defined by

$$p(z) = \frac{z^2 f'(z)}{f^2(z)}. \tag{2.4.10}$$

By a computation from (2.4.10), it follows that

$$1 + \beta\frac{zp'(z)}{p(z)} = 1 + \beta\left(\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)}\right).$$

Thus (2.4.9) becomes

$$1 + \beta\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez}.$$

The result now follows from Lemma 2.4.3. □

By taking $\beta = 1,\ A = 1-\alpha,\ B = 0$, $E = 0$ and $D = (1-\alpha)/(2-\alpha) \quad (0 \le \alpha < 1)$ in Theorem 2.4.3, we have the following result:

COROLLARY 2.4.11. [**26**, Theorem 2.4, p. 307] *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)}\right| < \frac{1-\alpha}{2-\alpha} \quad (0 \le \alpha < 1),$$

*then*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1 - \alpha.$$

## CHAPTER 3

# DIFFERENTIAL SUBORDINATION AND SUPERORDINATION FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

## 3.1. INTRODUCTION

The Schwarzian derivative $\{f,z\}$ of $f\in\mathcal{A}$ is defined by

$$\{f,z\}:=\left(\frac{f''(z)}{f'(z)}\right)'-\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Owa and Obradović [**69**] proved that if $f\in\mathcal{A}$ satisfies

$$\Re\left[\frac{1}{2}\left(1+\frac{zf''(z)}{f'(z)}\right)^2+z^2\{f,z\}\right]>0, \tag{3.1.1}$$

then $f\in C$. Miller and Mocanu [**58**] proved that if $f\in\mathcal{A}$ satisfies the condition

$$\Re\left[\left(1+\frac{zf''(z)}{f'(z)}\right)+\alpha z^2\{f,z\}\right]>0 \quad (\Re\alpha\geq 0), \tag{3.1.2}$$

or

$$\Re\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^2+z^2\{f,z\}\right]>0, \tag{3.1.3}$$

or

$$\Re\left[\left(1+\frac{zf''(z)}{f'(z)}\right)e^{z^2\{f,z\}}\right]>0, \tag{3.1.4}$$

then $f\in C$. More generally, Miller and Mocanu [**58**] found conditions on $\phi$ such that

$$\Re\left\{\phi\left(1+\frac{zf''(z)}{f'(z)},z^2\{f,z\};z\right)\right\}>0$$

implies $f\in C$. Miller and Mocanu [**58**] proved that if $f\in\mathcal{A}$ satisfies the condition

$$\Re\left[\alpha\left(\frac{zf'(z)}{f(z)}\right)+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)+\left(\frac{zf'(z)}{f(z)}\right)z^2\{f,z\}\right]>0 \quad (\alpha,\beta\in\mathcal{R}), \tag{3.1.5}$$

or

$$\Re\left[\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}+z^2\{f,z\}\right)\right]>-\frac{1}{2}, \tag{3.1.6}$$

then $f \in \mathcal{S}^*$. More generally, they found conditions on $\phi$ such that

$$\Re\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right)\right\} > 0$$

implies $f \in \mathcal{S}^*$.

Recently Miller and Mocanu [**59**] considered certain first and second order differential superordination. Using the results of Miller and Mocanu [**59**], Bulboaca considered certain classes of first order differential superordination [**13**] as well as superordination-preserving integral operators [**14**]. Ali *et al.* [**5**] used the results of Bulboaca [**13**] to obtain some sufficient conditions for normalized analytic functions $f$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where $q_1, q_2$ are given univalent functions in $U$. Our results includes several earlier results.

To prove our main results we need the following theorems.

THEOREM 3.1.1. [**58**, Theorem 2.3b, p. 28] *Let* $\psi \in \Psi_n[\Omega, q]$ *with* $q(0) = a$. *If the analytic function* $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ *satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,$$

*then* $p(z) \prec q(z)$.

THEOREM 3.1.2. [**59**, Theorem 1, p. 818] *Let* $\psi \in \Psi'_n[\Omega, q]$ *with* $q(0) = a$. *If* $p(z) \in \mathcal{Q}(a)$ *and* $\psi(p(z), zp'(z), z^2p''(z); z)$ *is univalent in* $U$, *then*

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) : z \in U\}$$

*implies* $q(z) \prec p(z)$.

In this chapter, we give an application of differential subordination and superordination to Schwarzian derivatives to obtain sufficient conditions for the functions $f \in \mathcal{A}$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z) \quad \text{or} \quad q_1(z) \prec 1+\frac{zf''(z)}{f'(z)} \prec q_2(z)$$

where $q_1, q_2$ are respectively the given analytic and univalent functions in $U$. In addition, subordination and superordination results for analytic functions associated with the *Dziok-Srivastava linear operator* and the *multiplier transformation* are obtained. These results are obtained by investigating appropriate class of admissible functions. Sandwich-type results are also obtained.

## 3.2. SCHWARZIAN DERIVATIVES AND STARLIKENESS

The following definition of a class of admissible functions is required in Theorem 3.2.1.

DEFINITION 3.2.1. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_S[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)} \quad (q(\zeta) \neq 0),$$

$$\Re\left\{\frac{2w + u^2 - 1 + 3(v-u)^2}{2(v-u)}\right\} \geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

THEOREM 3.2.1. *Let $\phi \in \Phi_S[\Omega, q]$. If $f \in \mathcal{A}$ satisfies*

(3.2.1) $$\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) : z \in U\right\} \subset \Omega,$$

*then*

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

PROOF. Define $p(z)$ by

(3.2.2) $$p(z) := \frac{zf'(z)}{f(z)}.$$

A simple computation yields

(3.2.3) $$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}.$$

Further computations yield

$$z^2\{f,z\} = \frac{zp'(z)+z^2p''(z)}{p(z)} - \frac{3}{2}\left[\frac{zp'(z)}{p(z)}\right]^2 + \frac{1-p^2(z)}{2}. \tag{3.2.4}$$

Define the transformations from $\mathcal{C}^3$ to $\mathcal{C}$ by

$$u = r,\ v = r + \frac{s}{r},\ w = \frac{s+t}{r} - \frac{3}{2}\left[\frac{s}{r}\right]^2 + \frac{1-r^2}{2}. \tag{3.2.5}$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, r+\frac{s}{r}, \frac{s+t}{r} - \frac{3}{2}\left[\frac{s}{r}\right]^2 + \frac{1-r^2}{2}; z\right). \tag{3.2.6}$$

The proof shall make use of Theorem 3.1.1. Using equations (3.2.2), (3.2.3) and (3.2.4), from (3.2.6), it follows that

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right). \tag{3.2.7}$$

Hence (3.2.1) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_S[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.1. Note that

$$\frac{t}{s} + 1 = \frac{2w + u^2 - 1 + 3(v-u)^2}{2(v-u)},$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 3.1.1, $p(z) \prec q(z)$ or

$$\frac{zf'(z)}{f(z)} \prec q(z). \qquad \square$$

If $\Omega \neq \mathcal{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of $U$ onto $\Omega$. In this case, the class $\Phi_S[h(U), q]$ is written as $\Phi_S[h, q]$. The following result is an immediate consequence of Theorem 3.2.1.

THEOREM 3.2.2. *Let $\phi \in \Phi_S[h, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \prec h(z), \tag{3.2.8}$$

*then*

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

The next result is an extension of Theorem 3.2.1 to the case where the behavior of $q(z)$ on $\partial U$ is not known.

COROLLARY 3.2.1. *Let $\Omega \subset \mathcal{C}$ and let $q(z)$ be univalent in $U$, $q(0) = 1$. Let $\phi \in \Phi_S[\Omega, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}$ and*

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \in \Omega, \tag{3.2.9}$$

*then*

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

PROOF. Theorem 3.2.1 yields $\frac{zf'(z)}{f(z)} \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$. $\square$

THEOREM 3.2.3. *Let $h(z)$ and $q(z)$ be univalent in $U$, with $q(0) = 1$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ satisfy one of the following conditions:*

(1) *$\phi \in \Phi_S[h, q_\rho]$ for some $\rho \in (0,1)$, or*
(2) *there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_S[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

*If $f \in \mathcal{A}$ satisfies (3.2.8), then*

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

PROOF. The proof is similar to the proof of [**58**, Theorem 2.3d, p. 30], and is therefore omitted. $\square$

The next theorem yields the best dominant of the differential subordination (3.2.8).

THEOREM 3.2.4. *Let $h(z)$ be univalent in $U$, and $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$. Suppose that the differential equation*

$$\phi\left(q(z), zq'(z), z^2q''(z); z\right) = h(z) \tag{3.2.10}$$

*has a solution $q(z)$ with $q(0) = 1$ and satisfy one of the following conditions:*

(1) *$q(z) \in \mathcal{Q}_1$ and $\phi \in \Phi_S[h, q]$,*

(2) $q(z)$ *is univalent in* $U$ *and* $\phi \in \Phi_S[h, q_\rho]$, *for some* $\rho \in (0,1)$, *or*

(3) $q(z)$ *is univalent in* $U$ *and there exists* $\rho_0 \in (0,1)$ *such that* $\phi \in \Phi_S[h_\rho, q_\rho]$, *for all* $\rho \in (\rho_0, 1)$.

*If* $f \in \mathcal{A}$ *satisfies (3.2.8), then*

$$\frac{zf'(z)}{f(z)} \prec q(z),$$

*and* $q(z)$ *is the best dominant.*

Proof. Following the same arguments as in the proof of [**58**, Theorem 2.3e, p. 31], from Theorems 3.2.2 and 3.2.3, it follows that $q(z)$ is a dominant. Since $q(z)$ satisfies (3.2.10), it is also a solution of (3.2.8) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant. $\square$

In the particular case $q(z) = 1 + Mz,\ M > 0$, in view of Definition 3.2.1, the class of admissible functions $\Phi_S[\Omega, q]$, denoted by $\Phi_S[\Omega, M]$, is described below.

Definition 3.2.2. Let $\Omega$ be a set in $\mathcal{C}$ and $M > 0$. The class of admissible functions $\Phi_S[\Omega, M]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ such that

(3.2.11)

$$\phi\left(1 + Me^{i\theta}, 1 + \frac{k+1+Me^{i\theta}}{1+Me^{i\theta}}Me^{i\theta}, \frac{\begin{array}{c}(M+e^{-i\theta})[2(k-1)M - (Me^{i\theta}+3)\\ \times M^2e^{i\theta} + 2Le^{-i\theta}] - 3k^2M^2\end{array}}{2(M+e^{-i\theta})^2}; z\right) \notin \Omega$$

whenever $z \in U,\ \theta \in \mathcal{R}$, $\Re\left(Le^{-i\theta}\right) \geq (k-1)kM$ for all real $\theta$ and $k \geq 1$.

Corollary 3.2.2. *Let* $\phi \in \Phi_S[\Omega, M]$. *If* $f \in \mathcal{A}$ *satisfies*

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) \in \Omega,$$

*then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M.$$

In the special case $\Omega = \{\omega : |\omega - 1| < M\}$, the class $\Phi_S[\Omega, M]$ is simply denoted by $\Phi_S[M]$.

COROLLARY 3.2.3. *Let* $\phi \in \Phi_S[M]$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) - 1\right| < M,$$

*then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M.$$

EXAMPLE 3.2.1. Let $\alpha \geq 2(M-1)$. If $f \in \mathcal{A}$ satisfies

$$\left|(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) - 1\right| < M,$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M.$$

This follows from Corollary 3.2.3, by taking $\phi(u,v,w;z) = (1-\alpha)u + \alpha v$ $(\alpha \geq 0)$.

EXAMPLE 3.2.2. If $0 < M \leq 2$ and $f \in \mathcal{A}$ satisfies

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1\right| < M,$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M.$$

This follows from Corollary 3.2.3, by taking $\phi(u,v,w;z) = v/u$.

COROLLARY 3.2.4. [**113**, Corollary 2, p. 583] *Let* $0 < M \leq 1$ *and* $\lambda \geq -2(1-M)$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\frac{z^2 f''(z)}{f(z)} + \lambda\left(\frac{zf'(z)}{f(z)} - 1\right)\right| < M(2+\lambda-M),$$

*then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M.$$

PROOF. The proof follows from Corollary 3.2.3, by taking $\phi(u,v,w;z) = u(v-1) + \lambda(u-1)$ $(\lambda \geq -2(1-M))$ and $\Omega = h(U)$ where $h(z) = M(\lambda+2-M)z$, $(0 < M \leq 1)$. ∎

Next, we apply Theorem 3.2.1 to the particular case where $q(U)$ is a half-plane $q(U) = \{w : \Re w > 0\} =: \Delta$. Let $q(z) := (1+z)/(1-z)$, then $q(0) = 1$, $E(q) = \{1\}$ and $q \in \mathcal{Q}_1$. Let $e^{i\theta} =: \zeta \in \partial U \setminus E(q)$, then $q(\zeta) = i\rho$, and $\zeta q'(\zeta) = -(1+\rho^2)/2$ where $\rho := \cot\frac{\theta}{2}$. The class of admissible functions $\Phi_S[\Omega, q]$, denoted by $\Phi_S[\Omega, \Delta]$, is described below.

Definition 3.2.3. Let $\Omega$ be a set in $\mathcal{C}$, $q(z) \in \mathcal{Q}_1$ and $q(0) = 1$. The class of admissible functions $\Phi_S[\Omega, \Delta]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi(i\rho, i\tau, \xi + i\eta; z) \notin \Omega \tag{3.2.12}$$

for all $z \in U$ and for all real $\rho, \tau, \xi$ and $\eta$ with

$$\rho\tau \geq \frac{1}{2}(1+3\rho^2), \quad \text{and } \rho\eta \geq 0.$$

Theorem 3.2.5. *Let $\phi \in \Phi_S[\Omega, \Delta]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) \in \Omega, \tag{3.2.13}$$

*then $f \in \mathcal{S}^*$.*

If $\Omega \neq \mathcal{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of $U$ onto $\Omega$. Let the function $h(z) = (1+z)/(1-z)$, clearly $h(U) = \Delta$. In this case, the class of admissible functions $\Phi_S[h(U), \Delta]$ is written as $\Phi_S[\Delta]$ and hence the following result is an immediate consequence of Theorem 3.2.5.

Corollary 3.2.5. [**58**, Theorem 4.6a, p. 244] *Let $\phi \in \Phi_S[\Delta]$. If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right)\right\} > 0, \tag{3.2.14}$$

*then $f \in \mathcal{S}^*$.*

A dual result of Theorem 3.2.1 for the differential superordination is given below.

DEFINITION 3.2.4. Let $\Omega$ be a set in $\mathcal{C}$, $q(z) \in \mathcal{H}$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_S[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z), \quad v = q(z) + \frac{zq'(z)}{mq(z)} \quad (q(z) \neq 0,\ zq'(z) \neq 0),$$
$$\Re\left\{\frac{2w + u^2 - 1 + 3(v-u)^2}{2(v-u)}\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

THEOREM 3.2.6. *Let $\phi \in \Phi'_S[\Omega, q]$. If $f \in \mathcal{A}$, $zf'(z)/f(z) \in \mathcal{Q}_1$ and*

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right)$$

*is univalent in $U$, then*

$$\Omega \subset \left\{\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) : z \in U\right\} \tag{3.2.15}$$

*implies*

$$q(z) \prec \frac{zf'(z)}{f(z)}.$$

PROOF. From (3.2.7) and (3.2.15), it follows that

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z);\right) : z \in U\right\}.$$

In view of (3.2.5), the admissibility condition for $\phi \in \Phi'_S[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 3.1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{zf'(z)}{f(z)}. \qquad \square$$

THEOREM 3.2.7. *Let $q(z) \in \mathcal{H}$, h(z) be analytic in $U$ and $\phi \in \Phi'_S[h, q] := \Phi'_S[h(U), q]$. If $f \in \mathcal{A}$,*

$$\frac{zf'(z)}{f(z)} \in \mathcal{Q}_1 \text{ and } \phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right)$$

*is univalent in $U$, then*

$$h(z) \prec \phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) \tag{3.2.16}$$

*implies*

$$q(z) \prec \frac{zf'(z)}{f(z)}.$$

Theorems 3.2.6 and 3.2.7 are used to obtain subordinants of differential superordination of the form (3.2.15) or (3.2.16). The following theorem proves the existence of the best subordinant of (3.2.16) for an appropriate $\phi$.

THEOREM 3.2.8. *Let h(z) be analytic in $U$ and $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$. Suppose that the differential equation*

$$\phi(q(z), zq'(z), z^2q''(z); z) = h(z) \tag{3.2.17}$$

*has a solution $q(z) \in \mathcal{Q}_1$. If $\phi \in \Phi'_S[h, q]$, $f \in \mathcal{A}$,*

$$\frac{zf'(z)}{f(z)} \in \mathcal{Q}_1 \text{ and } \phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right)$$

*is univalent in $U$, then*

$$h(z) \prec \phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right)$$

*implies*

$$q(z) \prec \frac{zf'(z)}{f(z)}$$

*and $q(z)$ is the best subordinant.*

PROOF. The proof is similar to the proof of Theorem 3.2.4, and is therefore omitted. □

Combining Theorems 3.2.2 and 3.2.7, the following sandwich-type theorem is obtained.

COROLLARY 3.2.6. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $U$, $h_2(z)$ be univalent function in $U$, $q_2(z) \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_S[h_2, q_2] \cap \Phi'_S[h_1, q_1]$. If $f \in \mathcal{A}$,*

$$\frac{zf'(z)}{f(z)} \in \mathcal{H} \cap \mathcal{Q}_1 \text{ and } \phi\left(\frac{zf'(z)}{f(z)},\ 1 + \frac{zf''(z)}{f'(z)},\ z^2\{f, z\}; z\right)$$

*is univalent in $U$, then*

$$h_1(z) \prec \phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \prec h_2(z)$$

*implies*

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z).$$

## 3.3. SCHWARZIAN DERIVATIVES AND CONVEXITY

The following definition of a class of admissible functions is required in Theorem 3.3.1.

DEFINITION 3.3.1. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{Sc}[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^2 \times U \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi\left(q(\zeta), k\zeta q'(\zeta) + \frac{1-q^2(\zeta)}{2}; z\right) \notin \Omega,$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

THEOREM 3.3.1. *Let $\phi \in \Phi_{Sc}[\Omega, q]$. If $f \in \mathcal{A}$ satisfies*

$$\left\{\phi\left(1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) : z \in U\right\} \subset \Omega, \tag{3.3.1}$$

*then*

$$1+\frac{zf''(z)}{f'(z)} \prec q(z).$$

PROOF. Define $p(z)$ by

$$p(z) := 1+\frac{zf''(z)}{f'(z)}. \tag{3.3.2}$$

A simple calculation yields

$$z^2\{f,z\} = zp'(z) + \frac{1-p^2(z)}{2}. \tag{3.3.3}$$

Define the transformations from $\mathcal{C}^2$ to $\mathcal{C}$ by

$$u = r,\ v = s + \frac{1-r^2}{2}. \tag{3.3.4}$$

Let

$$\psi(r,s;z)=\phi(u,v;z)=\phi\left(r,s+\frac{1-r^2}{2};z\right). \tag{3.3.5}$$

The proof shall make use of Theorem 3.1.1. Using equations (3.3.2) and (3.3.3), from (3.3.5), it follows that

$$\psi(p(z),zp'(z);z)=\phi\left(1+\frac{zf''(z)}{f'(z)},z^2\{f,z\};z\right). \tag{3.3.6}$$

Hence (3.3.1) becomes

$$\psi(p(z),zp'(z);z)\in\Omega.$$

In view of (3.3.5), the admissibility condition for $\phi\in\Phi_{Sc}[\Omega,q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.1. Hence $\psi\in\Psi[\Omega,q]$ and by Theorem 3.1.1, $p(z)\prec q(z)$ or

$$1+\frac{zf''(z)}{f'(z)}\prec q(z). \qquad \square$$

THEOREM 3.3.2. *Let* $\phi\in\Phi_{Sc}[h,q]:=\Phi_{Sc}[h(U),q]$. *If* $f\in\mathcal{A}$ *satisfies*

$$\phi\left(1+\frac{zf''(z)}{f'(z)},z^2\{f,z\};z\right)\prec h(z), \tag{3.3.7}$$

*then*

$$1+\frac{zf''(z)}{f'(z)}\prec q(z).$$

DEFINITION 3.3.2. Let $\Omega$ be a set in $\mathcal{C}$ and let $\Phi_{Sc}[\Omega,M]:=\Phi_{Sc}[\Omega,q]$ where $q(z)=1+Mz,\ M>0$. The class of admissible functions $\Phi_{Sc}[\Omega,M]$ consists of those functions $\phi:\mathcal{C}^2\times U\to\mathcal{C}$ such that

$$\phi\left(1+Me^{i\theta},\frac{2(k-1)-Me^{i\theta}}{2}Me^{i\theta};z\right)\notin\Omega \tag{3.3.8}$$

whenever $z\in U,\ \theta\in\mathcal{R}$ and $k\geq 1$.

COROLLARY 3.3.1. *Let* $\phi\in\Phi_{Sc}[\Omega,M]$. *If* $f\in\mathcal{A}$ *satisfies*

$$\phi\left(1+\frac{zf''(z)}{f'(z)},z^2\{f,z\};z\right)\in\Omega,$$

*then*

$$\left|\frac{zf''(z)}{f'(z)}\right|<M.$$

COROLLARY 3.3.2. *Let* $\phi \in \Phi_S[M] := \Phi_{Sc}[\Omega, M]$ *where* $\Omega = \{\omega : |\omega - 1| < M\}$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) - 1\right| < M,$$

*then*

$$\left|\frac{zf''(z)}{f'(z)}\right| < M.$$

EXAMPLE 3.3.1. If $0 < M \leq 4$ and $f \in \mathcal{A}$ satisfies

$$\left|\frac{zf''(z)}{f'(z)} + z^2\{f, z\}\right| < M, \quad \text{then} \quad \left|\frac{zf''(z)}{f'(z)}\right| < M.$$

This follows from Corollary 3.3.2 by taking $\phi(u, v; z) = u + v$.

Next, an application of Theorem 3.3.1 to the particular case corresponding to $q(U)$ being a half-plane $q(U) = \Delta$ is given.

DEFINITION 3.3.3. Let $\Omega$ be a set in $\mathcal{C}$. The class of admissible functions $\Phi_{Sc}[\Omega, \Delta]$ consists of those functions $\phi : \mathcal{C}^2 \times U \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi(i\rho, \eta; z) \notin \Omega \tag{3.3.9}$$

for all $z \in U$ and for all real $\rho$ and $\eta$ such that $\eta \leq 0$.

THEOREM 3.3.3. *Let* $\phi \in \Phi_{Sc}[\Omega, \Delta]$. *If* $f \in \mathcal{A}$ *satisfies*

$$\phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) \in \Omega, \tag{3.3.10}$$

*then* $f \in C$.

Let $h : U \to \mathcal{C}$ be defined by $h(z) = (1 + z)/(1 - z)$, then $h(U) = \Delta$. In this case, we denote the class $\Phi_{Sc}[h(U), \Delta]$ by $\Phi_{Sc}[\Delta]$ and hence the following result is an immediate consequence of Theorem 3.3.3.

COROLLARY 3.3.3. [**58**, Theorem 4.6b, p. 246] *Let* $\phi \in \Phi_{Sc}[\Delta]$. *If* $f \in \mathcal{A}$ *satisfies*

$$\Re\left\{\phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right)\right\} > 0, \tag{3.3.11}$$

*then* $f \in C$.

DEFINITION 3.3.4. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{H}$. The class of admissible functions $\Phi'_{Sc}[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^2 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi\left(q(z), \frac{zq'(z)}{m} + \frac{1-q^2(z)}{2}; \zeta\right) \in \Omega$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

THEOREM 3.3.4. *Let $\phi \in \Phi'_{Sc}[\Omega, q]$. If $f \in \mathcal{A}$, $1 + zf''(z)/f'(z) \in \mathcal{Q}_1$ and*

$$\phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right)$$

*is univalent in $U$, then*

$$\Omega \subset \left\{\phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) : z \in U\right\} \tag{3.3.12}$$

*implies*

$$q(z) \prec 1 + \frac{zf''(z)}{f'(z)}.$$

PROOF. From (3.3.6) and (3.3.12), it follows that

$$\Omega \subset \{\psi\left(p(z), zp'(z);\right) : z \in U\}.$$

In view of (3.3.4), the admissibility condition for $\phi \in \Phi'_{Sc}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 3.1.2, $q(z) \prec p(z)$ or

$$q(z) \prec 1 + \frac{zf''(z)}{f'(z)}. \qquad \square$$

For a simply connected domain $\Omega \neq \mathcal{C}$, let $h : U \to \mathcal{C}$ be the conformal mapping such that $\Omega = h(U)$. In this case, the class $\Phi'_{Sc}[h(U), q]$ is written as $\Phi'_{Sc}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.3.4.

THEOREM 3.3.5. *Let $q(z) \in \mathcal{H}$, h(z) be analytic in $U$ and $\phi \in \Phi'_{Sc}[h, q]$. If $f \in \mathcal{A}$,*

$$1 + \frac{zf''(z)}{f'(z)} \in \mathcal{Q}_1 \text{ and } \phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right)$$

*is univalent in $U$, then*

$$h(z) \prec \phi\left(1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \tag{3.3.13}$$

*implies*

$$q(z) \prec 1+\frac{zf''(z)}{f'(z)}.$$

Combining Theorems 3.3.2 and 3.3.5, the following sandwich-type theorem is obtained.

COROLLARY 3.3.4. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $U$, $h_2(z)$ be univalent function in $U$, $q_2(z) \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{Sc}[h_2, q_2] \cap \Phi'_{Sc}[h_1, q_1]$. If $f \in \mathcal{A}$,*

$$1+\frac{zf''(z)}{f'(z)} \in \mathcal{H} \cap \mathcal{Q}_1 \text{ and } \phi\left(\ 1+\frac{zf''(z)}{f'(z)},\ z^2\{f,z\}; z\right)$$

*is univalent in $U$, then*

$$h_1(z) \prec \phi\left(1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \prec h_2(z)$$

*implies*

$$q_1(z) \prec 1+\frac{zf''(z)}{f'(z)} \prec q_2(z).$$

## 3.4. SUBORDINATION RESULTS INVOLVING THE DZIOK-SRIVASTAVA LINEAR OPERATOR

Differential subordination of the Dziok-Srivastava linear operator is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

DEFINITION 3.4.1. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{Q}_0 \cap \mathcal{H}[0,p]$. The class of admissible functions $\Phi_H[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + (\alpha_1 - p)q(\zeta)}{\alpha_1} \quad (\alpha_1 \in \mathcal{C},\ \alpha_1 \neq 0, -1),$$

$$\Re\left\{\frac{\alpha_1(\alpha_1+1)w+(p-\alpha_1)(\alpha_1-p+1)u}{\alpha_1 v+(p-\alpha_1)u}-(2(\alpha_1-p)+1)\right\}\geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

$z\in U$, $\zeta\in\partial U\setminus E(q)$ and $k\geq p$.

THEOREM 3.4.1. *Let* $\phi\in\Phi_H[\Omega,q]$. *If* $f\in\mathcal{A}_p$ *satisfies*

$$\left\{\phi\left(H_p^{l,m}[\alpha_1]f(z),H_p^{l,m}[\alpha_1+1]f(z),H_p^{l,m}[\alpha_1+2]f(z);z\right):z\in U\right\}\subset\Omega, \tag{3.4.1}$$

*then*

$$H_p^{l,m}[\alpha_1]f(z)\prec q(z),\qquad (z\in U).$$

PROOF. Define the analytic function $p(z)$ in $U$ by

$$p(z):=H_p^{l,m}[\alpha_1]f(z). \tag{3.4.2}$$

In view of the relation

$$\alpha_1 H_p^{l,m}[\alpha_1+1]f(z)=z[H_p^{l,m}[\alpha_1]f(z)]'+(\alpha_1-p)H_p^{l,m}[\alpha_1]f(z), \tag{3.4.3}$$

a simple computation using (3.4.2), yields

$$H_p^{l,m}[\alpha_1+1]f(z)=\frac{zp'(z)+(\alpha_1-p)p(z)}{\alpha_1}. \tag{3.4.4}$$

Further computations yield

(3.4.5)

$$H_p^{l,m}[\alpha_1+2]f(z)=\frac{z^2p''(z)+2(\alpha_1-p+1)zp'(z)+(\alpha_1-p)(\alpha_1-p+1)p(z)}{\alpha_1(\alpha_1+1)}.$$

Define the transformations from $\mathcal{C}^3$ to $\mathcal{C}$ by

$$u=r,\ v=\frac{s+(\alpha_1-p)r}{\alpha_1},\ w=\frac{t+2(\alpha_1-p+1)s+(\alpha_1-p)(\alpha_1-p+1)r}{\alpha_1(\alpha_1+1)}. \tag{3.4.6}$$

Let

$$\begin{aligned}\psi(r,s,t;z)&=\phi(u,v,w;z)\\&=\phi\left(r,\frac{s+(\alpha_1-p)r}{\alpha_1},\frac{t+2(\alpha_1-p+1)s+(\alpha_1-p)(\alpha_1-p+1)r}{\alpha_1(\alpha_1+1)};z\right).\end{aligned} \tag{3.4.7}$$

The proof shall make use of Theorem 3.1.1. Using equations (3.4.2), (3.4.4) and (3.4.5), from (3.4.7), it follows that

(3.4.8)

$$\psi(p(z),zp'(z),z^2p''(z);z)=\phi\left(H_p^{l,m}[\alpha_1]f(z),H_p^{l,m}[\alpha_1+1]f(z),H_p^{l,m}[\alpha_1+2]f(z);z\right).$$

Hence (3.4.1) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_H[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.1. Note that

$$\frac{t}{s} + 1 = \frac{\alpha_1(\alpha_1+1)w + (p-\alpha_1)(\alpha_1-p+1)u}{\alpha_1 v + (p-\alpha_1)u} - (2(\alpha_1 - p)+1),$$

and hence $\psi \in \Psi_p[\Omega, q]$. By Theorem 3.1.1, $p(z) \prec q(z)$ or

$$H_p^{l,m}[\alpha_1]f(z) \prec q(z). \qquad \square$$

If $\Omega \neq \mathcal{C}$ is a simply connected domain and $h$ is the corresponding conformal mapping with $\Omega = h(U)$, we denote the class $\Phi_H[h(U), q]$ by $\Phi_H[h, q]$. The following result is an immediate consequence of Theorem 3.4.1.

THEOREM 3.4.2. *Let* $\phi \in \Phi_H[h, q]$. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right) \prec h(z), \tag{3.4.9}$$

*then*

$$H_p^{l,m}[\alpha_1]f(z) \prec q(z).$$

The following result is an extension of Theorem 3.4.1 to the case where the behavior of $q(z)$ on $\partial U$ is not known.

COROLLARY 3.4.1. *Let* $\Omega \subset \mathcal{C}$, $q(z)$ *be univalent in* $U$ *and* $q(0) = 0$. *Let* $\phi \in \Phi_H[\Omega, q_\rho]$ *for some* $\rho \in (0,1)$ *where* $q_\rho(z) = q(\rho z)$. *If* $f \in \mathcal{A}_p$ *and*

$$\phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right) \in \Omega,$$

*then*

$$H_p^{l,m}[\alpha_1]f(z) \prec q(z).$$

PROOF. Theorem 3.4.1 yields $H_p^{l,m}[\alpha_1]f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$. $\square$

THEOREM 3.4.3. *Let $h(z)$ and $q(z)$ be univalent in $U$, with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ satisfy one of the following conditions:*

(1) *$\phi \in \Phi_H[h, q_\rho]$ for some $\rho \in (0,1)$, or*
(2) *there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

*If $f \in \mathcal{A}_p$ satisfies (3.4.9), then*

$$H_p^{l,m}[\alpha_1]f(z) \prec q(z).$$

PROOF. The result is similar to the proof of Theorem 3.2.3 and is therefore omitted. □

The next theorem yields the best dominant of the differential subordination (3.4.9).

THEOREM 3.4.4. *Let $h(z)$ be univalent in $U$. Let $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$. Suppose that the differential equation*

$$\phi\left(q(z), zq'(z), z^2q''(z); z\right) = h(z) \tag{3.4.10}$$

*has a solution $q(z)$ with $q(0) = 0$ and satisfies one of the following conditions:*

(1) *$q(z) \in \mathcal{Q}_0$ and $\phi \in \Phi_H[h, q]$,*
(2) *$q(z)$ is univalent in $U$ and $\phi \in \Phi_H[h, q_\rho]$, for some $\rho \in (0,1)$, or*
(3) *$q(z)$ is univalent in $U$ and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.*

*If $f \in \mathcal{A}_p$ satisfies (3.4.9), then*

$$H_p^{l,m}[\alpha_1]f(z) \prec q(z),$$

*and $q(z)$ is the best dominant.*

PROOF. Following the same arguments as in the proof of [**58**, Theorem 2.3e, p. 31], from Theorems 3.4.2 and 3.4.3, it follows that $q(z)$ is a dominant. Since $q(z)$

satisfies (3.4.10), it is also a solution of (3.4.9) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant. □

In the particular case $q(z) = Mz,\ M > 0$, and in view of Definition 3.4.1, the class of admissible functions $\Phi_H[\Omega, q]$, denoted by $\Phi_H[\Omega, M]$, is described below.

DEFINITION 3.4.2. Let $\Omega$ be a set in $\mathcal{C}$ and $M > 0$. The class of admissible functions $\Phi_H[\Omega, M]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{k+\alpha_1-p}{\alpha_1}Me^{i\theta}, \frac{L+(\alpha_1-p+1)(2k+\alpha_1-p)Me^{i\theta}}{\alpha_1(\alpha_1+1)}; z\right) \notin \Omega \tag{3.4.11}$$

whenever $z \in U,\ \theta \in \mathcal{R}$, $\Re(Le^{-i\theta}) \geq (k-1)kM$ for all real $\theta$, $\alpha_1 \in \mathcal{C}$ $(\alpha_1 \neq 0, -1)$ and $k \geq p$.

COROLLARY 3.4.2. *Let $\phi \in \Phi_H[\Omega, M]$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right) \in \Omega,$$

*then*

$$|H_p^{l,m}[\alpha_1]f(z)| < M.$$

COROLLARY 3.4.3. *Let $\phi \in \Phi_H[M] := \Phi_H[\Omega, M]$ where $\Omega = \{\omega : |\omega| < M\}$. If $f \in \mathcal{A}_p$ satisfies*

$$|\phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right)| < M,$$

*then*

$$|H_p^{l,m}[\alpha_1]f(z)| < M.$$

COROLLARY 3.4.4. *If $\Re\alpha_1 \geq (p-k)/2,\ k \geq p$ and $f \in \mathcal{A}_p$ satisfies*

$$|H_p^{l,m}[\alpha_1+1]f(z)| < M,$$

*then*

$$|H_p^{l,m}[\alpha_1]f(z)| < M.$$

PROOF. This follows from Corollary 3.4.3, by taking $\phi(u, v, w; z) = v$. □

By taking $\Omega = U$ and $M = 1$ in Corollary 3.4.2, we have the following result:

COROLLARY 3.4.5. [**2**, Theorem 1, p. 269] *Let* $\phi \in \Phi_H[U] := \Phi_H[U,1]$. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\left|\phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right)\right| < 1,$$

*then*

$$|H_p^{l,m}[\alpha_1]f(z)| < 1.$$

COROLLARY 3.4.6. [**41**, Theorem 1, p. 230] *Let* $\phi \in \Phi_H[U]$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\phi\left(\mathcal{D}^\alpha f(z), \mathcal{D}^{\alpha+1} f(z), \mathcal{D}^{\alpha+2} f(z); z\right)\right| < 1,$$

*then*

$$|\mathcal{D}^\alpha f(z)| < 1$$

*where* $\mathcal{D}^\alpha f(z)$ *is the Ruscheweyh derivative operator in* $U$.

PROOF. The proof follows from Corollary 3.4.2, by taking $\Omega = U$, $p = 1$, $l = 2$, $m = 1$, $\alpha_1 = \alpha + 1$, $\alpha_2 = 1$ $(\alpha > -1)$, $\beta_1 = 1$ and $M = 1$. □

By taking $M = 1$ in Corollary 3.4.4, we have the following result:

COROLLARY 3.4.7. [**2**, Corollary 3, p. 271] *If* $\Re\alpha_1 \geq (p-k)/2$, $k \geq p$ *and* $f \in \mathcal{A}_p$ *satisfies*

$$|H_p^{l,m}[\alpha_1+1]f(z)| < 1,$$

*then*

$$|H_p^{l,m}[\alpha_1]f(z)| < 1.$$

COROLLARY 3.4.8. *Let* $M > 0$ *and* $0 \neq \alpha_1 \in \mathcal{C}$. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\left|H_p^{l,m}[\alpha_1+1]f(z) + (p/\alpha_1 - 1)H_p^{l,m}[\alpha_1]f(z)\right| < \frac{Mp}{|\alpha_1|}, \tag{3.4.12}$$

*then*

$$|H_p^{l,m}[\alpha_1]f(z)| < M.$$

Proof. Let $\phi(u,v,w;z) = v + (p/\alpha_1 - 1)u$ and $\Omega = h(U)$ where $h(z) = Mpz/|\alpha_1|$, $(M > 0)$. Since

$$\left|\phi\left(Me^{i\theta}, \frac{k+\alpha_1-p}{\alpha_1}Me^{i\theta}, \frac{L+(\alpha_1-p+1)(2k+\alpha_1-p)Me^{i\theta}}{\alpha_1(\alpha_1+1)}; z\right)\right| = \frac{kM}{|\alpha_1|} \geq \frac{Mp}{|\alpha_1|}$$

$z \in U,\ \theta \in \mathcal{R},\ \alpha_1 \in \mathcal{C}\ (\alpha_1 \neq 0, -1)$ and $k \geq p$, it follows that $\phi \in \Phi_H[\Omega, M]$. Hence by Corollary 3.4.2, the required result is obtained. $\square$

The differential equation

$$zq'(z) + (p/\alpha_1 - 1)q(z) = \frac{Mp}{|\alpha_1|}z$$

has a univalent solution $q(z) = Mz$. Theorem 3.4.4 shows that $q(z) = Mz$ is the best dominant of (3.4.12).

Note that

$$\begin{aligned}
H_p^{(2,1)}(1,1;1)f(z) &= f(z),\\
H_p^{(2,1)}(2,1;1)f(z) &= zf'(z) + (1-p)f(z),\\
H_p^{(2,1)}(3,1;1)f(z) &= \frac{1}{2}[z^2f''(z) + 2(2-p)zf'(z) + (1-p)(2-p)f(z)].
\end{aligned}$$

By taking $l = 2,\ m = 1,\ \alpha_1 = \alpha_2 = \beta_1 = 1$, (3.4.12) shows that for $f \in \mathcal{A}_p$, whenever

$$zf'(z) + p\left(\frac{1-\alpha_1}{\alpha_1}\right)f(z) \prec \frac{Mz}{|\alpha_1|}, \quad \text{then} \quad f(z) \prec Mz.$$

Definition 3.4.3. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{Q}_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = [k\zeta q'(\zeta) + (\alpha_1 - 1)q(\zeta)]/\alpha_1 \quad (\alpha_1 \in \mathcal{C},\ \alpha_1 \neq 0, -1),$$

$$\Re\left\{\frac{\alpha_1[(\alpha_1+1)w + (1-\alpha_1)u]}{v\alpha_1 + (1-\alpha_1)u} + 1 - 2\alpha_1\right\} \geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

THEOREM 3.4.5. *Let $\phi \in \Phi_{H,1}[\Omega, q]$. If $f \in \mathcal{A}_p$ satisfies*

$$\left\{\phi\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^{p-1}}; z\right) : z \in U\right\} \subset \Omega, \tag{3.4.13}$$

*then*

$$\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}} \prec q(z).$$

PROOF. Define the function $p(z)$ in $U$ by

$$p(z) := \frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}. \tag{3.4.14}$$

By making use of (3.4.3), a simple computation from (3.4.14) yields

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}} = \frac{1}{\alpha_1}\left(zp'(z) + (\alpha_1 - 1)p(z)\right). \tag{3.4.15}$$

Further computations yield

$$\frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^{p-1}} = \frac{1}{\alpha_1(\alpha_1+1)}\left(z^2p''(z) + 2\alpha_1 zp'(z) + \alpha_1(\alpha_1-1)p(z)\right). \tag{3.4.16}$$

Define the transformations from $\mathcal{C}^3$ to $\mathcal{C}$ by

$$u = r,\ v = \frac{s + (\alpha_1 - 1)r}{\alpha_1},\ w = \frac{t + 2\alpha_1 s + \alpha_1(\alpha_1 - 1)r}{\alpha_1(\alpha_1+1)}. \tag{3.4.17}$$

Let

(3.4.18)

$$\psi(r, s, t; z) = \phi(u, v, w; z) = \phi\left(r, \frac{s + (\alpha_1 - 1)r}{\alpha_1}, \frac{t + 2\alpha_1 s + \alpha_1(\alpha_1 - 1)r}{\alpha_1(\alpha_1+1)}; z\right).$$

The proof shall make use of Theorem 3.1.1. Using equations (3.4.14), (3.4.15) and (3.4.16), from (3.4.18), it follows that

(3.4.19)

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^{p-1}}; z\right).$$

Hence (3.4.13) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.1. Note that

$$\frac{t}{s} + 1 = \frac{\alpha_1[(\alpha_1+1)w + (1-\alpha_1)u]}{v\alpha_1 + (1-\alpha_1)u} + 1 - 2\alpha_1,$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 3.1.1, $p(z) \prec q(z)$ or

$$\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}} \prec q(z). \qquad \square$$

We let $\Phi_{H,1}[h, q] := \Phi_{H,1}[h(U), q]$, for any conformal mapping $h$ mapping $U$ onto $\Omega$, and $\Phi_{H,1}[\Omega, q] =: \Phi_{H,1}[\Omega, M]$ for the function $q(z) = Mz$ $(M > 0)$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.4.5.

THEOREM 3.4.6. *Let $\phi \in \Phi_{H,1}[h, q]$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^{p-1}}; z\right) \prec h(z), \tag{3.4.20}$$

*then*

$$\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}} \prec q(z).$$

DEFINITION 3.4.4. Let $\Omega$ be a set in $\mathcal{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,1}[\Omega, M]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{k+\alpha_1-1}{\alpha_1}Me^{i\theta}, \frac{L+\alpha_1(2k+\alpha_1-1)Me^{i\theta}}{\alpha_1(\alpha_1+1)}; z\right) \notin \Omega \tag{3.4.21}$$

whenever $z \in U,\ \theta \in \mathcal{R}$, $\Re(Le^{-i\theta}) \geq (k-1)kM$ for all real $\theta$, $\alpha_1 \in \mathcal{C}$ $(\alpha_1 \neq 0, -1)$ and $k \geq 1$.

COROLLARY 3.4.9. *Let $\phi \in \Phi_{H,1}[\Omega, M]$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^{p-1}}; z\right) \in \Omega,$$

*then*

$$\left|\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}\right| < M.$$

COROLLARY 3.4.10. *Let $\phi \in \Phi_{H,1}[M] := \Phi_{H,1}[\Omega, M]$ where $\Omega = \{\omega : |\omega| < M\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\phi\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^{p-1}}; z\right)\right| < M,$$

*then*

$$\left|\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}\right| < M.$$

COROLLARY 3.4.11. *If* $\Re\alpha_1 \geq 0$ *and* $f \in \mathcal{A}_p$ *satisfies*

$$\left|\frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}}\right| < M,$$

*then*

$$\left|\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}\right| < M.$$

PROOF. This follows from Corollary 3.4.10 by taking $\phi(u,v,w;z) = v$. □

COROLLARY 3.4.12. [**41**, Theorem 1, p. 230] *Let* $\phi \in \Phi_{H,1}[U] := \Phi_{H,1}[U,1]$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\phi\left(\mathcal{D}^{\alpha}f(z), \mathcal{D}^{\alpha+1}f(z), \mathcal{D}^{\alpha+2}f(z); z\right)\right| < 1,$$

*then*

$$|\mathcal{D}^{\alpha}f(z)| < 1$$

*where* $\mathcal{D}^{\alpha}$ *is the Ruscheweyh derivative operator in* $U$.

PROOF. The proof follows from Corollary 3.4.9, by taking $\Omega = U$, $p = 1$, $l = 2$, $m = 1$, $\alpha_1 = \alpha + 1$, $\alpha_2 = 1$ $(\alpha > -1)$, $\beta_1 = 1$ and $M = 1$. □

By taking $p = 1$, $l = 2$, $m = 1$, $\alpha_1 = \alpha + 1$, $\alpha_2 = 1$ $(\alpha > -1)$, $\beta_1 = 1$, $M = 1$ and $\phi(u,v,w;z) = v$ in Corollary 3.4.11, we have the following result:

COROLLARY 3.4.13. [**41**, Corollary 1, p. 231] *If* $\alpha > -1$ *and* $f \in \mathcal{A}$ *satisfies*

$$\left|\mathcal{D}^{\alpha+1}f(z)\right| < 1,$$

*then*

$$|\mathcal{D}^{\alpha}f(z)| < 1$$

*where* $\mathcal{D}^{\alpha}$ *is the Ruscheweyh derivative operator in* $U$.

COROLLARY 3.4.14. *If* $\delta \geq 0$ *and* $f \in \mathcal{A}$ *satisfies*

$$\left|\delta\left(\frac{zf''(z)}{f'(z)}+1\right) + (1-\delta)\frac{zf'(z)}{f(z)}\right| < 1, \quad \textit{then} \quad |f(z)| < 1. \tag{3.4.22}$$

PROOF. Let $\phi(u,v,w;z) = \delta(2w/v-1)+(1-\delta)v/u$ for all real $\delta \geq 0$, $p=1$, $l=2$, $m=1$, $\alpha_1=1$, $\alpha_2=1$, $\beta_1=1$, $M=1$ and $\Omega = h(U)$ where $h(z)=z$. To use Corollary 3.4.9, it must be verify that $\phi \in \Phi_{H,1}[\Omega, M] \equiv \Phi_{H,1}[U]$, or equivalently, that the admissible condition (3.4.21) is satisfied. This follows since

$$\begin{aligned}
&\left|\phi\left(Me^{i\theta}, \frac{k+\alpha_1-1}{\alpha_1}Me^{i\theta}, \frac{L+\alpha_1(2k+\alpha_1-1)Me^{i\theta}}{\alpha_1(\alpha_1+1)};z\right)\right| \\
&= \left|\delta\left(\frac{Le^{-i\theta}}{k}+1\right)+(1-\delta)k\right| \geq \delta+(1-\delta)k+\frac{\delta}{k}\Re(Le^{-i\theta}) \\
&\geq \delta+(1-\delta)k+\frac{\delta}{k}k(k-1) = k \geq 1,
\end{aligned}$$

$z\in U$, $\theta \in \mathcal{R}$, $\Re(Le^{-i\theta}) \geq k(k-1)$ and $k \geq 1$. Hence by Corollary 3.4.9, the required result is obtained. $\square$

DEFINITION 3.4.5. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{H,2}[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition $\phi(u,v,w;z) \notin \Omega$ whenever

$$u = q(\zeta),\ v = \frac{1}{\alpha_1+1}\left(1+\alpha_1 q(\zeta)+\frac{k\zeta q'(\zeta)}{q(\zeta)}\right) \quad (\alpha_1 \in \mathcal{C},\ \alpha_1 \neq 0,-1,-2,\ q(\zeta)\neq 0),$$

$$\Re\left\{\frac{\begin{array}{c}[(\alpha_1+1)(w-v)+w-1] \\ \times(1+\alpha_1)v\end{array}}{(1+\alpha_1)v-(1+\alpha_1 u)}+(1+\alpha_1)v-(2\alpha_1 u+1)\right\} \geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

THEOREM 3.4.7. *Let $\phi \in \Phi_{H,2}[\Omega, q]$. If $f \in \mathcal{A}$ satisfies*

(3.4.23)
$$\left\{\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)};z\right) : z \in U\right\} \subset \Omega,$$

*then*

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec q(z).$$

PROOF. Define the function $p(z)$ in $U$ by

(3.4.24)
$$p(z) := \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}.$$

Then a computation yields

$$\frac{zp'(z)}{p(z)} := \frac{z[H_p^{l,m}[\alpha_1+1]f(z)]'}{H_p^{l,m}[\alpha_1+1]f(z)} - \frac{z[H_p^{l,m}[\alpha_1]f(z)]'}{H_p^{l,m}[\alpha_1]f(z)}. \tag{3.4.25}$$

Use of (3.4.3) in (3.4.25), shows that

$$\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)} = \frac{1}{\alpha_1+1}\left(\alpha_1 p(z) + 1 + \frac{zp'(z)}{p(z)}\right). \tag{3.4.26}$$

Further computations show that

$$\begin{aligned} &\frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)} \\ = &\frac{1}{\alpha_1+2}\left(2+\alpha_1 p(z) + \frac{zp'(z)}{p(z)} + \frac{\alpha_1 zp'(z) + \frac{zp'(z)}{p(z)} - (\frac{zp'(z)}{p(z)})^2 + \frac{z^2p''(z)}{p(z)}}{1+\alpha_1 p(z) + \frac{zp'(z)}{p(z)}}\right). \end{aligned} \tag{3.4.27}$$

Define the transformations from $\mathcal{C}^3$ to $\mathcal{C}$ by

(3.4.28)

$$u=r,\ v=\frac{1}{\alpha_1+1}\left(1+\alpha_1 r+\frac{s}{r}\right),\ w=\frac{1}{\alpha_1+2}\left(2+\alpha_1 r+\frac{s}{r}+\frac{\alpha_1 s+\frac{s}{r}-(\frac{s}{r})^2+\frac{t}{r}}{1+\alpha_1 r+\frac{s}{r}}\right).$$

Let

$$\begin{aligned} &\psi(r,s,t;z) := \phi(u,v,w;z) \\ &= \phi\left(r, \frac{1}{\alpha_1+1}\left[\alpha_1 r+1+\frac{s}{r}\right], \frac{1}{\alpha_1+2}\left(2+\alpha_1 r+\frac{s}{r}+\frac{\alpha_1 s+\frac{s}{r}-(\frac{s}{r})^2+\frac{t}{r}}{1+\alpha_1 r+\frac{s}{r}}\right);z\right). \end{aligned} \tag{3.4.29}$$

The proof shall make use of Theorem 3.1.1. Using equations (3.4.24), (3.4.26) and (3.4.27), from (3.4.29), it follows that

$$\begin{aligned} &\psi(p(z), zp'(z), z^2p''(z); z) \\ = &\ \phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right). \end{aligned} \tag{3.4.30}$$

Hence (3.4.23) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{H,2}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.1. Note that

$$\frac{t}{s}+1 = \frac{[(\alpha_1+1)(w-v)+w-1](1+\alpha_1)v}{(1+\alpha_1)v-(1+\alpha_1 u)} + (1+\alpha_1)v - (2\alpha_1 u+1),$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 3.1.1, $p(z) \prec q(z)$ or

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec q(z). \qquad \square$$

For a simply connected domain $\Omega \neq \mathcal{C}$, let $h : U \to \mathcal{C}$ be the conformal mapping of $U$ onto $\Omega$. In this case, the class $\Phi_{H,2}[h(U), q]$ is written as $\Phi_{H,2}[h, q]$. In the particular case $q(z) = 1 + Mz,\ M > 0$, the class of admissible functions $\Phi_{H,2}[\Omega, q]$ is denoted by $\Phi_{H,2}[\Omega, M]$. The following result is an immediate consequence of Theorem 3.4.7.

THEOREM 3.4.8. *Let $\phi \in \Phi_{H,2}[h, q]$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right) \prec h(z), \tag{3.4.31}$$

*then*

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec q(z).$$

DEFINITION 3.4.6. Let $\Omega$ be a set in $\mathcal{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,2}[\Omega, M]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ such that

$$\begin{aligned}\phi\Big(1 + Me^{i\theta}, 1 + \frac{k + \alpha_1(1+Me^{i\theta})}{(1+\alpha_1)(1+Me^{i\theta})}Me^{i\theta},\ 1 + \frac{\alpha_1(1+Me^{i\theta}) + k}{(\alpha_1+2)(1+Me^{i\theta})}Me^{i\theta} \\ + \frac{(M+e^{-i\theta})[Le^{-i\theta} + kM(\alpha_1+1) + \alpha_1 kM^2 e^{i\theta}] - k^2M^2}{(\alpha_1+2)(M+e^{-i\theta})[\alpha_1 M^2 e^{i\theta} + (1+\alpha_1)e^{-i\theta} + M(1+2\alpha_1+k)]}; z\Big) \notin \Omega\end{aligned} \tag{3.4.32}$$

whenever $z \in U,\ \theta \in \mathcal{R}$, $\Re(Le^{-i\theta}) \geq (k-1)kM$ for all real $\theta$, $\alpha_1 \in \mathcal{C}$ $(\alpha_1 \neq 0, -1, -2)$ and $k \geq 1$.

COROLLARY 3.4.15. *Let $\phi \in \Phi_{H,2}[\Omega, M]$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right) \in \Omega,$$

*then*

$$\left|\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} - 1\right| < M.$$

COROLLARY 3.4.16. *Let $\phi \in \Phi_{H,2}[M] := \Phi_{H,2}[\Omega, M]$ where $\Omega = \{\omega : |\omega - 1| < M\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right) - 1\right| < M,$$

*then*

$$\left|\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}-1\right|<M.$$

Corollary 3.4.17. *If $M>0$, $\alpha_1\in\mathcal{C}$ $(\alpha_1\neq 0,-1)$ and $f\in\mathcal{A}_p$ satisfies*

$$\left|\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}-\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\right|<\frac{M^2}{|1+\alpha_1|(1+M)}, \tag{3.4.33}$$

*then*

$$\left|\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}-1\right|<M.$$

Proof. This follows from Corollary 3.4.15 by taking $\phi(u,v,w;z)=v-u$ and $\Omega=h(U)$ where

$$h(z)=\frac{M^2z}{|1+\alpha_1|(1+M)}.$$

Since

$$\begin{aligned}|\phi\left(u,v,w;z\right)| &= \left|-1-Me^{i\theta}+1+\frac{k+\alpha_1(1+Me^{i\theta})}{(1+\alpha_1)(1+Me^{i\theta}}Me^{i\theta}\right|\\ &= \frac{M}{|1+\alpha_1|}\left|\frac{k-1-Me^{i\theta}}{1+Me^{i\theta}}\right|\\ &\geq \frac{M}{|1+\alpha_1|}\left|\frac{k-1-M}{1+M}\right|\\ &\geq \frac{M}{|1+\alpha_1|}\left|\frac{1}{1+M}-1\right|\\ &= \frac{M^2}{|1+\alpha_1|(1+M)}.\end{aligned}$$

$z\in U$, $\theta\in\mathcal{R}$, $\alpha_1\in\mathcal{C}$ $(\alpha_1\neq 0,-1)$, $k\neq 1+M$ and $k\geq 1$, it follows that $\phi\in\Phi_{H,2}[\Omega,M]$. Corollary 3.4.15 now yeilds the required result. □

By taking $l=2,\ m=1,\ \alpha_1=\alpha_2=\beta_1=1$, (3.4.33) shows that for $f\in\mathcal{A}_p$, whenever

$$\frac{\frac{zf'(z)}{f(z)}\left[\frac{zf''(z)}{f'(z)}-2\frac{zf'(z)}{f(z)}+p\right]}{\frac{zf'(z)}{f(z)}-p+1}\prec -p+\frac{M^2}{1+M}z,\quad \text{then}\quad \frac{zf'(z)}{f(z)}\prec Mz+p.$$

Example 3.4.1. If $f\in\mathcal{A}$, then

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}+1-\frac{zf'(z)}{f(z)}\right)\right|<1\Rightarrow\left|\frac{zf'(z)}{f(z)}-1\right|<1.$$

This follows from Corollary 3.4.15 by taking $\phi(u,v,w;z) = u(2v-1-u)$, $p=1$, $l=2$, $m=1$, $\alpha_1 = 1,\ \alpha_2 = 1$, $\beta_1 = 1$, $M=1$ and $\Omega = h(U)$ where $h(z) = z$.

EXAMPLE 3.4.2. If $M > 0$ and $f \in \mathcal{A}$ satisfies

$$\left| \frac{\frac{zf''(z)}{f'(z)} + 1}{\frac{zf'(z)}{f(z)}} - 1 \right| < M,$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < M.$$

This follows from Corollary 3.4.16 by taking $\phi(u,v,w;z) = (2v-1)/u$, $p=1$, $l=2$, $m=1$, $\alpha_1 = 1,\ \alpha_2 = 1$ and $\beta_1 = 1$.

## 3.5. SUPERORDINATION OF THE DZIOK-SRIVASTAVA LINEAR OPERATOR

The dual problem of differential subordination, that is, differential superordination of the Dziok-Srivastava linear operator is investigated in this section. For this purpose, the class of admissible functions is given in the following definition.

DEFINITION 3.5.1. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{H}[0,p]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_H[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition $\phi(u,v,w;\zeta) \in \Omega$ whenever

$$u = q(z), \quad v = \frac{zq'(z) + m(\alpha_1 - p)q(z)}{m\alpha_1}, \quad (\alpha_1 \in \mathcal{C},\ \alpha_1 \neq 0, -1)$$

$$\Re\left\{ \frac{\alpha_1(\alpha_1+1)w + (p-\alpha_1)(\alpha_1 - p + 1)u}{\alpha_1 v + (p-\alpha_1)u} - [2(p-\alpha_1)+1] \right\} \leq \frac{1}{m} \Re\left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq p$.

THEOREM 3.5.1. *Let $\phi \in \Phi'_H[\Omega, q]$. If $f \in \mathcal{A}_p$, $H_p^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_0$ and*

$$\phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right)$$

*is univalent in $U$, then*

$$\Omega \subset \left\{ \phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right) : z \in U \right\} \tag{3.5.1}$$

*implies*

$$q(z) \prec H_p^{l,m}[\alpha_1]f(z).$$

PROOF. From (3.4.8) and (3.5.1), it follows that

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

In view of (3.4.6), the admissibility condition for $\phi \in \Phi'_H[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.2. Hence $\psi \in \Psi'_p[\Omega, q]$, and by Theorem 3.1.2, $q(z) \prec p(z)$ or

$$q(z) \prec H_p^{l,m}[\alpha_1]f(z). \qquad \square$$

THEOREM 3.5.2. *Let $h(z)$ be analytic in $U$ and $\phi \in \Phi'_H[h, q] := \Phi'_H[h(U), q]$ . If $f \in \mathcal{A}_p$, $H_p^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_0$ and*

$$\phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right)$$

*is univalent in $U$, then*

$$h(z) \prec \phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right) \tag{3.5.2}$$

*implies*

$$q(z) \prec H_p^{l,m}[\alpha_1]f(z).$$

Theorems 3.5.1 and 3.5.2 are used to obtain subordinants of differential superordination of the form (3.5.1) or (3.5.2). The following theorem proves the existence of the best subordinant of (3.5.2) for certain $\phi$.

THEOREM 3.5.3. *Let $h(z)$ be analytic in $U$ and $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$. Suppose that the differential equation*

$$\phi(q(z), zq'(z), z^2q''(z); z) = h(z)$$

*has a solution $q(z) \in \mathcal{Q}_0$. If $\phi \in \Phi'_H[h, q]$, $f \in \mathcal{A}_p$, $H_p^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_0$ and*

$$\phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right)$$

*is univalent in* $U$, *then*

$$h(z) \prec \phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right)$$

*implies*

$$q(z) \prec H_p^{l,m}[\alpha_1]f(z)$$

*and* $q(z)$ *is the best subordinant.*

PROOF. The result is similar to the proof of Theorem 3.4.4 and is therefore omitted. □

Combining Theorems 3.4.2 and 3.5.2, the following sandwich-type theorem is obtained.

COROLLARY 3.5.1. *Let* $h_1(z)$ *and* $q_1(z)$ *be analytic functions in* $U$, $h_2(z)$ *be univalent function in* $U$, $q_2(z) \in \mathcal{Q}_0$ *with* $q_1(0) = q_2(0) = 0$ *and* $\phi \in \Phi_H[h_2, q_2] \cap \Phi'_H[h_1, q_1]$. *If* $f \in \mathcal{A}_p$, $H_p^{l,m}[\alpha_1]f(z) \in \mathcal{H}[0,p] \cap \mathcal{Q}_0$ *and*

$$\phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right)$$

*is univalent in* $U$, *then*

$$h_1(z) \prec \phi\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z); z\right) \prec h_2(z)$$

*implies*

$$q_1(z) \prec H_p^{l,m}[\alpha_1]f(z) \prec q_2(z).$$

DEFINITION 3.5.2. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{H}_0$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition $\phi(u, v, w; \zeta) \in \Omega$ whenever

$$u = q(z), \quad v = \frac{zq'(z) + m(\alpha_1 - 1)q(z)}{m\alpha_1} \quad (\alpha_1 \in \mathcal{C},\ \alpha_1 \neq 0, -1),$$

$$\Re\left\{\frac{\alpha_1[(\alpha_1+1)w + (1-\alpha_1)u]}{\alpha_1 v + (1-\alpha_1)u} + (1 - 2\alpha_1)\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

A dual result of Theorem 3.4.5 for the differential superordination is given below.

THEOREM 3.5.4. *Let* $\phi \in \Phi'_{H,1}[\Omega, q]$. *If* $f \in \mathcal{A}_p$, $H_p^{l,m}[\alpha_1]f(z)/z^{p-1} \in \mathcal{Q}_0$ *and*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^{p-1}}; z\right)$$

*is univalent in* $U$, *then*

$$\Omega \subset \left\{\phi\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^{p-1}}; z\right) : z \in U\right\} \tag{3.5.3}$$

*implies*

$$q(z) \prec \frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}.$$

PROOF. From (3.4.19) and (3.5.3), it follows that

$$\Omega \subset \left\{\phi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

In view of (3.4.17), the admissibility condition for $\phi \in \Phi'_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 3.1.2, $q(z) \prec p(z)$ or

$$q(z) \prec H_p^{l,m}[\alpha_1]f(z). \qquad \square$$

THEOREM 3.5.5. *Let* $q(z) \in \mathcal{H}_0$, *h(z) is analytic on* $U$ *and* $\phi \in \Phi'_{H,1}[h, q] := \Phi'_{H,1}[h(U), q]$. *If* $f \in \mathcal{A}_p$, $H_p^{l,m}[\alpha_1]f(z)/z^{p-1} \in \mathcal{Q}_0$ *and*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^{p-1}}; z\right)$$

*is univalent in* $U$, *then*

$$h(z) \prec \phi\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^{p-1}}; z\right) \tag{3.5.4}$$

*implies*

$$q(z) \prec \frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}.$$

Combining Theorems 3.4.6 and 3.5.5, the following sandwich-type theorem is obtained.

Corollary 3.5.2. *Let* $h_1(z)$ *and* $q_1(z)$ *be analytic functions in* $U$, $h_2(z)$ *be univalent function in* $U$, $q_2(z) \in \mathcal{Q}_0$ *with* $q_1(0) = q_2(0) = 0$ *and* $\phi \in \Phi_{H,1}[h_2, q_2] \cap \Phi'_{H,1}[h_1, q_1]$. *If* $f \in \mathcal{A}_p$, $H_p^{l,m}[\alpha_1]f(z)/z^{p-1} \in \mathcal{H}_0 \cap \mathcal{Q}_0$ *and*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^{p-1}}; z\right)$$

*is univalent in* $U$, *then*

$$h_1(z) \prec \phi\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^{p-1}}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^{p-1}}; z\right) \prec h_2(z)$$

*implies*

$$q_1(z) \prec \frac{H_p^{l,m}[\alpha_1]f(z)}{z^{p-1}} \prec q_2(z).$$

Definition 3.5.3. Let $\Omega$ be a set in $\mathcal{C}$, $q(z) \neq 0$, $zq'(z) \neq 0$ and $q(z) \in \mathcal{H}$. The class of admissible functions $\Phi'_{H,2}[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition $\phi(u, v, w; \zeta) \in \Omega$ whenever

$$u = q(z),\ v = \frac{1}{\alpha_1 + 1}\left(1 + \alpha_1 q(z) + \frac{zq'(z)}{mq(z)}\right), \quad (\alpha_1 \in \mathcal{C},\ \alpha_1 \neq 0, -1, -2)$$

$$\Re\left\{\frac{\begin{array}{c}[(\alpha_1+1)(w-v)+w-1]\\ \times(1+\alpha_1)v\end{array}}{(1+\alpha_1)v-(1+\alpha_1 u)} + (1+\alpha_1)v - (2\alpha_1 u + 1)\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

Theorem 3.5.6. *Let* $\phi \in \Phi'_{H,2}[\Omega, q]$. *If* $f \in \mathcal{A}_p$, $\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \in \mathcal{Q}_1$ *and*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right)$$

*is univalent in* $U$, *then*

(3.5.5)
$$\Omega \subset \left\{\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right) : z \in U\right\}$$

*implies*

$$q(z) \prec \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}.$$

PROOF. From (3.4.30) and (3.5.5), it follows that

$$\Omega \subset \left\{\phi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

In view of (3.4.28), the admissibility condition for $\phi \in \Phi'_{H,2}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 3.1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}. \qquad \square$$

THEOREM 3.5.7. *Let $q(z) \in \mathcal{H}$, $h(z)$ be analytic in $U$ and $\phi \in \Phi'_{H,2}[h, q] := \Phi'_{H,2}[h(U), q]$. If $f \in \mathcal{A}_p$, $\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \in \mathcal{Q}_1$ and*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right)$$

*is univalent in $U$, then*

$$h(z) \prec \phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right) \tag{3.5.6}$$

*implies*

$$q(z) \prec \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}.$$

Combining Theorems 3.4.8 and 3.5.7, the following sandwich-type theorem is obtained.

COROLLARY 3.5.3. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $U$, $h_2(z)$ be univalent in $U$, $q_2(z) \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{H,2}[h_2, q_2] \cap \Phi'_{H,2}[h_1, q_1]$. If $f \in \mathcal{A}_p$, $\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \in \mathcal{H} \cap \mathcal{Q}_1$ and*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right)$$

*is univalent in $U$, then*

$$h_1(z) \prec \phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right) \prec h_2(z)$$

*implies*

$$q_1(z) \prec \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec q_2(z).$$

## 3.6. SUBORDINATION RESULTS INVOLVING THE MULTIPLIER TRANSFORMATION

Differential subordination of the multiplier transformation is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

Definition 3.6.1. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{Q}_0 \cap \mathcal{H}[0,p]$. The class of admissible functions $\Phi_I[\Omega,q]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition $\phi(u,v,w;z) \notin \Omega$ whenever

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + \lambda q(\zeta)}{\lambda + p},$$

$$\Re\left\{\frac{(\lambda+p)^2 w - \lambda^2 u}{(\lambda+p)v - \lambda u} - 2\lambda\right\} \geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq p$.

Theorem 3.6.1. *Let $\phi \in \Phi_I[\Omega,q]$. If $f \in \mathcal{A}_p$ satisfies*

$$\{\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right) : z \in U\} \subset \Omega, \tag{3.6.1}$$

*then*

$$I_p(n,\lambda)f(z) \prec q(z).$$

Proof. Define the analytic function $p(z)$ in $U$ by

$$p(z) := I_p(n,\lambda)f(z). \tag{3.6.2}$$

In view of the relation

$$(p+\lambda)I_p(n+1,\lambda)f(z) = z[I_p(n,\lambda)f(z)]' + \lambda I_p(n,\lambda)f(z), \tag{3.6.3}$$

and a simple computation from (3.6.2), yields

$$I_p(n+1,\lambda)f(z) = \frac{zp'(z) + \lambda p(z)}{\lambda + p}. \tag{3.6.4}$$

Further computations yield

$$I_p(n+2,\lambda)f(z) = \frac{z^2p''(z) + (2\lambda+1)zp'(z) + \lambda^2 p(z)}{(\lambda+p)^2}. \tag{3.6.5}$$

Define the transformations from $\mathcal{C}^3$ to $\mathcal{C}$ by

(3.6.6) $$u = r,\ v = \frac{s+\lambda r}{\lambda+p},\ w = \frac{t+(2\lambda+1)s+\lambda^2 r}{(\lambda+p)^2}.$$

Let

(3.6.7) $$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{s+\lambda r}{\lambda+p}, \frac{t+(2\lambda+1)s+\lambda^2 r}{(\lambda+p)^2}; z\right).$$

The proof shall make use of Theorem 3.1.1. Using equations (3.6.2), (3.6.4) and (3.6.5), from (3.6.7), it follows that

(3.6.8)

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right).$$

Hence (3.6.1) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_I[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.1. Note that

$$\frac{t}{s} + 1 = \frac{(\lambda+p)^2 w - \lambda^2 u}{(\lambda+p)v - \lambda u} - 2\lambda,$$

and hence $\psi \in \Psi_p[\Omega, q]$. By Theorem 3.1.1, $p(z) \prec q(z)$ or

$$I_p(n,\lambda)f(z) \prec q(z). \qquad \square$$

THEOREM 3.6.2. *Let $\phi \in \Phi_I[h,q] := \Phi_I[h(U), q]$. If $f \in \mathcal{A}_p$ satisfies*

(3.6.9) $$\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right) \prec h(z),$$

*then*

$$I_p(n,\lambda)f(z) \prec q(z).$$

The following result is an extension of Theorem 3.6.2 to the case where the behavior of $q(z)$ on $\partial U$ is not known.

COROLLARY 3.6.1. *Let $\Omega \subset \mathcal{C}$ and let $q(z)$ be univalent in $U$, $q(0) = 0$. Let $\phi \in \Phi_I[\Omega, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}_p$ and*

$$\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right) \in \Omega,$$

*then*

$$I_p(n,\lambda)f(z) \prec q(z).$$

PROOF. Theorem 3.6.1 yields $I_p(n,\lambda)f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$. □

THEOREM 3.6.3. *Let $h(z)$ and $q(z)$ be univalent in $U$, with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ satisfies one of the following conditions:*

(1) *$\phi \in \Phi_I[h, q_\rho]$ for some $\rho \in (0,1)$, or*
(2) *there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_I[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

*If $f \in \mathcal{A}_p$ satisfies (3.6.9), then*

$$I_p(n,\lambda)f(z) \prec q(z).$$

PROOF. The proof is similar to the proof of Theorem 3.4.3 and is therefore omitted. □

The next theorem yields the best dominant of the differential subordination (3.6.9).

THEOREM 3.6.4. *Let $h(z)$ be univalent in $U$. Let $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$. Suppose that the differential equation*

$$\phi\left(q(z), zq'(z), z^2q''(z); z\right) = h(z) \tag{3.6.10}$$

*has a solution $q(z)$ with $q(0) = 0$ and satisfies one of the following conditions:*

(1) *$q(z) \in \mathcal{Q}_0$ and $\phi \in \Phi_I[h, q]$,*
(2) *$q(z)$ is univalent in $U$ and $\phi \in \Phi_H[h, q_\rho]$, for some $\rho \in (0,1)$, or*
(3) *$q(z)$ is univalent in $U$ and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_I[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.*

*If $f \in \mathcal{A}_p$ satisfies (3.6.9), then*

$$I_p(n,\lambda)f(z) \prec q(z),$$

*and $q(z)$ is the best dominant.*

Proof. Following the same arguments as in the proof of [**58**, Theorem 2.3e, p. 31], from Theorems 3.6.2 and 3.6.3, it follows that $q(z)$ is a dominant. Since $q(z)$ satisfies (3.6.10), it is also a solution of (3.6.9) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant. $\square$

Definition 3.6.2. Let $\Omega$ be a set in $\mathcal{C}$ and let $\Phi_I[\Omega, M] := \Phi_I[\Omega, q]$ where $q(z) = Mz,\ M > 0$. The class of admissible functions $\Phi_I[\Omega, M]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{k+\lambda}{\lambda+p}Me^{i\theta}, \frac{L+((2\lambda+1)k+\lambda^2)Me^{i\theta}}{(\lambda+p)^2}; z\right) \notin \Omega \tag{3.6.11}$$

whenever $z \in U,\ \theta \in \mathcal{R}$, $\Re(Le^{-i\theta}) \geq (k-1)kM$ for all real $\theta$ and $k \geq p$.

Corollary 3.6.2. *Let $\phi \in \Phi_I[\Omega, M]$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right) \in \Omega,$$

*then*

$$|I_p(n,\lambda)f(z)| < M.$$

Corollary 3.6.3. *Let $\phi \in \Phi_I[M] := \Phi_I[\Omega, M]$ where $\Omega = \{\omega : |\omega| < M\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right)\right| < M,$$

*then*

$$|I_p(n,\lambda)f(z)| < M.$$

When $\Omega = U$ and $M = 1$, Corollary 3.6.2 reduces to the following result:

Corollary 3.6.4. [**2**, Theorem 2, p. 271] *Let $\phi \in \Phi_I[U] := \Phi_I[U, 1]$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right)\right| < 1,$$

*then*

$$|I_p(n,\lambda)f(z)| < 1.$$

When $\Omega = U$, $\lambda = a - 1$ $(a > 0)$, $p = 1$ and $M = 1$, Corollary 3.6.2 reduces to the following result:

Corollary 3.6.5. [**41**, Theorem 2, p. 231] *Let* $\phi \in \Phi_I[U]$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\phi\left(\mathcal{I}_a^{\lambda} f(z), \mathcal{I}_a^{\lambda-1} f(z), \mathcal{I}_a^{\lambda-2} f(z); z\right)\right| < 1,$$

*then*

$$|\mathcal{I}_a^{\lambda} f(z)| < 1$$

*where* $\mathcal{I}_a^{\lambda}$ *is the Komatu integral operator in* $U$.

When $\Omega = U$, $\lambda = 1$, $p = 1$ and $M = 1$, Corollary 3.6.2 reduces to the following result:

Corollary 3.6.6. [**7**, Theorem 1, p. 477] *Let* $\phi \in \Phi_I[U]$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\phi\left(P^{\alpha} f(z), P^{\alpha-1} f(z), P^{\alpha-2} f(z); z\right)\right| < 1,$$

*then*

$$|P^{\alpha} f(z)| < 1.$$

Corollary 3.6.7. *If* $M > 0$ *and* $f \in \mathcal{A}_p$ *satisfies*

$$\begin{aligned} &\left|(\lambda+p)^2 I_p(n+2,\lambda) f(z) - (\lambda+p) I_p(n+1-\lambda) f(z) - \lambda^2 I_p(n,\lambda) f(z)\right| \\ &\qquad < \left[(2p-1)\lambda + p(p-1)\right] Mz, \quad \text{then} \quad |I_p(n,\lambda) f(z)| < M. \end{aligned} \tag{3.6.12}$$

Proof. This follows from Corollary 3.6.2 by taking $\phi(u, v, w; z) = (\lambda+p)^2 w - (\lambda+p)v - \lambda^2 u$ and $\Omega = h(U)$ where $h(z) = \lambda M z$, $M > 0$. Since

$$\begin{aligned} &\left|\phi\left(Me^{i\theta}, \frac{k+\lambda}{\lambda+p} Me^{i\theta}, \frac{L + ((2\lambda+1)k+\lambda^2)Me^{i\theta}}{(\lambda+p)^2}; z\right)\right| \\ &= \left|L + ((2\lambda+1)k+\lambda^2)Me^{i\theta} - (k+\lambda)Me^{i\theta} - \lambda^2 Me^{i\theta}\right| \\ &= \left|L + (2k-1)\lambda Me^{i\theta}\right| \\ &\geq (2k-1)\lambda M + \Re(Le^{-i\theta}) \\ &\geq (2k-1)\lambda M + k(k-1)M \geq \left[(2p-1)\lambda + p(p-1)\right] M \end{aligned}$$

$z \in U,\ \theta \in \mathcal{R}$, $\Re(Le^{-i\theta}) \geq k(k-1)M$ and $k \geq p$, it follows that $\phi \in \Phi_I[\Omega, M]$. Hence by Corollary 3.6.2, the required result is obtained. $\square$

The differential equation

$$(\lambda+p)^2 z^2 q''(z) - (\lambda+p) z q'(z) - \lambda^2 q(z) = [(2p-1)\lambda + p(p-1)]\, Mz$$

has a univalent solution $q(z) = Mz$. Theorem 3.6.4 shows that $q(z) = Mz$ is the best dominant of (3.6.12).

Definition 3.6.3. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{Q}_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_{I,1}[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition $\phi(u, v, w; z) \notin \Omega$ whenever

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + (\lambda+p-1)q(\zeta)}{\lambda+p},$$

$$\Re\left\{\frac{(\lambda+p)^2 w - (\lambda+p-1)^2 u}{(\lambda+p)v - (\lambda+p-1)u} - 2(\lambda+p-1)\right\} \geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

Theorem 3.6.5. *Let $\phi \in \Phi_{I,1}[\Omega, q]$. If $f \in \mathcal{A}_p$ satisfies*

$$\left\{\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right) : z \in U\right\} \subset \Omega, \tag{3.6.13}$$

*then*

$$\frac{I_p(n,\lambda)f(z)}{z^{p-1}} \prec q(z).$$

Proof. Define the function $p(z)$ in $U$ by

$$p(z) := \frac{I_p(n,\lambda)f(z)}{z^{p-1}}. \tag{3.6.14}$$

By making use of (3.6.3), a simple computation from (3.6.14) yields

$$\frac{I_p(n+1,\lambda)f(z)}{z^{p-1}} = \frac{zp'(z) + (\lambda+p-1)p(z)}{\lambda+p}. \tag{3.6.15}$$

Further computations yield

$$\frac{I_p(n+2,\lambda)f(z)}{z^{p-1}} = \frac{z^2 p''(z) + [2(\lambda+p)-1]zp'(z) + (\lambda+p-1)^2 p(z)}{(\lambda+p)^2}. \tag{3.6.16}$$

Define the transformations from $\mathcal{C}^3$ to $\mathcal{C}$ by

(3.6.17) $$u = r,\ v = \frac{s + (\lambda + p - 1)r}{\lambda + p},\ w = \frac{t + [2(\lambda + p) - 1]s + (\lambda + p - 1)^2 r}{(\lambda + p)^2}.$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z)$$

(3.6.18) $$= \phi\left(r, \frac{s + (\lambda + p - 1)r}{\lambda + p}, \frac{t + [2(\lambda + p) - 1]s + (\lambda + p - 1)^2 r}{(\lambda + p)^2}; z\right).$$

The proof shall make use of Theorem 3.1.1. Using equations (3.6.14), (3.6.15) and (3.6.16), from (3.6.18), it follows that

(3.6.19)
$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(\frac{I_p(n, \lambda)f(z)}{z^{p-1}}, \frac{I_p(n + 1, \lambda)f(z)}{z^{p-1}}, \frac{I_p(n + 2, \lambda)f(z)}{z^{p-1}}; z\right).$$

Hence (3.6.13) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{I,1}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.1. Note that

$$\frac{t}{s} + 1 = \frac{(\lambda + p)^2 w - (\lambda + p - 1)^2 u}{(\lambda + p)v - (\lambda + p - 1)u} - 2(\lambda + p - 1),$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 3.1.1, $p(z) \prec q(z)$ or

$$I_p(n, \lambda)f(z) \prec q(z). \qquad \square$$

THEOREM 3.6.6. *Let* $\phi \in \Phi_{I,1}[h, q] := \Phi_{I,1}[h(U), q]$. *If* $f \in \mathcal{A}_p$ *satisfies*

(3.6.20) $$\phi\left(\frac{I_p(n, \lambda)f(z)}{z^{p-1}}, \frac{I_p(n + 1, \lambda)f(z)}{z^{p-1}}, \frac{I_p(n + 2, \lambda)f(z)}{z^{p-1}}; z\right) \prec h(z),$$

*then*

$$\frac{I_p(n, \lambda)f(z)}{z^{p-1}} \prec q(z).$$

DEFINITION 3.6.4. Let $\Omega$ be a set in $\mathcal{C}$ and let $\Phi_{I,1}[\Omega, M] := \Phi_{I,1}[\Omega, q]$ where $q(z) = Mz,\ M > 0$. The class of admissible functions $\Phi_{I,1}[\Omega, M]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ such that

(3.6.21)
$$\phi\left(Me^{i\theta}, \frac{k + \lambda + p - 1}{\lambda + p}Me^{i\theta}, \frac{L + [(2(\lambda + p) - 1)k + (\lambda + p - 1)^2]Me^{i\theta}}{(\lambda + p)^2}; z\right) \notin \Omega$$

whenever $z \in U,\ \theta \in \mathcal{R}$, $\Re(Le^{-i\theta}) \geq (k-1)kM$ for all real $\theta$ and $k \geq 1$.

Corollary 3.6.8. *Let* $\phi \in \Phi_{I,1}[\Omega, M]$. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right) \in \Omega,$$

*then*

$$\left|\frac{I_p(n,\lambda)f(z)}{z^{p-1}}\right| < M.$$

Corollary 3.6.9. *Let* $\phi \in \Phi_{I,1}[M] := \Phi_{I,1}[\Omega, M]$ *where* $\Omega = \{\omega : |\omega| < M\}$. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\left|\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right)\right| < M,$$

*then*

$$\left|\frac{I_p(n,\lambda)f(z)}{z^{p-1}}\right| < M.$$

When $\Omega = U$, $\lambda = a-1$ $(a > 0)$, $p = 1$ and $M = 1$, Corollary 3.6.8 reduces to the following result:

Corollary 3.6.10. [**41**, Theorem 2, p. 231] *Let* $\phi \in \Phi_{I,1}[U]$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\phi\left(\mathcal{I}_a^{\lambda} f(z), \mathcal{I}_a^{\lambda-1} f(z), \mathcal{I}_a^{\lambda-2} f(z); z\right)\right| < 1,$$

*then*

$$|\mathcal{I}_a^{\lambda} f(z)| < 1$$

*where* $\mathcal{I}_a^{\lambda}$ *is the Komatu integral operator in* $U$.

When $\Omega = U$, $\lambda = 1$, $p = 1$ and $M = 1$, Corollary 3.6.8 reduces to the following result:

Corollary 3.6.11. [**7**, Theorem 1, p. 477] *Let* $\phi \in \Phi_{I,1}[U]$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\phi\left(P^{\alpha} f(z), P^{\alpha-1} f(z), P^{\alpha-2} f(z); z\right)\right| < 1,$$

*then*

$$|P^{\alpha} f(z)| < 1.$$

Corollary 3.6.12. *If $f \in \mathcal{A}_p$, then,*

$$\left|\frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}\right| < M \Rightarrow \left|\frac{I_p(n,\lambda)f(z)}{z^{p-1}}\right| < M.$$

This follows from Corollary 3.6.9 by taking $\phi(u,v,w;z) = v$.

Corollary 3.6.13. *If $M > 0$ and $f \in \mathcal{A}_p$ satisfies*

$$\left|(\lambda+p)^2\frac{I_p(n+2,\lambda)f(z)}{z^{p-1}} + (\lambda+p)\frac{I_p(n+1,\lambda)f(z)}{z^{p-1}} - (\lambda+p-1)^2\frac{I_p(n,\lambda)f(z)}{z^{p-1}}\right|$$
$$< [3(\lambda+p)-1]M,$$

*then*

$$\left|\frac{I_p(n,\lambda)f(z)}{z^{p-1}}\right| < M. \tag{3.6.22}$$

Proof. This follows from Corollary 3.6.8 by taking $\phi(u,v,w;z) = (\lambda+p)^2 w + (\lambda+p)v - (\lambda+p-1)^2 u$ and $\Omega = h(U)$ where $h(z) = (3(\lambda+p)-1)Mz,\ M > 0$. To use Corollary 3.6.8, it must verify that $\phi \in \Phi_{I,1}[\Omega, M]$, that is, the admissible condition (3.6.21) is satisfied. This follows since

$$\left|\phi\left(Me^{i\theta}, \frac{k+\lambda+p-1}{\lambda+p}Me^{i\theta}, \frac{L+[(2(\lambda+p)-1)k+(\lambda+p-1)^2]Me^{i\theta}}{(\lambda+p)^2}; z\right)\right|$$
$$= \left|L + (2(\lambda+p)-1)kMe^{i\theta} + (k+\lambda+p-1)Me^{i\theta}\right|$$
$$= \left|L + [(2k+1)(\lambda+p)-1]Me^{i\theta}\right|$$
$$\geq [(2k+1)(\lambda+p)-1]M + \Re(Le^{-i\theta})$$
$$\geq [(2k+1)(\lambda+p)-1]M + k(k-1)M$$
$$\geq (3(\lambda+p)-1)M$$

$z \in U,\ \theta \in \mathcal{R}$, $\Re(Le^{-i\theta}) \geq k(k-1)M$ and $k \geq 1$. Hence by Corollary 3.6.8, the required result is obtained. $\square$

Definition 3.6.5. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{I,2}[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition $\phi(u,v,w;z) \notin \Omega$ whenever

$$u = q(\zeta),\ v = \frac{1}{\lambda+p}\left((\lambda+p)q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)}\right) \qquad (q(\zeta) \neq 0),$$

$$\Re\left\{\frac{(\lambda+p)v(w-v)}{v-u}-(\lambda+p)(2u-v)\right\}\geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

$z\in U$, $\zeta\in\partial U\setminus E(q)$ and $k\geq 1$.

THEOREM 3.6.7. *Let $\phi\in\Phi_{I,2}[\Omega,q]$. If $f\in\mathcal{A}_p$ satisfies*

(3.6.23)
$$\left\{\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)},\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)},\frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)};z\right):z\in U\right\}\subset\Omega,$$

*then*

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\prec q(z).$$

PROOF. Define the function $p(z)$ in $U$ by

$$p(z):=\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}. \tag{3.6.24}$$

By making use of (3.6.3) and (3.6.24), a simple computation yields

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}=\frac{1}{\lambda+p}\left[(\lambda+p)p(z)+\frac{zp'(z)}{p(z)}\right]. \tag{3.6.25}$$

Further computations yield

(3.6.26)
$$\frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}=p(z)+\frac{1}{\lambda+p}\left[\frac{zp'(z)}{p(z)}+\frac{(\lambda+p)zp'(z)+\frac{zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+\frac{z^2p''(z)}{p(z)}}{(\lambda+p)p(z)+\frac{zp'(z)}{p(z)}}\right].$$

Define the transformations from $\mathcal{C}^3$ to $\mathcal{C}$ by

$$u=r,\ v=r+\frac{1}{\lambda+p}\left(\frac{s}{r}\right),\ w=r+\frac{1}{\lambda+p}\left[\frac{s}{r}+\frac{(\lambda+p)s+\frac{s}{r}-(\frac{s}{r})^2+\frac{t}{r}}{(\lambda+p)r+\frac{s}{r}}\right]. \tag{3.6.27}$$

Let

$$\psi(r,s,t;z)=\phi(u,v,w;z) \tag{3.6.28}$$
$$=\phi\left(r,\frac{1}{\lambda+p}\left[(\lambda+p)r+\frac{s}{r}\right],\frac{1}{\lambda+p}\left[(\lambda+p)r+\frac{s}{r}+\frac{(\lambda+p)s+\frac{s}{r}-(\frac{s}{r})^2+\frac{t}{r}}{(\lambda+p)r+\frac{s}{r}}\right];z\right).$$

The proof shall make use of Theorem 3.1.1. Using equations (3.6.24), (3.6.25) and (3.6.26), from (3.6.28), it follows that

$$\psi(p(z),zp'(z),z^2p''(z);z)$$
$$=\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)},\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)},\frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)};z\right). \tag{3.6.29}$$

Hence (3.6.23) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{I,2}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.1. Note that

$$\frac{t}{s} + 1 = \frac{(\lambda + p)v(w - v)}{v - u} - (\lambda + p)(2u - v),$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 3.1.1, $p(z) \prec q(z)$ or

$$\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \prec q(z). \qquad \square$$

THEOREM 3.6.8. *Let* $\phi \in \Phi_{I,2}[h, q] := \Phi_{I,2}[h(U), q]$. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\phi\left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)}, \frac{I_p(n+2, \lambda)f(z)}{I_p(n+1, \lambda)f(z)}, \frac{I_p(n+3, \lambda)f(z)}{I_p(n+2, \lambda)f(z)}; z\right) \prec h(z), \tag{3.6.30}$$

*then*

$$\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \prec q(z).$$

DEFINITION 3.6.6. Let $\Omega$ be a set in $\mathcal{C}$ and let $\Phi_{I,2}[\Omega, M] := \Phi_{I,2}[\Omega, q]$ where $q(z) = 1 + Mz,\ M > 0$. The class of admissible functions $\Phi_{I,2}[\Omega, M]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ such that

$$\begin{aligned} &\phi\Bigg(1 + Me^{i\theta},\ 1 + \frac{k + (\lambda+p)(1 + Me^{i\theta})}{(\lambda+p)(1 + Me^{i\theta})}Me^{i\theta}, \\ &1 + \frac{k + (\lambda+p)(1 + Me^{i\theta})}{(\lambda+p)(1 + Me^{i\theta})}Me^{i\theta} + \frac{\begin{array}{c}(M + e^{-i\theta})[Le^{-i\theta} + [\lambda + p + 1]kM \\ +(\lambda+p)kM^2e^{i\theta}] - k^2M^2\end{array}}{\begin{array}{c}(\lambda+p)(M + e^{-i\theta})[(\lambda+p)e^{-i\theta} \\ +(2(\lambda+p)+k)M + (\lambda+p)M^2e^{i\theta}]\end{array}}; z\Bigg) \notin \Omega \end{aligned} \tag{3.6.31}$$

$z \in U,\ \theta \in \mathcal{R}$, $\Re(Le^{-i\theta}) \geq (k-1)kM$ for all real $\theta$ and $k \geq 1$.

COROLLARY 3.6.14. *Let* $\phi \in \Phi_{I,2}[\Omega, M]$. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\phi\left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)}, \frac{I_p(n+2, \lambda)f(z)}{I_p(n+1, \lambda)f(z)}, \frac{I_p(n+3, \lambda)f(z)}{I_p(n+2, \lambda)f(z)}; z\right) \in \Omega,$$

*then*

$$\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \prec 1 + Mz.$$

COROLLARY 3.6.15. *Let* $\phi \in \Phi_{I,2}[M] := \Phi_{I,2}[\Omega, M]$ *where* $\Omega = \{\omega : |\ \omega - 1| < M\}$. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\left|\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) - 1\right| < M,$$

*then*

$$\left|\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} - 1\right| < M.$$

COROLLARY 3.6.16. *If* $M > 0$ *and* $f \in \mathcal{A}_p$ *satisfies*

$$\left|\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} - \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right| < \frac{M}{(\lambda+p)(1+M)},$$

*then*

$$\left|\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} - 1\right| < M.$$

This follows from Corollary 3.6.14 by taking $\phi(u, v, w; z) = v - u$ and $\Omega = h(U)$ where $h(z) = Mz/(\lambda + p)(1 + M)$.

## 3.7. SUPERORDINATION OF THE MULTIPLIER TRANSFORMATION

The dual problem of differential subordination, that is, the differential superordination of the multiplier transformation is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

DEFINITION 3.7.1. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{H}[0, p]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_I[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition $\phi(u, v, w; \zeta) \in \Omega$ whenever

$$u = q(z), \quad v = \frac{(zq'(z)/m) + \lambda q(z)}{\lambda + p},$$

$$\Re\left\{\frac{(\lambda+p)^2 w - \lambda^2 u}{(\lambda+p)v - \lambda u} - 2\lambda\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq p$

THEOREM 3.7.1. *Let* $\phi \in \Phi'_I[\Omega, q]$. *If* $f \in \mathcal{A}_p$, $I_p(n, \lambda)f(z) \in \mathcal{Q}_0$ *and*

$$\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right)$$

*is univalent in $U$, then*

(3.7.1) $$\Omega \subset \{\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right) : z \in U\}$$

*implies*

$$q(z) \prec I_p(n,\lambda)f(z).$$

PROOF. From (3.6.8) and (3.7.1) , it follows that

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

In view of (3.6.6), the admissibility condition for $\phi \in \Phi'_I[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.2. Hence $\psi \in \Psi'_p[\Omega, q]$, and by Theorem 3.1.2, $q(z) \prec p(z)$ or

$$q(z) \prec I_p(n,\lambda)f(z). \qquad \square$$

THEOREM 3.7.2. *Let $q(z) \in \mathcal{H}[0,p]$, h(z) is analytic on $U$ and $\phi \in \Phi'_I[h,q] := \Phi'_I[h(U),q]$ . If $f \in \mathcal{A}_p$, $I_p(n,\lambda)f(z) \in \mathcal{Q}_0$ and*

$$\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right)$$

*is univalent in $U$, then*

(3.7.2) $$h(z) \prec \phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right)$$

*implies*

$$q(z) \prec I_p(n,\lambda)f(z).$$

Theorems 3.7.1 and 3.7.2 are used to obtain subordinants of differential superordination of the form (3.7.1) or (3.7.2). The following theorem proves the existence of the best subordinant of (3.7.2) for certain $\phi$.

THEOREM 3.7.3. *Let $h(z)$ be analytic in $U$ and $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$. Suppose that the differential equation*

(3.7.3) $$\phi(q(z), zq'(z), z^2q''(z); z) = h(z)$$

*has a solution $q(z) \in \mathcal{Q}_0$. If $\phi \in \Phi'_I[h,q]$, $f \in \mathcal{A}_p$, $I_p(n,\lambda)f(z) \in \mathcal{Q}_0$ and*

$$\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right)$$

*is univalent in $U$, then*

$$h(z) \prec \phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right)$$

*implies*

$$q(z) \prec I_p(n,\lambda)f(z),$$

*and $q(z)$ is the best subordinant.*

Proof. The proof is similar to the proof of Theorem 3.6.4 and is therefore omitted. □

Combining Theorems 3.6.2 and 3.7.2, the following sandwich-type theorem is obtained.

Corollary 3.7.1. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $U$, $h_2(z)$ be univalent function in $U$, $q_2(z) \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_I[h_2, q_2] \cap \Phi_I'[h_1, q_1]$. If $f \in \mathcal{A}_p$, $I_p(n,\lambda)f(z) \in \mathcal{H}[0,p] \cap \mathcal{Q}_0$ and*

$$\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right)$$

*is univalent in $U$, then*

$$h_1(z) \prec \phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z); z\right) \prec h_2(z),$$

*implies*

$$q_1(z) \prec I_p(n,\lambda)f(z) \prec q_2(z).$$

Definition 3.7.2. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{H}_0$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi_{I,1}'[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition $\phi(u,v,w;\zeta) \in \Omega$ whenever

$$u = q(z), \quad v = \frac{(zq'(z)/m) + (\lambda+p-1)q(z)}{\lambda+p},$$

$$\Re\left\{\frac{(\lambda+p)^2 w - (\lambda+p-1)^2 u}{(\lambda+p)v - (\lambda+p-1)u} - 2(\lambda+p-1)\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

THEOREM 3.7.4. *Let* $\phi \in \Phi'_{I,1}[\Omega, q]$. *If* $f \in \mathcal{A}_p$, $I_p(n,\lambda)f(z)/z^{p-1} \in \mathcal{Q}_0$ *and*

$$\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right)$$

*is univalent in* $U$, *then*

$$\Omega \subset \left\{\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right) : z \in U\right\} \tag{3.7.4}$$

*implies*

$$q(z) \prec \frac{I_p(n,\lambda)f(z)}{z^{p-1}}.$$

PROOF. From (3.6.19) and (3.7.4), it follows that

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

In view of (3.6.17), the admissibility condition for $\phi \in \Phi'_{I,1}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 3.1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{I_p(n,\lambda)f(z)}{z^{p-1}}. \qquad \square$$

THEOREM 3.7.5. *Let* $q(z) \in \mathcal{H}_0$, *h(z) is analytic on* $U$ *and* $\phi \in \Phi'_{I,1}[h, q] := \Phi'_{I,1}[h(U), q]$. *If* $f \in \mathcal{A}_p$, $I_p(n,\lambda)f(z)/z^{n-1} \in \mathcal{Q}_0$ *and*

$$\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right)$$

*is univalent in* $U$, *then*

$$h(z) \prec \phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right) \tag{3.7.5}$$

*implies*

$$q(z) \prec \frac{I_p(n,\lambda)f(z)}{z^{p-1}}.$$

Combining Theorems 3.6.6 and 3.7.5, the following sandwich-type theorem is obtained.

Corollary 3.7.2. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $U$, $h_2(z)$ be univalent in $U$, $q_2(z) \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_{I,1}[h_2, q_2] \cap \Phi'_{I,1}[h_1, q_1]$. If $f \in \mathcal{A}_p$, $\frac{I_p(n,\lambda)f(z)}{z^{p-1}} \in \mathcal{H}_0 \cap \mathcal{Q}_0$ and*

$$\phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right)$$

*is univalent in $U$, then*

$$h_1(z) \prec \phi\left(\frac{I_p(n,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+1,\lambda)f(z)}{z^{p-1}}, \frac{I_p(n+2,\lambda)f(z)}{z^{p-1}}; z\right) \prec h_2(z),$$

*implies*

$$q_1(z) \prec \frac{I_p(n,\lambda)f(z)}{z^{p-1}} \prec q_2(z).$$

Definition 3.7.3. Let $\Omega$ be a set in $\mathcal{C}$, $q(z) \neq 0$, $zq'(z) \neq 0$ and $q(z) \in \mathcal{H}$. The class of admissible functions $\Phi'_{I,2}[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition $\phi(u, v, w; \zeta) \in \Omega$ whenever

$$u = q(z),\ v = \frac{1}{\lambda+p}\left((\lambda+p)q(z) + \frac{zq'(z)}{mq(z)}\right),$$

$$\Re\left\{\frac{(\lambda+p)v(w-v)}{v-u} - (\lambda+p)(2u-v)\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

Theorem 3.7.6. *Let $\phi \in \Phi'_{I,2}[\Omega, q]$. If $f \in \mathcal{A}_p$, $\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \in \mathcal{Q}_1$ and*

$$\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right)$$

*is univalent in $U$, then*

$$\Omega \subset \left\{\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) : z \in U\right\} \tag{3.7.6}$$

*implies*

$$q(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}.$$

Proof. From (3.6.29) and (3.7.6 ), it follows that

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

In view of (3.6.28), the admissibility condition for $\phi \in \Phi'_{I,2}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 3.1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}. \qquad \square$$

THEOREM 3.7.7. *Let $h(z)$ be analytic in $U$ and $\phi \in \Phi'_{I,2}[h, q] := \Phi'_{I,1}[h(U), q]$. If $f \in \mathcal{A}_p$, $\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \in \mathcal{Q}_1$ and*

$$\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right)$$

*is univalent in $U$, then*

$$h(z) \prec \phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right), \tag{3.7.7}$$

*implies*

$$q(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}.$$

Combining Theorems 3.6.8 and 3.7.7, the following sandwich-type theorem is obtained.

COROLLARY 3.7.3. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $U$, $h_2(z)$ be univalent in $U$, $q_2(z) \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{I,2}[h_2, q_2] \cap \Phi'_{I,2}[h_1, q_1]$. If $f \in \mathcal{A}_p$, $\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \in \mathcal{H} \cap \mathcal{Q}_1$ and*

$$\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right)$$

*is univalent in $U$, then*

$$h_1(z) \prec \phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) \prec h_2(z)$$

*implies*

$$q_1(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec q_2(z).$$

## CHAPTER 4

# DIFFERENTIAL SUBORDINATION AND SUPERORDINATION OF MEROMORPHIC FUNCTIONS

## 4.1. INTRODUCTION

Let $\Sigma_p$ denotes the class of all $p$-valent functions of the form

(4.1.1)
$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (z \in U^* := \{z \in \mathcal{C} : 0 < |z| < 1\},\ p \in \mathcal{N} := \{1, 2, \cdots\})$$

and let $\Sigma_1 := \Sigma$. For two functions $f(z)$ given by (4.1.1) and $g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k$, the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$(f * g)(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{4.1.2}$$

For $\alpha_j \in \mathcal{C} \quad (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathcal{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_l)_k}{(\beta_1)_k \ldots (\beta_m)_k} \frac{z^k}{k!}$$

$$(l \leq m + 1; l, m \in \mathcal{N}_0 := \{0, 1, 2, \ldots\})$$

where $(a)_n$ is the Pochhammer symbol defined by (1.2.2). Corresponding to the function

$$h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z^{-p}\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Liu-Srivastava operator **[52]** $H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \Sigma_p \to \Sigma_p$ is defined by the Hadamard product

$$\begin{aligned} H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) &:= h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \frac{(\alpha_1)_{k+p} \ldots (\alpha_l)_{k+p}}{(\beta_1)_{k+p} \ldots (\beta_m)_{k+p}} \frac{a_k z^k}{(k+p)!}. \end{aligned} \tag{4.1.3}$$

For convenience, we write

$$H_p^{l,m}[\alpha_1]f(z) := H_p^{(l,m)}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)f(z).$$

Special cases of the Liu-Srivastava linear operator include the meromorphic analogue of the Carlson-Shaffer linear operator $\mathcal{L}_p(a,c) := H_p^{(2,1)}(1,a;c)$ studied by Liu [**50**], the operator $D^{n+1} := \mathcal{L}_p(n+p,1)$ investigated by Yang [**114**] (which is analogous to the Ruscheweyh derivative operator), and the operator $J_{c,p} := \mathcal{L}_p(c,c+1)$ studied by Uralegaddi and Somanatha [**110**]. Note that

$$J_{c,p}f = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt \quad (c > 0).$$

Motivated by the operator studied by Aouf and Hossen [**8**] (see also [**18, 50, 75**]), define the *multiplier transformation* $I_p(n,\lambda)$ on $\Sigma_p$ by the following infinite series

$$I_p(n,\lambda)f(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{k+\lambda}{\lambda-p}\right)^n a_k z^k \quad (\lambda > p). \tag{4.1.4}$$

In this chapter, differential subordination and superordination results are obtained for meromorphic functions in the punctured unit disk that are associated with the *Liu-Srivastava linear operator* and the *multiplier transformation*. These results are obtained by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained.

## 4.2. SUBORDINATION INEQUALITIES ASSOCIATED WITH THE LIU-SRIVASTAVA LINEAR OPERATOR

In this section, the differential subordination of the Liu-Srivastava linear operator is investigated. For this purpose, the class of admissible functions is given in the following definition.

DEFINITION 4.2.1. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Theta_H[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega \tag{4.2.1}$$

whenever

$$u = q(\zeta),\ v = \frac{k\zeta q'(\zeta) + \alpha_1 q(\zeta)}{\alpha_1}, \quad (\alpha_1 \in \mathcal{C},\ \alpha_1 \neq 0, -1)$$

$$\Re\left\{\frac{(\alpha_1+1)(w-u)}{v-u} - (2\alpha_1+1)\right\} \geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

THEOREM 4.2.1. *Let* $\phi \in \Theta_H[\Omega, q]$. *If* $f(z) \in \Sigma_p$ *satisfies*

(4.2.2)
$$\left\{\phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z); z\right) : z \in U\right\} \subset \Omega,$$

*then*

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z).$$

PROOF. Define the analytic function $p(z)$ in $U$ by

(4.2.3)
$$p(z) := z^p H_p^{l,m}[\alpha_1]f(z).$$

In view of the relation

(4.2.4)
$$\alpha_1 H_p^{l,m}[\alpha_1+1]f(z) = z[H_p^{l,m}[\alpha_1]f(z)]' + (\alpha_1+p)H_p^{l,m}[\alpha_1]f(z),$$

a simple computation from (4.2.3) yields

(4.2.5)
$$z^p H_p^{l,m}[\alpha_1+1]f(z) = \frac{1}{\alpha_1}\left[\alpha_1 p(z) + zp'(z)\right].$$

Further computations yield

(4.2.6)
$$z^p H_p^{l,m}[\alpha_1+2]f(z) = \frac{1}{\alpha_1(\alpha_1+1)}\left[z^2 p''(z) + 2(\alpha_1+1)zp'(z) + \alpha_1(\alpha_1+1)p(z)\right].$$

Define the transformations from $\mathcal{C}^3$ to $\mathcal{C}$ by

(4.2.7)
$$u = r,\ v = \frac{\alpha_1 r + s}{\alpha_1},\ w = \frac{t + 2(\alpha_1+1)s + \alpha_1(\alpha_1+1)r}{\alpha_1(\alpha_1+1)}.$$

Let

(4.2.8)
$$\psi(r,s,t;z) := \phi(u,v,w;z) = \phi\left(r,\ \frac{\alpha_1 r + s}{\alpha_1}, \frac{t + 2(\alpha_1+1)s + \alpha_1(\alpha_1+1)r}{\alpha_1(\alpha_1+1)}; z\right).$$

The proof shall make use of Theorem 3.1.1. Using equations (4.2.3), (4.2.5) and (4.2.6), from (4.2.8), it follows that

$$\begin{aligned} \text{(4.2.9)} \qquad & \psi(p(z), zp'(z), z^2p''(z); z) \\ &= \phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z); z\right). \end{aligned}$$

Hence (4.2.2) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Theta_H[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.1. Note that

$$\frac{t}{s} + 1 = \frac{(\alpha_1+1)(w-u)}{v-u} - (2\alpha_1+1),$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 3.1.1, $p(z) \prec q(z)$ or

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z). \qquad \square$$

Theorem 4.2.2. *Let $q(z)$ be an analytic function with $q(0) = 1$. Let $\phi \in \Theta_H[h, q] := \Theta_H[h(U), q]$. If $f(z) \in \Sigma_p$ satisfies*

$$\text{(4.2.10)} \qquad \phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z); z\right) \prec h(z),$$

*then*

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z).$$

The following result is an extension of Theorem 4.2.1 to the case where the behavior of $q(z)$ on $\partial U$ is not known.

Corollary 4.2.1. *Let $\Omega \subset \mathcal{C}$ and let $q(z)$ be univalent in $U$, $q(0) = 1$. Let $\phi \in \Theta_H[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f(z) \in \Sigma_p$ and*

$$\text{(4.2.11)} \qquad \phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z); z\right) \in \Omega,$$

*then*

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z).$$

Proof. Theorem 4.2.1 yields $z^p H_p^{l,m}[\alpha_1]f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$. $\square$

THEOREM 4.2.3. *Let $h(z)$ and $q(z)$ be univalent in $U$, with $q(0) = 1$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ satisfies one of the following conditions:*

(1) *$\phi \in \Theta_H[h, q_\rho]$ for some $\rho \in (0,1)$, or*
(2) *there exists $\rho_0 \in (0,1)$ such that $\phi \in \Theta_H[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

*If $f(z) \in \Sigma_p$ satisfies (4.2.10), then*

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z).$$

PROOF. The result is similar to the proof of Theorem 3.4.3 and is therefore omitted. □

The next theorem yields the best dominant of the differential subordination (4.2.10).

THEOREM 4.2.4. *Let $h(z)$ be univalent in $U$, and $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$. Suppose the differential equation*

$$\phi\left(q(z), zq'(z), z^2 q''(z); z\right) = h(z) \tag{4.2.12}$$

*has a solution $q(z)$ with $q(0) = 1$ and satisfies one of the following conditions:*

(1) *$q(z) \in \mathcal{Q}_1$ and $\phi \in \Theta_H[h, q]$*
(2) *$q(z)$ is univalent in $U$ and $\phi \in \Theta_H[h, q_\rho]$, for some $\rho \in (0,1)$, or*
(3) *$q(z)$ is univalent in $U$ and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Theta_H[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.*

*If $f(z) \in \Sigma_p$ satisfies (4.2.10), then*

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z),$$

*and $q(z)$ is the best dominant.*

PROOF. Following the same arguments as in the proof of [**58**, Theorem 2.3e, p. 31], from Theorems 4.2.2 and 4.2.3, it follows that $q(z)$ is a dominant. Since $q(z)$ satisfies

(4.2.12), it is also a solution of (4.2.10) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant. $\square$

Definition 4.2.2. Let $\Omega$ be a set in $\mathcal{C}$ and let $\Theta_H[\Omega, M] := \Theta_H[\Omega, q]$ where $q(z) = 1 + Mz,\ M > 0$. The class of admissible functions $\Theta_H[\Omega, M]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ such that

$$\phi\left(1 + Me^{i\theta},\ 1 + \frac{k + \alpha_1}{\alpha_1} Me^{i\theta},\ 1 + \frac{L + (2k + \alpha_1)(1 + \alpha_1)Me^{i\theta}}{\alpha_1(1 + \alpha_1)}; z\right) \notin \Omega \tag{4.2.13}$$

whenever $z \in U,\ \theta \in \mathcal{R}$, $\Re\left(Le^{-i\theta}\right) \geq (k-1)kM$ for all real $\theta$, $\alpha_1 \in \mathcal{C}$ $(\alpha_1 \neq 0, -1)$ and $k \geq 1$.

Corollary 4.2.2. *Let $\phi \in \Theta_H[\Omega, M]$. If $f(z) \in \Sigma_p$ satisfies*

$$\phi\left(z^p H_p^{l,m}[\alpha_1] f(z), z^p H_p^{l,m}[\alpha_1 + 1] f(z), z^p H_p^{l,m}[\alpha_1 + 2] f(z); z\right) \in \Omega,$$

*then*

$$|z^p H_p^{l,m}[\alpha_1] f(z) - 1| < M.$$

Corollary 4.2.3. *Let $\phi \in \Theta_H[M] := \Theta_H[\Omega, M]$ where $\Omega = \{\omega : |\omega - 1| < M\}$. If $f(z) \in \Sigma_p$ satisfies*

$$|\phi\left(z^p H_p^{l,m}[\alpha_1] f(z), z^p H_p^{l,m}[\alpha_1 + 1] f(z), z^p H_p^{l,m}[\alpha_1 + 2] f(z); z\right) - 1| < M,$$

*then*

$$|z^p H_p^{l,m}[\alpha_1] f(z) - 1| < M.$$

Corollary 4.2.4. *If $\Re\alpha_1 \geq -1/2$ and $f(z) \in \Sigma_p$ satisfies*

$$|z^p H_p^{l,m}[\alpha_1 + 1] f(z) - 1| < M,$$

*then*

$$|z^p H_p^{l,m}[\alpha_1] f(z) - 1| < M.$$

Proof. This follows from Corollary 4.2.3 by taking $\phi(u, v, w; z) = v$. $\square$

Corollary 4.2.5. *Let $M > 0$ and $0 \neq \alpha_1 \in \mathcal{C}$. If $f(z) \in \Sigma_p$ satisfies*

$$\left|z^p H_p^{l,m}[\alpha_1 + 1] f(z) - z^p H_p^{l,m}[\alpha_1] f(z)\right| < \frac{M}{|\alpha_1|}, \quad \textit{then} \quad |z^p H_p^{l,m}[\alpha_1] f(z) - 1| < M. \tag{4.2.14}$$

PROOF. Let $\phi(u,v,w;z) = v - u$ and $\Omega = h(U)$ where $h(z) = Mz/|\alpha_1|$, $M > 0$. Since

$$\left|\phi\left(1 + Me^{i\theta},\ 1 + \frac{k+\alpha_1}{\alpha_1}Me^{i\theta},\ 1 + \frac{L + (2k+\alpha_1)(1+\alpha_1)Me^{i\theta}}{\alpha_1(1+\alpha_1)};z\right)\right| = \frac{kM}{|\alpha_1|} \geq \frac{M}{|\alpha_1|}$$

$z \in U,\ \theta \in \mathcal{R},\ \alpha_1 \in \mathcal{C}\ (\alpha_1 \neq 0, -1)$, and $k \geq 1$, it follows that $\phi \in \Theta_H[\Omega, M]$. Hence by Corollary 4.2.2, the required result is obtained. $\square$

The differential equation

$$zq'(z) - q(z) = \frac{M}{|\alpha_1|}z \qquad (|\alpha_1| < M)$$

has a univalent solution $q(z) = 1 + Mz$. Theorem 4.2.4 shows that $q(z) = 1 + Mz$ is the best dominant of (4.2.14).

Note that

$$\begin{aligned}
H_p^{(2,1)}(1,1;1)f(z) &= f(z)\\
H_p^{(2,1)}(2,1;1)f(z) &= zf'(z) + (1+p)f(z)\\
H_p^{(2,1)}(3,1;1)f(z) &= \frac{1}{2}[z^2f''(z) + 2(p+2)zf'(z) + (p+1)(p+2)f(z)].
\end{aligned}$$

By taking $l = 2,\ m = 1,\ \alpha_1 = \alpha_2 = \beta_1 = 1$, (4.2.14) shows that for $f(z) \in \Sigma_p$, whenever

$$z^p[zf'(z) + pf(z)] \prec Mz, \quad \text{then} \quad z^pf(z) \prec 1 + Mz.$$

DEFINITION 4.2.3. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Theta_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(\zeta),\ v = \frac{1}{\alpha_1 + 1}\left(1 + \alpha_1 q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)}\right)\ (\alpha_1 \in \mathcal{C},\ \alpha_1 \neq 0, -1, -2,\ q(\zeta) \neq 0),$$

$$\Re\left\{\frac{\begin{array}{c}[(\alpha_1+2)w - 1 - (1+\alpha_1)v)]\\ \times(1+\alpha_1)v\end{array}}{(1+\alpha_1)v - (1+\alpha_1 u)} + (1+\alpha_1)v - (1+2\alpha_1 u)\right\} \geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

THEOREM 4.2.5. *Let $\phi \in \Theta_{H,1}[\Omega, q]$. If $f(z) \in \Sigma_p$ satisfies*

(4.2.15)
$$\left\{\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right) : z \in U\right\} \subset \Omega,$$

*then*

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec q(z).$$

PROOF. Define the function $p(z)$ in $U$ by

$$p(z) := \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}. \tag{4.2.16}$$

A simple computation using (4.2.16), yields

$$\frac{zp'(z)}{p(z)} := \frac{z[H_p^{l,m}[\alpha_1+1]f(z)]'}{H_p^{l,m}[\alpha_1+1]f(z)} - \frac{z[H_p^{l,m}[\alpha_1]f(z)]'}{H_p^{l,m}[\alpha_1]f(z)}. \tag{4.2.17}$$

Use of (4.2.4) in (4.2.17) yields

$$\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)} = \frac{1}{\alpha_1+1}\left(\alpha_1 p(z) + 1 + \frac{zp'(z)}{p(z)}\right). \tag{4.2.18}$$

Differentiating logarithmically (4.2.18), and further computations show that

(4.2.19)
$$\begin{aligned} &\frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)} \\ &= \frac{1}{\alpha_1+2}\left(2+\alpha_1 p(z) + \frac{zp'(z)}{p(z)} + \frac{\alpha_1 zp'(z) + \frac{zp'(z)}{p(z)} - (\frac{zp'(z)}{p(z)})^2 + \frac{z^2p''(z)}{p(z)}}{1+\alpha_1 p(z) + \frac{zp'(z)}{p(z)}}\right). \end{aligned}$$

Define the transformations from $\mathcal{C}^3$ to $\mathcal{C}$ by

(4.2.20)
$$u = r,\ v = \frac{1}{\alpha_1+1}\left(1+\alpha_1 r + \frac{s}{r}\right),\ w = \frac{1}{\alpha_1+2}\left(2+\alpha_1 r + \frac{s}{r} + \frac{\alpha_1 s + \frac{s}{r} - (\frac{s}{r})^2 + \frac{t}{r}}{1+\alpha_1 r + \frac{s}{r}}\right).$$

Let

(4.2.21) $\psi(r,s,t;z) := \phi(u,v,w;z)$
$$= \phi\left(r, \frac{1}{\alpha_1+1}\left[\alpha_1 r + 1 + \frac{s}{r}\right], \frac{1}{\alpha_1+2}\left(2+\alpha_1 r + \frac{s}{r} + \frac{\alpha_1 s + \frac{s}{r} - (\frac{s}{r})^2 + \frac{t}{r}}{1+\alpha_1 r + \frac{s}{r}}\right); z\right).$$

Using the equations (4.2.16), (4.2.18) and (4.2.19), from (4.2.21), it follows that

(4.2.22)
$$\begin{aligned} &\psi(p(z), zp'(z), z^2p''(z); z) \\ &= \phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right). \end{aligned}$$

Hence (4.2.15) implies $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$. The proof is completed if it can be shown that the admissibility condition for $\phi \in \Theta_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.1. For this purpose, note that

$$\begin{aligned}
\frac{s}{r} &= (\alpha_1+1)v - (1+\alpha_1 r),\\
\frac{t}{r} &= (1+\alpha_1)v[(\alpha_1+2)w - 1 - (1+\alpha_1)v)] - \frac{s}{r}\left[(1+\alpha_1)v - \frac{2s}{r}\right],
\end{aligned}$$

and thus $$\frac{t}{s}+1 = \frac{\begin{matrix}[(\alpha_1+2)w-1-(1+\alpha_1)v]\\ \times(1+\alpha_1)v\end{matrix}}{(1+\alpha_1)v-(1+\alpha_1 u)} + (1+\alpha_1)v - (1+2\alpha_1 u).$$

Hence $\psi \in \Psi[\Omega, q]$ and by Theorem 3.1.1, $p(z) \prec q(z)$ or

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec q(z). \qquad \square$$

THEOREM 4.2.6. *Let* $\phi \in \Theta_{H,1}[h, q] := \Theta_{H,1}[h(U), q]$ *with* $q(0) = 1$. *If* $f(z) \in \Sigma_p$ *satisfies*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right) \prec h(z), \tag{4.2.23}$$

*then*

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec q(z).$$

DEFINITION 4.2.4. Let $\Omega$ be a set in $\mathcal{C}$ and let $\Theta_{H,1}[\Omega, M] := \Theta_{H,1}[\Omega, q]$ where $q(z) = 1 + Mz,\ M > 0$. The class of admissible functions $\Theta_{H,1}[\Omega, M]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ such that

$$\begin{aligned}
&\phi\Big(1+Me^{i\theta}, 1 + \frac{k+\alpha_1(1+Me^{i\theta})}{(1+\alpha_1)(1+Me^{i\theta})}Me^{i\theta}, 1 + \frac{k+\alpha_1(1+Me^{i\theta})}{(2+\alpha_1)(1+Me^{i\theta})}Me^{i\theta}\\
&\quad + \frac{(M+e^{-i\theta})[Le^{-i\theta} + kM(1+\alpha_1+\alpha_1 Me^{i\theta})] - k^2M^2}{(\alpha_1+2)(M+e^{-i\theta})[(M+e^{-i\theta})(1+\alpha_1+\alpha_1 Me^{i\theta}) + kM]}; z\Big) \notin \Omega
\end{aligned} \tag{4.2.24}$$

whenever $z \in U,\ \theta \in \mathcal{R}$, $\Re(Le^{-i\theta}) \geq kM(k-1)$ for all real $\theta$, $\alpha_1 \in \mathcal{C}$ $(\alpha_1 \neq 0, -1, -2)$ and $k \geq 1$.

COROLLARY 4.2.6. *Let* $\phi \in \Theta_{H,1}[\Omega, M]$. *If* $f(z) \in \Sigma_p$ *satisfies*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right) \in \Omega,$$

*then*

$$\left|\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} - 1\right| < M.$$

Corollary 4.2.7. *Let* $\phi \in \Theta_{H,1}[M] := \Theta_{H,1}[\Omega, M]$ *where* $\Omega = \{\omega : |\omega - 1| < M\}$. *If* $f(z) \in \Sigma_p$ *satisfies*

$$\left|\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right) - 1\right| < M,$$

*then*

$$\left|\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} - 1\right| < M.$$

Corollary 4.2.8. *Let* $M > 0$, $\alpha_1 \in \mathcal{C}$ $(\alpha_1 \neq 0, -1)$, *and* $f(z) \in \Sigma_p$ *satisfies*

$$\left|\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)} - \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\right| \quad < \quad \frac{M^2}{|1+\alpha_1|(1+M)},$$

*then*

$$\left|\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} - 1\right| < M.$$

Proof. This follows from Corollary 4.2.6 by taking $\phi(u, v, w; z) = v - u$ and $\Omega = h(U)$ where

$$h(z) = \frac{M^2 z}{|1+\alpha_1|(1+M)} \quad (M > 0).$$

Indeed, since

$$\begin{aligned}
|\phi(u, v, w; z)| &= \left|1 + \frac{k + \alpha_1(1+Me^{i\theta})}{(1+\alpha_1)(1+Me^{i\theta})}Me^{i\theta} - 1 - Me^{i\theta}\right| \\
&= \frac{M}{|1+\alpha_1|}\left|\frac{k - 1 - Me^{i\theta}}{1+Me^{i\theta}}\right| \\
&\geq \frac{M}{|1+\alpha_1|}\left|\frac{k-1-M}{1+M}\right| \\
&\geq \frac{M}{|1+\alpha_1|}\left|\frac{1}{1+M} - 1\right| \\
&= \frac{M^2}{|1+\alpha_1|(1+M)},
\end{aligned}$$

$z \in U$, $\theta \in \mathcal{R}$, $\alpha_1 \in \mathcal{C}$ $(\alpha_1 \neq 0, -1)$, $k \neq 1+M$ and $k \geq 1$, it follows that $\phi \in \Theta_{H,1}[M]$. Hence by Corollary 4.2.6, the required result is obtained. □

For $l = 2,\ m = 1,\ \alpha_1 = \alpha_2 = \beta_1 = 1$, Corollary 4.2.8 reduces to:

EXAMPLE 4.2.1. If $f(z) \in \Sigma_p$, then

$$\frac{\frac{zf'(z)}{f(z)}\left[\frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)} - p\right]}{\frac{zf'(z)}{f(z)} + p + 1} \prec p + \frac{M^2}{1+M}z \Rightarrow \frac{zf'(z)}{f(z)} \prec Mz - p.$$

## 4.3. SUPERORDINATION OF THE LIU-SRIVASTAVA LINEAR OPERATOR

The dual problem of differential subordination, that is, the differential superordination of the Liu-Srivastava linear operator is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

DEFINITION 4.3.1. Let $\Omega$ be a set in $\mathcal{C}$, $q(z) \in \mathcal{H}$ with $zq'(z) \neq 0$. The class of admissible functions $\Theta'_H[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z),\ v = \frac{zq'(z) + m\alpha_1 q(z)}{m\alpha_1}, \quad (\alpha_1 \in \mathcal{C},\ \alpha_1 \neq 0, -1)$$
$$\Re\left\{\frac{(\alpha_1+1)(w-u)}{v-u} - (2\alpha_1+1)\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

THEOREM 4.3.1. *Let $\phi \in \Theta'_H[\Omega, q]$. If $f(z) \in \Sigma_p$, $z^p H_p^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_1$ and*

$$\phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z); z\right)$$

*is univalent in $U$, then*

(4.3.1)

$$\Omega \subset \left\{\phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z); z\right) : z \in U\right\}$$

*implies*

$$q(z) \prec z^p H_p^{l,m}[\alpha_1]f(z).$$

PROOF. Let $p(z)$ be defined by (4.2.3) and $\psi$ by (4.2.8). Since $\phi \in \Theta'_H[\Omega, q]$, from (4.2.9) and (4.3.1), it follows that

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

In view of (4.2.7), the admissibility condition for $\phi \in \Theta'_H[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 3.1.2, $q(z) \prec p(z)$ or

$$q(z) \prec z^p H_p^{l,m}[\alpha_1]f(z). \qquad \square$$

THEOREM 4.3.2. *Let $q(z) \in \mathcal{H}$, h(z) be analytic in $U$ and $\phi \in \Theta'_H[h, q] := \Theta'_H[h(U), q]$. If $f(z) \in \Sigma_p$, $z^p H_p^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_1$ and*

$$\phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z); z\right)$$

*is univalent in $U$, then*

$$h(z) \prec \phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z); z\right) \tag{4.3.2}$$

*implies*

$$q(z) \prec z^p H_p^{l,m}[\alpha_1]f(z).$$

Theorems 4.3.1 and 4.3.2 can only be used to obtain subordinants of differential superordination of the form (4.3.1) or (4.3.2). The following theorem proves the existence of the best subordinant of (4.3.2) for an appropriate $\phi$.

THEOREM 4.3.3. *Let h(z) be analytic in $U$ and $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$. Suppose that the differential equation*

$$\phi(q(z), zq'(z), z^2q''(z); z) = h(z)$$

*has a solution $q(z) \in \mathcal{Q}_1$. If $\phi \in \Theta'_H[h, q]$, $f(z) \in \Sigma_p$, $z^p H_p^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_1$ and*

$$\phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z); z\right)$$

*is univalent in $U$, then*

$$h(z) \prec \phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z); z\right)$$

*implies*

$$q(z) \prec z^p H_p^{l,m}[\alpha_1]f(z)$$

*and $q(z)$ is the best subordinant.*

PROOF. The proof is similar to the proof of Theorem 4.2.4 and is therefore omitted. $\square$

Combining Theorems 4.2.2 and 4.3.2, the following sandwich-type theorem is obtained.

COROLLARY 4.3.1. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $U$, $h_2(z)$ be univalent in $U$ and $q_2(z) \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Theta_H[h_2, q_2] \cap \Theta'_H[h_1, q_1]$. If $f(z) \in \Sigma_p$, $z^p H_p^{l,m}[\alpha_1]f(z) \in \mathcal{H} \cap \mathcal{Q}_1$ and*

$$\phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z); z\right)$$

*is univalent in $U$, then*

$$h_1(z) \prec \phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z); z\right) \prec h_2(z)$$

*implies*

$$q_1(z) \prec z^p H_p^{l,m}[\alpha_1]f(z) \prec q_2(z).$$

DEFINITION 4.3.2. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z) \in \mathcal{H}$ with $zq'(z) \neq 0$. The class of admissible functions $\Theta'_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z),\ v = \frac{1}{\alpha_1+1}\left(1 + \alpha_1 q(z) + \frac{zq'(z)}{mq(z)}\right) \quad (\alpha_1 \in \mathcal{C},\ \alpha_1 \neq 0, -1, -2,\ q(z) \neq 0)$$

$$\Re\left\{\frac{\begin{array}{c}[(\alpha_1+2)w - 1 - (1+\alpha_1)v)] \\ \times(1+\alpha_1)v\end{array}}{(1+\alpha_1)v - (1+\alpha_1 u)} + (1+\alpha_1)v - (1+2\alpha_1 u)\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

THEOREM 4.3.4. *Let $\phi \in \Theta'_{H,1}[\Omega, q]$. If $f(z) \in \Sigma_p$, $\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \in \mathcal{Q}_1$ and*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right)$$

*is univalent in $U$, then*

(4.3.3)
$$\Omega \subset \left\{ \phi \left( \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z \right) : z \in U \right\}$$

*implies*

$$q(z) \prec \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}.$$

PROOF. Let $p(z)$ be defined by (4.2.16) and $\psi$ by (4.2.21). Since $\phi \in \Theta'_{H,1}[\Omega, q]$, from (4.2.22) and (4.3.3), it follows that

$$\Omega \subset \left\{ \psi \left( p(z), zp'(z), z^2p''(z); z \right) : z \in U \right\}.$$

In view of (4.2.20), the admissibility condition for $\phi \in \Theta'_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 3.1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}. \qquad \square$$

THEOREM 4.3.5. *Let $q(z) \in \mathcal{H}$, h(z) be analytic in $U$ and $\phi \in \Theta'_{H,1}[h, q] := \Theta'_{H,1}[h(U), q]$. If $f(z) \in \Sigma_p$, $\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \in \mathcal{Q}_1$ and*

$$\phi \left( \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z \right)$$

*is univalent in $U$, then*

(4.3.4)
$$h(z) \prec \phi \left( \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z \right)$$

*implies*

$$q(z) \prec \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}.$$

Combining Theorems 4.2.6 and 4.3.5, the following sandwich-type theorem is obtained.

COROLLARY 4.3.2. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $U$, $h_2(z)$ be univalent in $U$ and $q_2(z) \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Theta_{H,1}[h_2, q_2] \cap \Theta'_{H,1}[h_1, q_1]$.*

*If* $f(z) \in \Sigma_p$, $\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \in \mathcal{H} \cap \mathcal{Q}_1$, *and*

$$\phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right)$$

*is univalent in* $U$, *then*

$$h_1(z) \prec \phi\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}; z\right) \prec h_2(z)$$

*implies*

$$q_1(z) \prec \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \prec q_2(z).$$

## 4.4. SUBORDINATION INEQUALITIES ASSOCIATED WITH THE MULTIPLIER TRANSFORMATION

Differential subordination of the multiplier transformation is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

Definition 4.4.1. Let $\Omega$ be a set in $\mathcal{C}$, $q(z) \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Theta_I[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition

(4.4.1) $$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta),\ v = \frac{1}{\lambda - p}\left[(\lambda - p)q(\zeta) + k\zeta q'(\zeta)\right],$$
$$\Re\left\{\frac{(\lambda - p)(w + u - 2v)}{v - u}\right\} \geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

Theorem 4.4.1. *Let* $\phi \in \Theta_I[\Omega, q]$. *If* $f(z) \in \Sigma_p$ *satisfies*

(4.4.2) $$\{\phi\left(z^p I_p(n,\lambda)f(z), z^p I_p(n+1,\lambda)f(z), z^p I_p(n+2,\lambda)f(z); z\right) : z \in U\} \subset \Omega,$$

*then*

$$z^p I_p(n,\lambda)f(z) \prec q(z).$$

PROOF. Define the analytic function $p(z)$ in $U$ by

$$p(z) := z^p I_p(n,\lambda)f(z). \tag{4.4.3}$$

In view of the relation

$$(\lambda - p)I_p(n+1,\lambda)f(z) = z[I_p(n,\lambda)f(z)]' + \lambda I_p(n,\lambda)f(z), \tag{4.4.4}$$

a simple computation from (4.4.3) yields

$$z^p I_p(n+1,\lambda)f(z) = \frac{1}{\lambda - p}\{(\lambda - p)p(z) + zp'(z)\}. \tag{4.4.5}$$

Further computations show that

(4.4.6)

$$z^p I_p(n+2,\lambda)f(z) = \frac{1}{(\lambda - p)^2}\left\{(\lambda - p)^2 p(z) + (2(\lambda - p)+1)zp'(z) + z^2 p''(z)\right\}.$$

Define the transformations from $\mathcal{C}^3$ to $\mathcal{C}$ by

$$u = r, \quad v = \frac{(\lambda - p)r + s}{\lambda - p}, \quad w = \frac{(\lambda - p)^2 r + (2(\lambda - p)+1)s + t}{(\lambda - p)^2}. \tag{4.4.7}$$

Let

(4.4.8)

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{(\lambda - p)r + s}{\lambda - p}, \frac{(\lambda - p)^2 r + (2(\lambda - p)+1)s + t}{(\lambda - p)^2}; z\right).$$

Using equations (4.4.3), (4.4.5) and (4.4.6), from (4.4.8), it follows that

$$\begin{aligned} &\psi(p(z), zp'(z), z^2p''(z); z) \\ = \;&\phi\left(z^p I_p(n,\lambda)f(z), z^p I_p(n+1,\lambda)f(z), z^p I_p(n+2,\lambda)f(z); z\right). \end{aligned} \tag{4.4.9}$$

Hence (4.4.2) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Theta_I[h,q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.1. Note that

$$\frac{t}{s} + 1 = \frac{(\lambda - p)(w + u - 2v)}{v - u}.$$

Hence $\psi \in \Psi[\Omega, q]$. By Theorem 3.1.1, $p(z) \prec q(z)$ or

$$z^p I_p(n,\lambda)f(z) \prec q(z). \qquad \square$$

THEOREM 4.4.2. *Let $q(z)$ be an analytic function with $q(0) = 1$. Let $\phi \in \Theta_I[h, q] := \Theta_I[h(U), q]$. If $f(z) \in \Sigma_p$ satisfies*

$$\phi\left(z^p I_p(n,\lambda)f(z), z^p I_p(n+1,\lambda)f(z), z^p I_p(n+2,\lambda)f(z); z\right) \prec h(z), \tag{4.4.10}$$

*then*

$$z^p I_p(n,\lambda)f(z) \prec q(z).$$

Our next result is an extension of Theorem 4.4.1 to the case where the behavior of $q(z)$ on $\partial U$ is not known.

COROLLARY 4.4.1. *Let $\Omega \subset \mathcal{C}$ and let $q(z)$ be univalent in $U$, $q(0) = 1$. Let $\phi \in \Theta_I[\Omega, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If $f(z) \in \Sigma_p$ satisfies*

$$\phi\left(z^p I_p(n,\lambda)f(z), z^p I_p(n+1,\lambda)f(z), z^p I_p(n+2,\lambda)f(z); z\right) \in \Omega, \tag{4.4.11}$$

*then*

$$z^p I_p(n,\lambda)f(z) \prec q(z).$$

PROOF. Theorem 4.4.1 yields $z^p I_p(n,\lambda)f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$. $\square$

THEOREM 4.4.3. *Let $h(z)$ and $q(z)$ be univalent in $U$, with $q(0) = 1$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ satisfies one of the following conditions:*

(1) *$\phi \in \Theta_I[h, q_\rho]$ for some $\rho \in (0,1)$, or*
(2) *there exists $\rho_0 \in (0,1)$ such that $\phi \in \Theta_I[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

*If $f(z) \in \Sigma_p$ satisfies (4.4.10), then*

$$z^p I_p(n,\lambda)f(z) \prec q(z).$$

PROOF. The result is similar to the proof of Theorem 3.4.3 and is therefore omitted. $\square$

The following theorem yields the best dominant of the differential subordination (4.4.10).

THEOREM 4.4.4. *Let $h(z)$ be univalent in $U$. Let $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$. Suppose that the differential equation*

$$\phi\left(q(z), zq'(z), z^2q''(z); z\right) = h(z) \tag{4.4.12}$$

*has a solution $q(z)$ with $q(0) = 1$ and satisfies one of the following conditions:*

(1) $q(z) \in \mathcal{Q}_1$ *and* $\phi \in \Theta_I[h, q]$,
(2) $q(z)$ *is univalent in* $U$ *and* $\phi \in \Theta_I[h, q_\rho]$ *for some* $\rho \in (0, 1)$, *or*
(3) $q(z)$ *is univalent in* $U$ *and there exists* $\rho_0 \in (0, 1)$ *such that* $\phi \in \Theta_I[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

*If $f(z) \in \Sigma_p$ satisfies (4.4.10), then*

$$z^pI_p(n, \lambda)f(z) \prec q(z)$$

*and $q(z)$ is the best dominant.*

PROOF. The proof is similar to the proof of Theorem 4.2.4 and is therefore omitted.

□

DEFINITION 4.4.2. Let $\Omega$ be a set in $\mathcal{C}$ and let $\Theta_I[\Omega, M] := \Theta_I[\Omega, q]$ where $q(z) = 1 + Mz,\ M > 0$. The class of admissible functions $\Theta_I[\Omega, M]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ such that

(4.4.13)
$$\phi\left(1 + Me^{i\theta}, 1 + Me^{i\theta} + \frac{k}{\lambda - p}Me^{i\theta}, 1 + Me^{i\theta} + \frac{L + [2(\lambda - p) + 1]kMe^{i\theta}}{(\lambda - p)^2}; z\right) \notin \Omega$$

whenever $z \in U,\ \theta \in \mathcal{R},\ \Re(Le^{-i\theta}) \geq kM(k - 1)$ for all real $\theta$ and $k \geq 1$.

COROLLARY 4.4.2. *Let $\phi \in \Theta_I[\Omega, M]$. If $f(z) \in \Sigma_p$ satisfies*

$$\phi\left(z^pI_p(n, \lambda)f(z), z^pI_p(n + 1, \lambda)f(z), z^pI_p(n + 2, \lambda)f(z); z\right) \in \Omega,$$

*then*

$$z^pI_p(n, \lambda)f(z) \prec 1 + Mz.$$

COROLLARY 4.4.3. *Let* $\phi \in \Theta_I[M] := \Theta_I[\Omega, M]$ *where* $\Omega = \{\omega : |\omega - 1| < M\}$. *If* $f(z) \in \Sigma_p$ *satisfies*

$$|\phi\left(z^p I_p(n,\lambda)f(z), z^p I_p(n+1,\lambda)f(z), z^p I_p(n+2,\lambda)f(z); z\right) - 1| < M,$$

*then*

$$|z^p I_p(n,\lambda)f(z) - 1| < M.$$

EXAMPLE 4.4.1. If $f(z) \in \Sigma_p$ satisfies

$$|z^p I_p(n+1,\lambda)f(z) - 1| < M,$$

then

$$|z^p I_p(n,\lambda)f(z) - 1| < M.$$

This follows from Corollary 4.4.3, by taking $\phi(u, v, w; z) = v$.

COROLLARY 4.4.4. *If* $f(z) \in \Sigma_p$, $z^p I_p(n,\lambda)f(z) \in \mathcal{H}$, *then*

$$|z^p I_p(n+1,\lambda)f(z) - z^p I_p(n,\lambda)f(z)| < \frac{M}{\lambda - p} \Rightarrow |z^p I_p(n,\lambda)f(z) - 1| < M. \tag{4.4.14}$$

PROOF. Let $\phi(u, v, w; z) = v - u$ and $\Omega = h(U)$ where $h(z) = Mz/(\lambda - p), (M > 0)$. Since

$$\left|\phi\left(1 + Me^{i\theta}, 1 + Me^{i\theta} + \frac{k}{\lambda - p}Me^{i\theta}, 1 + Me^{i\theta} + \frac{L + [2(\lambda - p) + 1]kMe^{i\theta}}{(\lambda - p)^2}; z\right)\right|$$
$$= \frac{k}{\lambda - p}M \geq \frac{M}{\lambda - p},$$

$z \in U,\ \theta \in \mathcal{R}$ and $k \geq 1$, it follows that $\phi \in \Theta_I[\Omega, M]$. Hence by Corollary 4.4.2, the required result is obtained. □

The differential equation

$$zq'(z) - q(z) = \frac{Mz}{\lambda - p} \qquad (\lambda - p < M)$$

has a univalent solution $q(z) = 1 + Mz$. Theorem 4.4.4, shows that $q(z) = 1 + Mz$ is the best dominant of (4.4.14).

EXAMPLE 4.4.2. Let $M>0$, and $z^pI_p(n,\lambda)f(z)\in\mathcal{H}$. If $f(z)\in\Sigma_p$ satisfies

$$z^pI_p(n+2,\lambda)f(z)-z^pI_p(n+1,\lambda)f(z)\prec\frac{M(\lambda-p+1)z}{(\lambda-p)^2},$$

then

$$|z^pI_p(n,\lambda)f(z)-1|<M.$$

This follows from Corollary 4.4.2, by taking $\phi(u,v,w;z)=w-v$ and $\Omega=h(U)$ where

$$h(z)=\frac{M(\lambda-p+1)}{(\lambda-p)^2}z\quad(M>0).$$

DEFINITION 4.4.3. Let $\Omega$ be a set in $\mathcal{C}$ and $q(z)\in\mathcal{Q}_1\cap\mathcal{H}$. The class of admissible functions $\Theta_{I,1}[\Omega,q]$ consists of those functions $\phi:\mathcal{C}^3\times U\to\mathcal{C}$ that satisfy the admissibility condition

$$\phi(u,v,w;z)\notin\Omega$$

whenever

$$u=q(\zeta),\ v=\frac{1}{\lambda-p}\left((\lambda-p)q(\zeta)+\frac{k\zeta q'(\zeta)}{q(\zeta)}\right)\quad(q(\zeta)\neq0),$$

$$\Re\left\{\frac{(\lambda-p)v(w-v)}{v-u}-(\lambda-p)(2u-v)\right\}\geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

$z\in U$, $\zeta\in\partial U\setminus E(q)$ and $k\geq1$.

THEOREM 4.4.5. *Let $\phi\in\Theta_{I,1}[\Omega,q]$. If $f(z)\in\Sigma_p$ satisfies*

(4.4.15)
$$\left\{\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)},\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)},\frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)};z\right):z\in U\right\}\subset\Omega,$$

*then*

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\prec q(z).$$

PROOF. Define the function $p(z)$ in $U$ by

$$p(z):=\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}.\tag{4.4.16}$$

By making use of (4.4.4) and (4.4.16), a simple computation yields

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}=\frac{1}{\lambda-p}\left[(\lambda-p)p(z)+\frac{zp'(z)}{p(z)}\right].\tag{4.4.17}$$

Further computations show that

$$\frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}$$

$$= p(z) + \frac{1}{\lambda-p}\left[\frac{zp'(z)}{p(z)} + \frac{(\lambda-p)zp'(z) + \frac{zp'(z)}{p(z)} - (\frac{zp'(z)}{p(z)})^2 + \frac{z^2p''(z)}{p(z)}}{(\lambda-p)p(z) + \frac{zp'(z)}{p(z)}}\right]. \tag{4.4.18}$$

Define the transformations from $\mathcal{C}^3$ to $\mathcal{C}$ by

$$u = r,\ v = r + \frac{1}{\lambda-p}\left(\frac{s}{r}\right),$$

$$w = r + \frac{1}{\lambda-p}\left[\frac{s}{r} + \frac{(\lambda-p)s + \frac{s}{r} - (\frac{s}{r})^2 + \frac{t}{r}}{(\lambda-p)r + \frac{s}{r}}\right]. \tag{4.4.19}$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) \tag{4.4.20}$$

$$= \phi\left(r, r + \frac{1}{\lambda-p}\left(\frac{s}{r}\right),\ r + \frac{1}{\lambda-p}\left[\frac{s}{r} + \frac{(\lambda-p)s + \frac{s}{r} - (\frac{s}{r})^2 + \frac{t}{r}}{(\lambda-p)r + \frac{s}{r}}\right]; z\right).$$

Using equations (4.4.16), (4.4.17) and (4.4.18), from (4.4.20), it follows that

$$\psi(p(z), zp'(z), z^2p''(z); z) \tag{4.4.21}$$

$$= \phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right).$$

Hence (4.4.15) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.1. For this purpose, note that

$$\begin{aligned} \frac{s}{r} &= (\lambda-p)(v-r), \\ \frac{t}{r} &= (\lambda-p)^2 v(w-v) - \frac{s}{r}\left[(\lambda-p)v + 1 - \frac{2s}{r}\right], \\ \text{and thus}\quad \frac{t}{s}+1 &= (\lambda-p)\left[\frac{v(w-v)}{v-u} - (2u-v)\right]. \end{aligned}$$

Hence $\psi \in \Psi[\Omega, q]$ and by Theorem 3.1.1, $p(z) \prec q(z)$ or

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec q(z). \qquad \square$$

Theorem 4.4.6. *Let $\phi \in \Theta_{I,1}[h,q] := \Theta_{I,1}[h(U),q]$ with $q(0)=1.$ If $f(z) \in \Sigma_p$ satisfies*

$$\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) \prec h(z), \tag{4.4.22}$$

*then*

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec q(z).$$

Definition 4.4.4. Let $\Omega$ be a set in $\mathcal{C}$ and let $\Theta_{I,1}[\Omega,M] := \Theta_{I,1}[\Omega,q]$ where $q(z) = 1+Mz,\ M>0.$ The class of admissible functions $\Theta_{I,1}[\Omega,M]$ consists of those functions $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$ such that

(4.4.23)

$$\begin{aligned}\phi\Big(1+Me^{i\theta},\ 1+\frac{k+(\lambda-p)(1+Me^{i\theta})}{(\lambda-p)(1+Me^{i\theta})}Me^{i\theta},\ 1+\frac{k+(\lambda-p)(1+Me^{i\theta})}{(\lambda-p)(1+Me^{i\theta})}Me^{i\theta}\\ +\frac{(M+e^{-i\theta})\left[\big((\lambda-p)(1+Me^{i\theta})+1\big)kM+Le^{-i\theta}\right]-k^2M^2}{(\lambda-p)(1+Me^{i\theta})[(\lambda-p)(M+e^{-i\theta})^2+kMe^{-i\theta}]}; z\Big) \notin \Omega\end{aligned}$$

whenever $z \in U,\ \theta \in \mathcal{R}$, $\Re\left\{Le^{-i\theta}\right\} \geq kM(k-1)$ for all real $\theta$ and $k \geq 1$.

Corollary 4.4.5. *Let $\phi \in \Theta_{I,1}[\Omega,M].$ If $f(z) \in \Sigma_p$ satisfies*

$$\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) \in \Omega,$$

*then*

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec 1+Mz.$$

Corollary 4.4.6. *Let $\phi \in \Theta_{I,1}[M] := \Theta_{I,1}[\Omega,M]$ where $\Omega = \{\omega : |\omega-1| < M\}.$ If $f(z) \in \Sigma_p$ satisfies*

$$\left|\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) - 1\right| < M,$$

*then*

$$\left|\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} - 1\right| < M.$$

Corollary 4.4.7. *If $f(z) \in \Sigma_p$ satisfies*

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} - \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec \frac{M}{(\lambda-p)(1+M)}z, \tag{4.4.24}$$

*then*

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec 1+Mz.$$

This follows from Corollary 4.4.5, by taking $\phi(u, v, w; z) = v - u$ and $\Omega = h(U)$ where $h(z) = Mz/(\lambda - p)(1 + M)$.

By taking $M = 1$, Corollary 4.4.7 reduces to:

EXAMPLE 4.4.3. If $f(z) \in \Sigma_p$ satisfies

$$\left| \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} - \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \right| < \frac{1}{2(\lambda - p)},$$

then

$$\left| \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} - 1 \right| < 1.$$

## 4.5. SUPERORDINATION OF THE MULTIPLIER TRANSFORMATION

The dual problem of differential subordination, that is, differential superordination of the multiplier transformation is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

DEFINITION 4.5.1. Let $\Omega$ be a set in $\mathcal{C}$, $q(z) \in \mathcal{H}$ and $zq'(z) \neq 0$. The class of admissible functions $\Theta'_I[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z), \;\; v = \frac{1}{\lambda - p}\left[(\lambda - p)q(z) + \frac{zq'(z)}{m}\right],$$
$$\Re\left\{\frac{(\lambda - p)(w + u - 2v)}{v - u}\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

THEOREM 4.5.1. *Let* $\phi \in \Theta'_I[\Omega, q]$. *If* $f(z) \in \Sigma_p$, $z^p I_p(n, \lambda)f(z) \in \mathcal{Q}_1$ *and*

$$\phi\left(z^p I_p(n,\lambda)f(z), z^p I_p(n+1,\lambda)f(z), z^p I_p(n+2,\lambda)f(z); z\right)$$

*is univalent in* $U$, *then*

(4.5.1) $$\Omega \subset \{\phi\left(z^p I_p(n,\lambda)f(z), z^p I_p(n+1,\lambda)f(z), z^p I_p(n+2,\lambda)f(z); z\right) : z \in U\}$$

*implies*

$$q(z) \prec z^p I_p(n,\lambda)f(z).$$

PROOF. From (4.4.9) and (4.5.1), it follows that

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

In view of (4.4.7), the admissibility condition for $\phi \in \Theta_I'[\Omega, q]$, is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 3.1.2, $q(z) \prec p(z)$ or

$$q(z) \prec z^p I_p(n, \lambda) f(z). \qquad \square$$

THEOREM 4.5.2. *Let $q(z) \in \mathcal{H}$, h(z) be analytic in $U$ and $\phi \in \Theta_I'[h, q] := \Theta_I'[h(U), q]$. If $f(z) \in \Sigma_p$, $z^p I_p(n, \lambda) f(z) \in \mathcal{Q}_1$ and*

$$\phi\left(z^p I_p(n, \lambda) f(z), z^p I_p(n+1, \lambda) f(z), z^p I_p(n+2, \lambda) f(z); z\right)$$

*is univalent in $U$, then*

$$h(z) \prec \phi\left(z^p I_p(n, \lambda) f(z), z^p I_p(n+1, \lambda) f(z), z^p I_p(n+2, \lambda) f(z); z\right) \tag{4.5.2}$$

*implies*

$$q(z) \prec z^p I_p(n, \lambda) f(z).$$

Theorems 4.5.1 and 4.5.2 are used to obtain subordinants of differential superordination of the form (4.5.1) or (4.5.2). The following theorem proves the existence of the best subordinant of (4.5.2) for appropriate $\phi$.

THEOREM 4.5.3. *Let h(z) be analytic in $U$ and $\phi : \mathcal{C}^3 \times U \to \mathcal{C}$. Suppose that the differential equation*

$$\phi(q(z), zq'(z), z^2q''(z); z) = h(z)$$

*has a solution $q(z) \in \mathcal{Q}_1$. If $\phi \in \Theta_I'[h, q]$, $f(z) \in \Sigma_p$, $z^p I_p(n, \lambda) f(z) \in \mathcal{Q}_1$ and*

$$\phi\left(z^p I_p(n, \lambda) f(z), z^p I_p(n+1, \lambda) f(z), z^p I_p(n+2, \lambda) f(z); z\right)$$

*is univalent in $U$, then*

$$h(z) \prec \phi\left(z^p I_p(n, \lambda) f(z), z^p I_p(n+1, \lambda) f(z), z^p I_p(n+2, \lambda) f(z); z\right)$$

*implies*

$$q(z) \prec z^p I_p(n, \lambda) f(z),$$

*and $q(z)$ is the best subordinant.*

PROOF. The proof is similar to the proof of Theorem 4.4.4 and is therefore omitted. $\square$

Combining Theorems 4.4.2 and 4.5.2, the following sandwich-type theorem is obtained.

COROLLARY 4.5.1. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $U$, $h_2(z)$ be univalent function in $U$, and $q_2(z) \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Theta_I[h_2, q_2] \cap \Theta'_I[h_1, q_1]$. If $f(z) \in \Sigma_p$, $z^p I_p(n,\lambda) f(z) \in \mathcal{H} \cap \mathcal{Q}_1$ and*

$$\phi\left(z^p I_p(n,\lambda) f(z), z^p I_p(n+1,\lambda) f(z), z^p I_p(n+2,\lambda) f(z); z\right)$$

*is univalent in $U$, then*

$$h_1(z) \prec \phi\left(z^p I_p(n,\lambda) f(z), z^p I_p(n+1,\lambda) f(z), z^p I_p(n+2,\lambda) f(z); z\right) \prec h_2(z)$$

*implies*

$$q_1(z) \prec z^p I_p(n,\lambda) f(z) \prec q_2(z).$$

DEFINITION 4.5.2. Let $\Omega$ be a set in $\mathcal{C}$, $q(z) \neq 0,\ zq'(z) \neq 0$ and $q(z) \in \mathcal{H}$. The class of admissible functions $\Theta'_{I,1}[\Omega, q]$ consists of those functions $\phi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z),\ v = \frac{1}{\lambda - p}\left((\lambda - p) q(z) + \frac{zq'(z)}{mq(z)}\right),$$

$$\Re\left\{\frac{(\lambda - p) v (w - v)}{v - u} - (\lambda - p)(2u - v)\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

A dual result of Theorem 4.4.5 for the differential superordination is given below.

THEOREM 4.5.4. *Let* $\phi \in \Theta'_{I,1}[\Omega, q]$ *and* $q(z) \in \mathcal{H}$. *If* $f(z) \in \Sigma_p$, $\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \in \mathcal{Q}_1$, *and*

$$\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right)$$

*is univalent in* $U$, *then*

$$\Omega \subset \left\{\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) : z \in U\right\} \tag{4.5.3}$$

*implies*

$$q(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}.$$

PROOF. From (4.4.21) and (4.5.3), it follows that

$$\Omega \subset \left\{\phi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in U\right\}.$$

In view of (4.4.19), the admissibility condition for $\phi$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 3.1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}. \qquad \square$$

THEOREM 4.5.5. *Let* $q(z) \in \mathcal{H}$, $h(z)$ *be analytic in* $U$ *and* $\phi \in \Theta'_{I,1}[h, q] := \Theta'_{I,1}[h(U), q]$. *If* $f(z) \in \Sigma_p$, $\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \in \mathcal{Q}_1$, *and*

$$\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right)$$

*is univalent in* $U$, *then*

$$h(z) \prec \phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) \tag{4.5.4}$$

*implies*

$$q(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}.$$

Combining Theorems 4.4.6 and 4.5.5, the following sandwich-type theorem is obtained.

COROLLARY 4.5.2. *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $U$, $h_2(z)$ be univalent in $U$, and $q_2(z) \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Theta_{I,1}[h_2, q_2] \cap \Theta'_{I,1}[h_1, q_1]$. If $f(z) \in \Sigma_p$, $\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \in \mathcal{H} \cap \mathcal{Q}_1$, and*

$$\phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right)$$

*is univalent in $U$, then*

$$h_1(z) \prec \phi\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}; z\right) \prec h_2(z)$$

*implies*

$$q_1(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec q_2(z).$$

# CHAPTER 5

# SUBORDINATION BY CONVEX FUNCTIONS

## 5.1. INTRODUCTION

Let $f(z)$ be given by (1.1.1) and a fixed function $g(z)$ be given by

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \tag{5.1.1}$$

with $g_n \geq g_2 > 0$ $(n \geq 2)$, $\gamma < 1$, and let

$$T_g(\gamma) := \left\{ f(z) \in \mathcal{A} : \sum_{n=2}^{\infty} |a_n g_n| \leq 1 - \gamma \right\}.$$

The class $T_g(\gamma)$ includes as its special cases, various other classes that were considered in several earlier works. In particular, for $\gamma = \alpha$ and $g_n = n - \alpha$, the class $T_g(\gamma)$ becomes $TS^*(\alpha)$ of starlike functions of order $\alpha$ with negative coefficients introduced by Silverman [**98**]. For $\gamma = \alpha$ and $g_n = n(n-\alpha)$, the class $T_g(\gamma)$ reduces to $TC(\alpha)$ introduced by Silverman [**98**]. For these classes, Silverman [**98**] proved that $TS^*(\alpha) \subseteq S^*(\alpha)$ and $TC(\alpha) \subseteq C(\alpha)$.

By using convolution, Ruscheweyh [**92**] defined the operator

$$D^\alpha f(z) := \frac{z}{(1-z)^{\alpha+1}} * f(z), \quad (\alpha > -1).$$

Let $R_\alpha(\beta)$ denote the class of functions $f(z)$ in $\mathcal{A}$ that satisfy the inequality

$$\Re \frac{D^{\alpha+1} f(z)}{D^\alpha f(z)} > \frac{\alpha + 2\beta}{2(\alpha+1)}, \quad (\alpha \geq 0, \ 0 \leq \beta < 1, \ z \in U).$$

Al-Amiri [**3**] called the functions in this class as prestarlike functions of order $\alpha$ and type $\beta$. Let $H_\alpha(\beta)$ denote the class of functions $f(z)$ given by (1.1.1) whose coefficients satisfy the condition

$$\sum_{n=2}^{\infty} (2n + \alpha - 2\beta) C(\alpha, n) |a_n| \leq 2 + \alpha - 2\beta, \quad (\alpha \geq 0, \ 0 \leq \beta < 1), \tag{5.1.2}$$

where

$$C(\alpha,n) := \prod_{k=2}^{n} \frac{(k+\alpha-1)}{(n-1)!}, \quad (n=2,3,\cdots).$$

Al-Amiri [**3**] proved that $H_\alpha(\beta) \subseteq R_\alpha(\beta)$. By taking $g_n = (2n+\alpha-2\beta)C(\alpha,n)$ and $\gamma = 2\beta - 1 - \alpha$, we see that $H_\alpha(\beta) := T_g(\gamma)$.

For functions in the class $H_\alpha(\beta)$, Attiya [**9**] proved the following:

THEOREM 5.1.1. [**9**, Theorem 2.1, p. 3] *If $f(z) \in H_\alpha(\beta)$ and $h(z) \in \mathcal{C}$, then*

$$\frac{(4+\alpha-2\beta)(1+\alpha)}{2\left[\alpha+(2+\alpha)(3+\alpha-2\beta)\right]}(f*h)(z) \prec h(z) \tag{5.1.3}$$

*and*

$$\Re(f(z)) > -\frac{\alpha+(2+\alpha)(3+\alpha-2\beta))}{(4+\alpha-2\beta)(1+\alpha)}. \tag{5.1.4}$$

*The constant factor*

$$\frac{(4+\alpha-2\beta)(1+\alpha)}{2\left[\alpha+(2+\alpha)(3+\alpha-2\beta)\right]} \tag{5.1.5}$$

*in the subordination result (5.1.3) cannot be replaced by a larger number.*

Owa and Srivastava [**72**] as well as Owa and Nishiwaki [**68**] studied the subclasses $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$ consisting of functions $f \in \mathcal{A}$ satisfying

$$\sum_{n=2}^{\infty} \left[n-\lambda+|n+\lambda-2\alpha|\right]|a_n| \le 2(\alpha-1), \quad (\alpha>1, 0\le\lambda\le 1)$$

and

$$\sum_{n=2}^{\infty} n\left[n-\lambda+|n+\lambda-2\alpha|\right]|a_n| \le 2(\alpha-1), \quad (\alpha>1, 0\le\lambda\le 1)$$

respectively. These are special cases of $T_g(\gamma)$, with $g_n = n-\lambda+|n+\lambda-2\alpha|$, $\gamma = 3-2\alpha$, and $g_n = n\left(n-\lambda+|n+\lambda-2\alpha|\right)$, $\gamma = 3-2\alpha$, respectively. For the class $\mathcal{M}^*(\alpha)$, and $\mathcal{N}^*(\alpha)$, Srivastava and Attiya [**102**] proved the following:

THEOREM 5.1.2. [**102**, Theorem 1, p. 3] *Let $f(z) \in \mathcal{M}^*(\alpha)$. Then for any function $h(z) \in C$ and $z \in U$, we have*

$$\frac{2-\lambda+|2+\lambda-2\alpha|}{2\left[2\alpha-\lambda+|2+\lambda-2\alpha|\right]}(f*h)(z) \prec h(z) \tag{5.1.6}$$

*and*

$$\Re(f(z)) > -\frac{2\alpha-\lambda+|2+\lambda-2\alpha|}{[(2-\lambda)+|2+\lambda-2\alpha|]}. \tag{5.1.7}$$

*The constant factor*

$$\frac{2-\lambda+|2+\lambda-2\alpha|}{2\,[2\alpha-\lambda+|2+\lambda-2\alpha|]} \tag{5.1.8}$$

*in the subordination result (5.1.6) cannot be replaced by a larger number.*

THEOREM 5.1.3. [**102**, Theorem 2, p. 5] *Let $f(z) \in \mathcal{N}^*(\alpha)$. Then for any function $h(z) \in C$ and $z \in U$, we have*

$$\frac{2-\lambda+|2+\lambda-2\alpha|}{2\,[\alpha+1-\lambda+|2+\lambda-2\alpha|]}(f*h)(z) \prec h(z) \tag{5.1.9}$$

*and*

$$\Re(f(z)) > -\frac{\alpha+1-\lambda+|2+\lambda-2\alpha|}{[(2-\lambda)+|2+\lambda-2\alpha|]}.$$

*The constant factor*

$$\frac{2-\lambda+|2+\lambda-2\alpha|}{2\,[\alpha+1-\lambda+|2+\lambda-2\alpha|]}$$

*in the subordination result (5.1.9) cannot be replaced by a larger number.*

In this chapter, Theorems 5.1.1 and 5.1.2 are unified for the class $T_g(\gamma)$. Relevant connections of these results with several earlier investigations are also indicated.

A sequence $(b_n)_1^\infty$ of complex numbers is said to be a *subordinating factor sequence,* if for every convex univalent function $f(z)$ given by (1.1.1), then

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z).$$

The following result on subordinating factor sequence is needed to obtain the main result.

THEOREM 5.1.4. [**112**, Theorem 2, p. 690] *A sequence $(b_n)_1^\infty$ of complex numbers is a subordinating factor sequence if and only if*

$$\Re\left(1+2\sum_{n=1}^{\infty} b_n z^n\right) > 0.$$

Theorem 5.2.1. *If $f(z) \in T_g(\gamma)$ and $h(z) \in C$, then*

$$\frac{g_2}{2(g_2+1-\gamma)}(f*h)(z) \prec h(z) \tag{5.2.1}$$

*and*

$$\Re(f(z)) > -\frac{g_2+1-\gamma}{g_2} \quad (z \in U). \tag{5.2.2}$$

*The constant factor*

$$\frac{g_2}{2(g_2+1-\gamma)}$$

*in the subordination result (5.2.1) cannot be replaced by a larger number.*

Proof. Let $G(z) = z + \sum_{n=2}^{\infty} g_2 z^n$. Since $T_g(\gamma) \subseteq T_G(\gamma)$, it is enough to prove the result for the class $T_G(\gamma)$. Let $f(z) \in T_G(\gamma)$ and suppose that

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in C.$$

In this case,

$$\frac{g_2}{2(g_2+1-\gamma)}\left(f*h\right)(z) = \frac{g_2}{2(g_2+1-\gamma)}\left(z + \sum_{n=2}^{\infty} c_n a_n z^n\right).$$

Observe that the subordination result (5.2.1) holds true if

$$\left(\frac{g_2}{2(g_2+1-\gamma)}a_n\right)_1^{\infty}$$

is a subordinating factor sequence (with of course, $a_1 = 1$). In view of Theorem 5.1.4, this is equivalent to the condition that

$$\Re\left\{1 + \sum_{n=1}^{\infty} \frac{g_2}{g_2+1-\gamma} a_n z^n\right\} > 0. \tag{5.2.3}$$

Since $g_n \geq g_2 > 0$ for $n \geq 2$,

$$\begin{aligned}
&\Re\left\{1 + \frac{g_2}{g_2 + 1 - \gamma}\sum_{n=1}^{\infty} a_n z^n\right\} \\
&= \Re\left\{1 + \frac{g_2}{g_2 + 1 - \gamma} z + \frac{1}{g_2 + 1 - \gamma}\sum_{n=2}^{\infty} g_2 a_n z^n\right\} \\
&\geq 1 - \left\{\frac{g_2}{g_2 + 1 - \gamma} r + \frac{1}{g_2 + 1 - \gamma}\sum_{n=2}^{\infty} |g_2 a_n| r^n\right\} \\
&> 1 - \left\{\frac{g_2}{g_2 + 1 - \gamma} r + \frac{1 - \gamma}{g_2 + 1 - \gamma} r\right\} \\
&> 0 \quad (|z| = r < 1).
\end{aligned}$$

Thus (5.2.3) holds true in $U$, and proves (5.2.1). The inequality (5.2.2) follows by taking $h(z) = z/(1 - z)$ in (5.2.1).

Now consider the function

$$F(z) = z - \frac{1 - \gamma}{g_2} z^2 \quad (\gamma < 1).$$

Clearly $F(z) \in T_g(\gamma)$. For this function $F(z)$, (5.2.1) becomes

$$\frac{g_2}{2(g_2 + 1 - \gamma)} F(z) \prec \frac{z}{1 - z}.$$

It is easily verified that

$$\min\left\{\Re\left(\frac{g_2}{2(g_2 + 1 - \gamma)} F(z)\right)\right\} = -\frac{1}{2} \quad (z \in U).$$

Therefore the constant

$$\frac{g_2}{2(g_2 + 1 - \gamma)}$$

cannot be replaced by any larger one. □

Corollary 5.2.1. *If $f(z) \in TS^*(\alpha)$ and $h(z) \in C$, then*

$$\frac{2 - \alpha}{2(3 - 2\alpha)} (f * h)(z) \prec h(z) \tag{5.2.4}$$

*and*

$$\Re(f(z)) > -\frac{3 - 2\alpha}{2 - \alpha} \quad (z \in U).$$

*The constant factor*

$$\frac{2 - \alpha}{2(3 - 2\alpha)}$$

*in the subordination result (5.2.4) cannot be replaced by a larger number.*

REMARK 5.2.1. The case $\alpha = 0$ in Corollary 5.2.1 was obtained by Singh [**100**].

COROLLARY 5.2.2. *If $f(z) \in TC(\alpha)$ and $h(z) \in C$, then*

$$\frac{2-\alpha}{5-3\alpha}(f * h)(z) \prec h(z) \tag{5.2.5}$$

*and*

$$\Re(f(z)) > -\frac{5-3\alpha}{2(2-\alpha)} \quad (z \in U).$$

*The constant factor*

$$\frac{2-\alpha}{5-3\alpha}$$

*in the subordination result (5.2.5) cannot be replaced by a larger one.*

REMARK 5.2.2. Theorem 5.1.1 is obtained by taking $\gamma = 2\beta - 1 - \alpha$ and

$$g_n = (2n + \alpha - 2\beta) \prod_{k=2}^{n} \frac{(k + \alpha - 1)}{(n-1)!} \quad (n = 2, 3 \cdots, \quad \alpha > 0, \quad 0 \leq \beta < 1)),$$

in Theorem 5.2.1. Similarly by putting $\gamma = 3 - 2\alpha$ and

$$g_n = n - \lambda + |n + \lambda - 2\alpha| \quad (n = 2, 3 \cdots, \quad \alpha > 1, \quad 0 \leq \lambda \leq 1),$$

in Theorem 5.2.1 yields Theorem 5.1.2. By taking $\gamma = 3 - 2\alpha$ and

$$g_n = n(n - \lambda + |n + \lambda - 2\alpha|) \quad (n = 2, 3 \cdots, \quad \alpha > 1, \quad 0 \leq \lambda \leq 1),$$

in Theorem 5.2.1 yields Theorem 5.1.3.

## CHAPTER 6

# COEFFICIENT BOUNDS FOR $p$-VALENT FUNCTIONS

## 6.1. INTRODUCTION

Let $\varphi$ be an analytic function with positive real part in the unit disk $U$ with $\varphi(0) = 1$ and $\varphi'(0) > 0$, and maps $U$ onto a region starlike with respect to $1$ and symmetric with respect to the real axis. Define the class $S^*_{b,p}(\varphi)$ to be the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ satisfying

$$1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) \prec \varphi(z) \quad (z \in U \quad \text{and} \quad b \in \mathcal{C} \setminus \{0\}).$$

As special cases, let

$$S^*_p(\varphi) := S^*_{1,p}(\varphi), \quad S^*_b(\varphi) := S^*_{b,1}(\varphi), \quad S^*(\varphi) := S^*_{1,1}(\varphi).$$

For a fixed analytic function $g \in \mathcal{A}_p$ with positive coefficients, define the class $S^*_{b,p,g}(\varphi)$ to be the class of all functions $f \in \mathcal{A}_p$ satisfying $f * g \in S^*_{b,p}(\varphi)$. This class includes as special cases several other classes studied in the literature. For example, when $g(z) = z^p + \sum_{n=p+1}^{\infty} \frac{n}{p} z^n$, the class $S^*_{b,p,g}(\varphi)$ reduces to the class $C_{b,p}(\varphi)$ consisting of functions $f \in \mathcal{A}_p$ satisfying

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \quad (z \in U \quad \text{and} \quad b \in \mathcal{C} \setminus \{0\}).$$

The classes $S^*(\varphi)$ and $C(\varphi) := C_{1,1}(\varphi)$ were introduced and studied by Ma and Minda [**54**].

Define the class $R_{b,p}(\varphi)$ to be the class of all functions $f \in \mathcal{A}_p$ satisfying

$$1 + \frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}} - 1\right) \prec \varphi(z) \quad (z \in U \quad \text{and} \quad b \in \mathcal{C} \setminus \{0\}),$$

and for a fixed function $g$ with positive coefficients, let $R_{b,p,g}(\varphi)$ be the class of all functions $f \in \mathcal{A}_p$ satisfying $f * g \in R_{b,p}(\varphi)$.

Several authors [**36, 45, 73, 87, 84, 83**] have studied the classes of analytic functions defined by using the expression $\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}$. We shall also consider a class defined by the corresponding quantity for $p$-valent functions. Define the class $S_p^*(\alpha, \varphi)$ to be the class of all functions $f \in \mathcal{A}_p$ satisfying

$$\frac{1+\alpha(1-p)}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \quad (z \in U \quad \text{and} \quad \alpha \geq 0).$$

Note that $S_p^*(0, \varphi)$ is the class $S_p^*(\varphi)$ and $S^*(\alpha, \varphi) := S_1^*(\alpha, \varphi)$. Let $S_{p,g}^*(\alpha, \varphi)$ be the class of all functions $f \in \mathcal{A}_p$ for which $f * g \in S_p^*(\alpha, \varphi)$. Also let $S_{p,g}^*(\varphi) := S_{p,g}^*(0, \varphi)$.

Let $M_p(\alpha, \varphi)$ be the class of $p$-valent $\alpha$-convex functions with respect to $\varphi$ satisfying

$$\frac{1-\alpha}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \quad (z \in U \quad \text{and} \quad \alpha \geq 0).$$

Let $M(\alpha, \varphi) := M_1(\alpha, \varphi)$. Further, let $L_p^M(\alpha, \varphi)$ be the class of functions $f \in \mathcal{A}_p$ satisfying

$$\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \quad (z \in U \quad \text{and} \quad \alpha \geq 0).$$

Functions in this class are called logarithmic $p$-valent $\alpha$-convex functions with respect to $\varphi$.

In this chapter, Fekete-Szegö inequalities and bounds for the coefficient $a_{p+3}$ for the classes $S_p^*(\varphi)$ and $S_{p,g}^*(\varphi)$ are obtained. These results are then extended to the other classes defined earlier.

Let $\Upsilon$ be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + \cdots \tag{6.1.1}$$

in the unit disk $U$ satisfying the condition $|w(z)| < 1$. For the proof of the main results, the following lemmas are needed.

LEMMA 6.1.1. *If $w \in \Upsilon$, then*

$$|w_2 - tw_1^2| \leq \begin{cases} -t & \text{if } t \leq -1 \\ 1 & \text{if } -1 \leq t \leq 1 \\ t & \text{if } t \geq 1 \end{cases}$$

*When $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $w(z) = z\frac{\lambda+z}{1+\lambda z}(0 \le \lambda \le 1)$ or one of its rotations while for $t = 1$, equality holds if and only if $w(z) = -z\frac{\lambda+z}{1+\lambda z}$ $(0 \le \lambda \le 1)$ or one of its rotations.*

*Also the sharp upper bound above can be improved as follows when $-1 < t < 1$:*

$$|w_2 - tw_1^2| + (t+1)|w_1|^2 \le 1 \quad (-1 < t \le 0)$$

*and*

$$|w_2 - tw_1^2| + (1-t)|w_1|^2 \le 1 \quad (0 < t < 1).$$

Lemma 6.1.1 is a reformulation of a lemma by Ma and Minda [**54**].

LEMMA 6.1.2. [**39**, inequality 7, p. 10] *If $w \in \Upsilon$, then for any complex number $t$,*

$$|w_2 - tw_1^2| \le \max\{1;\ |t|\}.$$

*The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.*

LEMMA 6.1.3. [**82**] *If $w \in \Upsilon$ then for any real numbers $q_1$ and $q_2$, the following sharp estimate holds:*

$$|w_3 + q_1w_1w_2 + q_2w_1^3| \le H(q_1, q_2) \tag{6.1.2}$$

*where*

$$H(q_1,q_2) = \begin{cases} 1 & \text{for } (q_1,q_2) \in D_1 \cup D_2 \\ |q_2| & \text{for } (q_1,q_2) \in \cup_{k=3}^{7} D_k \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_8 \cup D_9 \\ \frac{1}{3}q_2\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{10} \cup D_{11} - \{\pm 2, 1\} \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{12} \end{cases}$$

*The extremal functions, up to rotations, are of the form*

$$w(z) = z^3, \quad w(z) = z, \quad w(z) = w_0(z) = \frac{z([(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1] - \varepsilon_1\varepsilon_2 z)}{1 - [(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2]z},$$

$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z}, \quad w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2 z}$$

$$|\varepsilon_1| = |\varepsilon_2| = 1, \quad \varepsilon_1 = t_0 - e^{\frac{-i\theta_0}{2}}(a \mp b), \quad \varepsilon_2 = -e^{\frac{-i\theta_0}{2}}(ia \pm b),$$

$$a = t_0 \cos\frac{\theta_0}{2}, \quad b = \sqrt{1 - t_0^2 \sin^2\frac{\theta_0}{2}}, \quad \lambda = \frac{b \pm a}{2b}$$

$$t_0 = \left[\frac{2q_2(q_1^2+2) - 3q_1^2}{3(q_2-1)(q_1^2-4q_2)}\right]^{\frac{1}{2}}, \quad t_1 = \left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}},$$

$$t_2 = \left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}}, \quad \cos\frac{\theta_0}{2} = \frac{q_1}{2}\left[\frac{q_2(q_1^2+8) - 2(q_1^2+2)}{2q_2(q_1^2+2) - 3q_1^2}\right].$$

*The sets $D_k, \quad k = 1, 2, \cdots, 12$, are defined as follows*

$$\begin{aligned}
D_1 &= \{(q_1, q_2) : |q_1| \le \tfrac{1}{2}, |q_2| \le 1\}, \\
D_2 &= \{(q_1, q_2) : \tfrac{1}{2} \le |q_1| \le 2, \tfrac{4}{27}(|q_1|+1)^3 - (|q_1|+1) \le q_2 \le 1\}, \\
D_3 &= \{(q_1, q_2) : |q_1| \le \tfrac{1}{2}, q_2 \le -1\}, \\
D_4 &= \{(q_1, q_2) : |q_1| \ge \tfrac{1}{2}, q_2 \le -\tfrac{2}{3}(|q_1|+1)\}, \\
D_5 &= \{(q_1, q_2) : |q_1| \le 2, q_2 \ge 1\}, \\
D_6 &= \{(q_1, q_2) : 2 \le |q_1| \le 4, q_2 \ge \tfrac{1}{12}(q_1^2+8)\}, \\
D_7 &= \{(q_1, q_2) : |q_1| \ge 4, q_2 \ge \tfrac{2}{3}(|q_1|-1)\}, \\
D_8 &= \{(q_1, q_2) : \tfrac{1}{2} \le |q_1| \le 2, -\tfrac{2}{3}(|q_1|+1) \le q_2 \le \tfrac{4}{27}(|q_1|+1)^3 - (|q_1|+1)\}, \\
D_9 &= \{(q_1, q_2) : |q_1| \ge 2, -\tfrac{2}{3}(|q_1|+1) \le q_2 \le \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\}, \\
D_{10} &= \{(q_1, q_2) : 2 \le |q_1| \le 4, \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \le q_2 \le \tfrac{1}{12}(q_1^2+8)\}, \\
D_{11} &= \{(q_1, q_2) : |q_1| \ge 4, \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \le q_2 \le \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\}, \\
D_{12} &= \{(q_1, q_2) : |q_1| \ge 4, \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4} \le q_2 \le \tfrac{2}{3}(|q_1|-1)\}.
\end{aligned}$$

## 6.2. COEFFICIENT BOUNDS

By making use of the Lemmas 6.1.1–6.1.3, the following bounds for the class $S_p^*(\varphi)$ is obtained.

THEOREM 6.2.1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, *and*

$$\sigma_1 := \frac{B_2 - B_1 + pB_1^2}{2pB_1^2}, \quad \sigma_2 := \frac{B_2 + B_1 + pB_1^2}{2pB_1^2}, \quad \sigma_3 := \frac{B_2 + pB_1^2}{2pB_1^2}.$$

*If* $f(z)$ *given by (1.1.2) belongs to* $S_p^*(\varphi)$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} \frac{p}{2}\left(B_2 + (1-2\mu)pB_1^2\right) & \text{if} \quad \mu \leq \sigma_1, \\ \frac{pB_1}{2} & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{p}{2}\left(B_2 + (1-2\mu)pB_1^2\right) & \text{if} \quad \mu \geq \sigma_2. \end{cases} \tag{6.2.1}$$

*Further, if* $\sigma_1 \leq \mu \leq \sigma_3$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{1}{2pB_1}\left(1 - \frac{B_2}{B_1} + (2\mu - 1)pB_1\right)|a_{p+1}|^2 \leq \frac{pB_1}{2}. \tag{6.2.2}$$

*If* $\sigma_3 \leq \mu \leq \sigma_2$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{1}{2pB_1}\left(1 + \frac{B_2}{B_1} - (2\mu - 1)pB_1\right)|a_{p+1}|^2 \leq \frac{pB_1}{2}. \tag{6.2.3}$$

*For any complex number* $\mu$,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{pB_1}{2}\max\left\{1; \left|\frac{B_2}{B_1} + (1-2\mu)\, pB_1\right|\right\}. \tag{6.2.4}$$

*Further,*

$$|a_{p+3}| \leq \frac{pB_1}{3} H(q_1, q_2) \tag{6.2.5}$$

*where* $H(q_1, q_2)$ *is as defined in Lemma 6.1.3,*

$$q_1 := \frac{4B_2 + 3pB_1^2}{2B_1} \text{ and } q_2 := \frac{2B_3 + 3pB_1B_2 + p^2B_1^3}{2B_1}.$$

*These results are sharp.*

PROOF. If $f(z) \in S_p^*(\varphi)$, then there is an analytic function

$$w(z) = w_1 z + w_2 z^2 + \cdots \in \Upsilon$$

such that

$$\frac{zf'(z)}{pf(z)} = \varphi(w(z)). \tag{6.2.6}$$

Since

$$\frac{zf'(z)}{pf(z)} = 1 + \frac{a_{p+1}}{p} z + \left(-\frac{a_{p+1}^2}{p} + \frac{2a_{p+2}}{p}\right) z^2 + \left(\frac{3a_{p+3}}{p} - \frac{3}{p}a_{p+1}a_{p+2} + \frac{a_{p+1}^3}{p}\right) z^3 + \cdots,$$

the equation (6.2.6) yields

$$a_{p+1} = pB_1 w_1, \tag{6.2.7}$$

$$a_{p+2} = \frac{1}{2}\left\{pB_1 w_2 + p(B_2 + pB_1^2)w_1^2\right\} \tag{6.2.8}$$

and

$$a_{p+3} = \frac{pB_1}{3}\left\{w_3 + \frac{4B_2 + 3pB_1^2}{2B_1}w_1 w_2 + \frac{2B_3 + 3pB_1B_2 + p^2B_1^3}{2B_1}w_1^3\right\}. \tag{6.2.9}$$

The equations (6.2.7) and (6.2.8) yields

$$a_{p+2} - \mu a_{p+1}^2 = \frac{pB_1}{2}\left\{w_2 - v w_1^2\right\} \tag{6.2.10}$$

where

$$v := \left[pB_1(2\mu - 1) - \frac{B_2}{B_1}\right].$$

The results (6.2.1)–(6.2.3) are established by an application of Lemma 6.1.1, while inequality (6.2.4) follows from Lemma 6.1.2 while (6.2.5) from Lemma 6.1.3.

To show that the bounds in (6.2.1)–(6.2.3) are sharp, let the functions $K_{\varphi n}$ $(n = 2, 3, \ldots)$ be defined by

$$\frac{zK'_{\varphi n}(z)}{pK_{\varphi n}(z)} = \varphi(z^{n-1}), \quad K_{\varphi n}(0) = 0 = [K_{\varphi n}]'(0) - 1$$

and the functions $F_\lambda$ and $G_\lambda$ $(0 \leq \lambda \leq 1)$ be defined by

$$\frac{zF'_\lambda(z)}{pF_\lambda(z)} = \varphi\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \quad F_\lambda(0) = 0 = F'_\lambda(0) - 1$$

and

$$\frac{zG'_\lambda(z)}{pG_\lambda(z)} = \varphi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right), \quad G_\lambda(0) = 0 = G'_\lambda(0) - 1.$$

Clearly the functions $K_{\varphi n}, F_\lambda, G_\lambda \in S_p^*(\varphi)$. We write $K_\varphi := K_{\varphi 2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then equality holds if and only if $f$ is $K_\varphi$ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then equality holds if and only if $f$ is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$ then equality holds if and only if $f$ is $F_\lambda$ or one of its rotations. Equality holds for $\mu = \sigma_2$ if and only if $f$ is $G_\lambda$ or one of its rotations. $\square$

COROLLARY 6.2.1. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, and let*

$$\sigma_1 := \frac{g_{p+1}^2}{g_{p+2}} \frac{B_2 - B_1 + pB_1^2}{2pB_1^2}, \quad \sigma_2 := \frac{g_{p+1}^2}{g_{p+2}} \frac{B_2 + B_1 + pB_1^2}{2pB_1^2}, \quad \sigma_3 := \frac{g_{p+1}^2}{g_{p+2}} \frac{B_2 + pB_1^2}{2pB_1^2}.$$

*If $f(z)$ given by (1.1.2) belongs to $S^*_{p,g}(\varphi)$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \begin{cases} \frac{p}{2g_{p+2}} \left(B_2 + \left(1 - 2\frac{g_{p+2}}{g_{p+1}^2}\mu\right) B_1^2\right) & \text{if} \quad \mu \le \sigma_1, \\ \frac{pB_1}{2g_{p+2}} & \text{if} \quad \sigma_1 \le \mu \le \sigma_2, \\ -\frac{p}{2g_{p+2}} \left(B_2 + \left(1 - 2\frac{g_{p+2}}{g_{p+1}^2}\mu\right) B_1^2\right) & \text{if} \quad \mu \ge \sigma_2. \end{cases}$$

*Further, if $\sigma_1 \le \mu \le \sigma_3$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{g_{p+1}^2}{2g_{p+2}pB_1} \left(1 - \frac{B_2}{B_1} + \left(2\frac{g_{p+2}}{g_{p+1}^2}\mu - 1\right) B_1\right) |a_{p+1}|^2 \le \frac{pB_1}{2g_{p+2}}.$$

*If $\sigma_3 \le \mu \le \sigma_2$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{g_{p+1}^2}{2g_{p+2}pB_1} \left(1 + \frac{B_2}{B_1} - \left(2\frac{g_{p+2}}{g_{p+1}^2}\mu - 1\right) B_1\right) |a_{p+1}|^2 \le \frac{pB_1}{2g_{p+2}}.$$

*For any complex number $\mu$,*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{pB_1}{2g_{p+2}} \max\left\{1; \left|\frac{B_2}{B_1} + \left(1 - 2\frac{g_{p+2}}{g_{p+1}^2}\mu\right) B_1\right|\right\}.$$

*Further,*

$$|a_{p+3}| \le \frac{pB_1}{3g_{p+3}} H(q_1, q_2) \tag{6.2.11}$$

*where $H(q_1, q_2)$ is as defined in Lemma 6.1.3,*

$$q_1 := \frac{4B_2 + 3pB_1^2}{2B_1} \text{ and } q_2 := \frac{2B_3 + 3pB_1B_2 + p^2B_1^3}{2B_1}.$$

*These results are sharp.*

THEOREM 6.2.2. *Let $\varphi$ be as in Theorem 6.2.1. If $f(z)$ given by (1.1.2) belongs to $S^*_{b,p,g}(\varphi)$, then for any complex number $\mu$, we have*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{p|b|B_1}{2g_{p+2}} \max\left\{1; \left|\frac{B_2}{B_1} + bp\left(1 - 2\frac{g_{p+1}^2}{g_{p+2}}\mu\right) B_1\right|\right\}.$$

*The result is sharp.*

PROOF. The proof is similar to the proof of Theorem 6.2.1. $\square$

For the class $R_{b,p}(\varphi)$, the following result is obtained.

THEOREM 6.2.3. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *If* $f(z)$ *given by (1.1.2) belongs to* $R_{b,p}(\varphi)$*, then for any complex number* $\mu$*,*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq |\gamma| \max\{1; |v|\}, \tag{6.2.12}$$

*where*

$$v := \mu \frac{pbB_1(p+2)}{(p+1)^2} - \frac{B_2}{B_1} \quad \text{and} \quad \gamma := \frac{bpB_1}{2+p}.$$

*Further,*

$$|a_{p+3}| \leq \frac{|b|pB_1}{3+p} H(q_1, q_2) \tag{6.2.13}$$

*where* $H(q_1, q_2)$ *is as defined in Lemma 6.1.3,*

$$q_1 := \frac{2B_2}{B_1} \text{ and } q_2 := \frac{B_3}{B_1}.$$

*These results are sharp.*

PROOF. A computation shows that

$$1 + \frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}} - 1\right) = 1 + \frac{p+1}{bp} a_{p+1} z + \frac{p+2}{bp} a_{p+2} z^2 + \frac{p+3}{bp} a_{p+3} z^3 + \cdots.$$

Thus

$$a_{p+2} - \mu a_{p+1}^2 = \frac{bpB_1}{p+2}\left\{w_2 - v w_1^2\right\} = \gamma\left\{w_2 - v w_1^2\right\} \tag{6.2.14}$$

where $v := \left[\frac{bpB_1\mu(2+p)}{(p+1)^2} - \frac{B_2}{B_1}\right]$ and $\gamma := \frac{bpB_1}{p+2}$. The result now follows from Lemma 6.1.2 and Lemma 6.1.3. □

REMARK 6.2.1. When $p = 1$ and

$$\varphi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B \leq A \leq 1),$$

inequality (6.2.12) reduces to the inequality [**23**, Theorem 4, p. 894]

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{|b|B_1}{3} \max\left\{1; \left|\frac{4B + 3b\mu B_1}{4}\right|\right\}.$$

For the class $R_{b,p,g}(\varphi)$, the following result is obtained.

THEOREM 6.2.4. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *If* $f(z)$ *given by (1.1.2) belongs to* $R_{b,p}(\varphi)$, *then for any complex number* $\mu$,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq |\gamma| \max\{1; |v|\}$$

*where*

$$v := \frac{\mu g_{p+1}^2}{g_{p+2}} \frac{pbB_1(p+2)}{(p+1)^2} - \frac{B_2}{B_1} \quad and \quad \gamma := \frac{bpB_1}{g_{p+2}(2+p)}.$$

*Further,*

$$|a_{p+3}| \leq \frac{|b|pB_1}{(3+p)g_{p+3}} H(q_1, q_2) \tag{6.2.15}$$

*where* $H(q_1, q_2)$ *is as defined in Lemma 6.1.3,*

$$q_1 := \frac{2B_2}{B_1} \text{ and } q_2 := \frac{B_3}{B_1}.$$

*These results are sharp.*

For the class $S_p^*(\alpha, \varphi)$, the following coefficient bounds are obtained.

THEOREM 6.2.5. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *Further let*

$$\sigma_1 := \frac{(1+\alpha(p+1))[(1+\alpha(p+1))(B_2 - B_1) + pB_1^2]}{2pB_1^2},$$

$$\sigma_2 := \frac{(1+\alpha(p+1))[(1+\alpha(p+1))(B_2 + B_1) + pB_1^2]}{2pB_1^2},$$

$$\sigma_3 := \frac{(1+\alpha(p+1))[(1+\alpha(p+1))B_2 + pB_1^2]}{2pB_1^2},$$

$$v := \frac{2\mu pB_1(p\alpha + 2\alpha + 1)}{(1+\alpha(p+1))^2} - \frac{pB_1}{1+\alpha(p+1)} - \frac{B_2}{B_1}, \text{ and } \gamma := \frac{pB_1}{2(1+\alpha(p+2))}.$$

*If* $f(z)$ *given by (1.1.2) belongs to* $S_p^*(\alpha, \varphi)$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} -\gamma v & \text{if } \mu \leq \sigma_1, \\ \gamma & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ \gamma v & \text{if } \mu \geq \sigma_2. \end{cases} \tag{6.2.16}$$

*Further, if* $\sigma_1 \leq \mu \leq \sigma_3$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{(1+\alpha(p+1))^2\gamma}{p^2 B_1^2}(v+1)|a_{p+1}|^2 \leq \gamma.$$

*If $\sigma_3 \le \mu \le \sigma_2$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{(1+\alpha(p+1))^2\gamma}{p^2 B_1^2}(1-v)|a_{p+1}|^2 \le \gamma.$$

*For any complex number $\mu$,*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \gamma \max\{1; |v|\}. \tag{6.2.17}$$

*Further,*

$$|a_{p+3}| \le \frac{pB_1}{\alpha(5-p-p^2)+3} H(q_1, q_2) \tag{6.2.18}$$

*where $H(q_1, q_2)$ is as defined in Lemma 6.1.3,*

$$q_1 := \left[\frac{2B_2}{B_1} + \frac{B_1^2 p(\alpha(3p+5)+3)}{2(\alpha(p+1)+1)(\alpha(p+2)+1)}\right],$$

*and*

$$q_2 := \left[\frac{B_3}{B_1} + \frac{B_2 p(\alpha(3p+5)+3) + p^2 B_1^2(p\alpha+\alpha+1)}{2(\alpha(p+1)+1)(\alpha(p+2)+1)}\right].$$

*These results are sharp.*

Proof. If $f(z) \in S_p^*(\alpha, \varphi)$. It is easily shown that

$$\begin{aligned}
&\frac{1+\alpha(1-p)}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\frac{z^2 f''(z)}{f(z)} \\
&:= 1 + \frac{(1+\alpha(p+1))}{p} a_{p+1} z + \left(\frac{(1+\alpha(p+2))}{p} 2a_{p+2} - \frac{(1+\alpha(p+1))}{p} a_{p+1}^2\right) z^2 \\
&+ \left(\frac{(3+\alpha(5-p-p^2))}{p} a_{p+3} - \frac{(3+\alpha(3p+5))}{p} a_{p+1}a_{p+2} + \frac{(1+\alpha(p+1))}{p} a_{p+1}^3\right) z^3 \\
&+ \cdots .
\end{aligned}$$

The proof can now be completed using similar arguments as in the proof of Theorem 6.2.1.

□

For the class $S_{p,g}^*(\alpha, \varphi)$, the following coefficient bounds are obtained.

Theorem 6.2.6. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, and let*

$$\sigma_1 := \frac{g_{p+1}^2(1+\alpha(p+1))[(1+\alpha(p+1))(B_2 - B_1) + pB_1^2]}{2pB_1^2 g_{p+2}},$$

$$\sigma_2 := \frac{g_{p+1}^2(1+\alpha(p+1))[(1+\alpha(p+1))(B_2 + B_1) + pB_1^2]}{2pB_1^2 g_{p+2}},$$

$$\sigma_3 := \frac{g_{p+1}^2(1+\alpha(p+1))[(1+\alpha(p+1))B_2+pB_1^2]}{2pB_1^2 g_{p+2}},$$

$$v := \frac{2g_{p+2}\mu p B_1(p\alpha+2\alpha+1)}{g_{p+1}^2(1+\alpha(p+1))^2} - \frac{pB_1}{1+\alpha(p+1)} - \frac{B_2}{B_1}, \text{ and } \gamma := \frac{pB_1}{2g_{p+2}(1+\alpha(p+2))}.$$

*If $f(z)$ given by 1.1.2 belongs to $S_p^*(\alpha,\varphi)$, then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \begin{cases} -\gamma v & \text{if} \quad \mu \leq \sigma_1, \\ \gamma & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \gamma v & \text{if} \quad \mu \geq \sigma_2. \end{cases}$$

*Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{(1+\alpha(p+1))^2 g_{p+1}^2\gamma}{p^2B_1^2}(v+1)|a_{p+1}|^2 \leq \gamma.$$

*If $\sigma_3 \leq \mu \leq \sigma_2$, then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{(1+\alpha(p+1))^2 g_{p+1}^2\gamma}{p^2B_1^2}(1-v)|a_{p+1}|^2 \leq \gamma.$$

*For any complex number $\mu$,*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \gamma \max\{1;|v|\}.$$

*Further,*

(6.2.19)
$$|a_{p+3}| \leq \frac{pB_1}{g_{p+3}[\alpha(5-p-p^2)+3]}H(q_1,q_2)$$

*where $H(q_1,q_2)$ is as defined in Lemma 6.1.3,*

$$q_1 := \left[\frac{2B_2}{B_1} + \frac{B_1^2p(\alpha(3p+5)+3)}{2(\alpha(p+1)+1)(\alpha(p+2)+1)}\right]$$

*and*

$$q_2 := \left[\frac{B_3}{B_1} + \frac{B_2p(\alpha(3p+5)+3)+p^2B_1^2(p\alpha+\alpha+1)}{2(\alpha(p+1)+1)(\alpha(p+2)+1)}\right].$$

*These results are sharp.*

When $p=1$ and

$$\varphi(z) = \frac{1+z(1-2\beta)}{1-z} \quad (0 \leq \beta < 1),$$

(6.2.16) and (6.2.17) of Theorem 6.2.5 reduces to the following result:

Corollary 6.2.2. [**36**, Theorem 4 and 3, p. 95] *Let* $\varphi(z) = \frac{1+z(1-2\beta)}{1-z}$ $(0 \leq \beta < 1)$. *If* $f(z)$ *given by (1.1.1) belongs to* $S^*(\alpha,\varphi)$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} -\gamma v & \text{if } \mu \leq \sigma_1, \\ \gamma & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ \gamma v & \text{if } \mu \geq \sigma_2, \end{cases} \tag{6.2.20}$$

*and for any complex number* $\mu$,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \gamma \max\{1; |v|\} \tag{6.2.21}$$

*where*

$$\begin{aligned} \sigma_1 &:= 2(1+\alpha(p+1)), \\ \sigma_2 &:= \frac{(1+\alpha(p+1))[1+\alpha+p(1+\alpha-\beta)]}{2p(1-\beta)}, \\ \sigma_3 &:= \frac{(1+\alpha(p+1))[1+\alpha(p+1)+2p(1-\beta)]}{4p(1-\beta)}, \\ \text{and } \ v &:= \frac{2p(1-\beta)[2\mu(\alpha(p+2)+1)-(1+\alpha(p+1))]}{(1+\alpha(p+1))^2}, \\ \gamma &:= \frac{p(1-\beta)}{1+\alpha(p+2)}. \end{aligned}$$

For the class $L_p^M(\alpha,\varphi)$, the following coefficient bounds are obtained.

Theorem 6.2.7. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *Let*

$$\begin{aligned} \sigma_1 &:= \frac{\gamma_3(B_2 - B_1) + \gamma_2 B_1^2}{\gamma_1 B_1^2}, \\ \sigma_2 &:= \frac{\gamma_3(B_2 + B_1) + \gamma_2 B_1^2}{\gamma_1 B_1^2}, \\ \sigma_3 &:= \frac{\gamma_3 B_2 + \gamma_2 B_1^2}{\gamma_1 B_1^2}, \\ \gamma_1 &:= 4p^2(p+2-2\alpha), \\ \gamma_2 &:= (1-\alpha)[2p(p+1)^2 + \alpha] + 2\alpha p^3, \end{aligned}$$

$$\gamma_3 := 2(p+1-\alpha)^2,$$
$$v := \frac{\gamma_1\mu-\gamma_2}{\gamma_3}B_1 - \frac{B_2}{B_1},$$
$$T_1 := -\frac{(1-\alpha)[3p(p+1)(p+2)+4\alpha]+3\alpha p^3}{p^4},$$

*and*

$$T_2 := \frac{1-\alpha}{6p^6}\left((p+1)^3[6p(p+\alpha)+\alpha(\alpha+1)] - p\alpha[6p^3+p^2(19+\alpha)+3p(3+\alpha)+3\alpha]\right)+\frac{\alpha}{p}.$$

*If $f(z)$ given by (1.1.2) belongs to $L_p^M(\alpha,\varphi)$, then*

$$(6.2.22)\qquad \left|a_{p+2}-\mu a_{p+1}^2\right| \le \begin{cases} \frac{-p^2B_1v}{2(p+2-2\alpha)} & \text{if} \quad \mu\le\sigma_1, \\ \frac{p^2B_1}{2(p+2-2\alpha)} & \text{if} \quad \sigma_1\le\mu\le\sigma_2, \\ \frac{p^2B_1v}{2(p+2-2\alpha)} & \text{if} \quad \mu\ge\sigma_2. \end{cases}$$

*Further, if $\sigma_1\le\mu\le\sigma_3$, then*

$$(6.2.23)\qquad \left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{\gamma_3}{B_1\gamma_1}(1+v)|a_{p+1}|^2 \le \frac{p^2B_1}{2(p+2-2\alpha)}.$$

*If $\sigma_3\le\mu\le\sigma_2$, then*

$$(6.2.24)\qquad \left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{\gamma_3}{B_1\gamma_1}(1-v)|a_{p+1}|^2 \le \frac{p^2B_1}{2(p+2-2\alpha)}.$$

*For any complex number $\mu$,*

$$(6.2.25)\qquad \left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p^2B_1}{2(p+2-2\alpha)}\max\{1;|v|\}.$$

*Further,*

$$|a_{p+3}| \le \frac{p^2B_1}{3(p-3\alpha+3)}H(q_1,q_2)$$

*where $H(q_1,q_2)$ is as defined in Lemma 6.1.3,*

$$q_1 := \frac{2B_2}{B_1} - \frac{T_1p^4B_1}{2(p-\alpha+1)(p-2\alpha+2)}$$
$$\text{and}\quad q_2 := \frac{B_3}{B_1} - \frac{T_1p^4(\gamma_3B_2+\gamma_2B_1^2)}{2\gamma_3(p-\alpha+1)(p-2\alpha+2)} - \frac{T_2p^6B_1^2}{(p-\alpha+1)^3}.$$

*These results are sharp.*

PROOF. The proof is similar to the proof of Theorem 6.2.1. □

The final result is on the coefficient bounds for functions in $M_p(\alpha,\varphi)$.

THEOREM 6.2.8. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *Let*

$$\begin{aligned}
\sigma_1 &:= \frac{p(1+p\alpha)^2(B_2 - B_1) + (\alpha + p(p+2\alpha))B_1^2}{2p(p+2\alpha)B_1^2}, \\
\sigma_2 &:= \frac{p(1+p\alpha)^2(B_2 + B_1) + (\alpha + p(p+2\alpha))B_1^2}{2p(p+2\alpha)B_1^2}, \\
\sigma_3 &:= \frac{p(1+p\alpha)^2 B_2 + (\alpha + p(p+2\alpha))B_1^2}{2p(p+2\alpha)B_1^2}, \\
\text{and } v &:= \frac{[p(p+2\alpha)(2\mu-1) - \alpha]B_1}{2p(p+2\alpha)} - \frac{B_2}{B_1}.
\end{aligned}$$

*If* $f(z)$ *given by (1.1.2) belongs to* $M_p(\alpha, \varphi)$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} \frac{-p^2 B_1 v}{2(p+2\alpha)} & \text{if} \quad \mu \leq \sigma_1, \\ \frac{p^2 B_1}{2(p+2\alpha)} & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{p^2 B_1 v}{2(p+2\alpha)} & \text{if} \quad \mu \geq \sigma_2. \end{cases} \tag{6.2.26}$$

*Further, if* $\sigma_1 \leq \mu \leq \sigma_3$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{(1+p\alpha)^2}{2(p+2\alpha)B_1}(1+v)|a_{p+1}|^2 \leq \frac{p^2 B_1}{2(p+2\alpha)}. \tag{6.2.27}$$

*If* $\sigma_3 \leq \mu \leq \sigma_2$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{(1+p\alpha)^2}{2(p+2\alpha)B_1}(1-v)|a_{p+1}|^2 \leq \frac{p^2 B_1}{2(p+2\alpha)}. \tag{6.2.28}$$

*For any complex number* $\mu$,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p^2 B_1}{2(p+2\alpha)} \max\{1; |v|\}. \tag{6.2.29}$$

*Further,*

$$|a_{p+3}| \leq \frac{p^2 B_1}{3(p+3\alpha)} H(q_1, q_2),$$

*where* $H(q_1, q_2)$ *is as defined in Lemma 6.1.3,*

$$q_1 := \frac{2B_2}{B_1} + \frac{3(p^2 + 3p\alpha + 2\alpha)}{2(1+p\alpha)(p+2\alpha)} B_1,$$

*and*

$$q_2 := \frac{B_3}{B_1} + \frac{3(p^2+3p\alpha+2\alpha)}{2(1+p\alpha)(p+2\alpha)} B_2 + \frac{(p^4 + 5\alpha p^3 + 3p^2\alpha(2\alpha+1) + p\alpha(9\alpha-2) + 2\alpha^2)}{2p(1+p\alpha)^3(p+2\alpha)} B_1^2.$$

*These results are sharp.*

Proof. For $f(z) \in M_p(\alpha, \varphi)$, a computation shows that

$$\begin{aligned}
&\frac{1-\alpha}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \\
&= 1 + \frac{(1+p\alpha)a_{p+1}}{p}z + \left(-\frac{(p^2+2p\alpha+\alpha)a_{p+1}^2}{p^3} + \frac{2(p+2\alpha)a_{p+2}}{p^2}\right)z^2 \\
&+ \left(\frac{3(p+3\alpha)}{p^2}a_{p+3} - \frac{3(p^2+3p\alpha+2\alpha)}{p^3}a_{p+1}a_{p+2} + \frac{p^3+3p^2\alpha+3p\alpha+\alpha}{p^4}a_{p+1}^3\right)z^3 \\
&+ \cdots .
\end{aligned}$$

The remaining part of the proof is similar to the proof of Theorem 6.2.1. $\square$

When $p = 1$ and $\varphi(z) = ((1+z)/(1-z))^\beta \quad (\alpha \geq 0, 0 < \beta \leq 1)$, (6.2.26) and (6.2.29) of Theorem 6.2.8 reduce to the following result:

Corollary 6.2.3. [**21**, Theorem 2.1 and 2.2, p. 23] *Let* $\varphi(z) = ((1+z)/(1-z))^\beta$ $(\alpha \geq 0, 0 < \beta \leq 1)$. *Let*

$$\begin{aligned}
\sigma_1 &:= \frac{(1+\alpha)^2(\beta-1) + 2(1+3\alpha)\beta}{4(1+2\alpha)\beta}, \\
\sigma_2 &:= \frac{(1+\alpha)^2(\beta+1) + 2(1+3\alpha)\beta}{4(1+2\alpha)\beta}, \\
\text{and } \ v &:= \frac{[2(1+2\alpha)(\mu-1)-\alpha]\beta}{1+2\alpha}.
\end{aligned}$$

*If $f(z)$ given by (1.1.1) belongs to $M(\alpha, \varphi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{-\beta v}{1+2\alpha} & \text{if} \quad \mu \leq \sigma_1, \\ \frac{\beta}{1+2\alpha} & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{\beta v}{1+2\alpha} & \text{if} \quad \mu \geq \sigma_2. \end{cases}$$

*For any complex number $\mu$,*

$$|a_3 - \mu a_2^2| \leq \frac{\beta}{1+2\alpha}\max\{1; |v|\}.$$

*These results are sharp.*

When $p = 1$ and $\alpha = 1$, (6.2.26)–(6.2.28) of Theorem 6.2.8 reduce to the following result:

COROLLARY 6.2.4. [**54**, Theorem 3 and Remark, p. 7] *If $f(z)$ given by (1.1.1) belongs to $C(\varphi)$, then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{-(3\mu-2)B_1^2+2B_2}{12} & \textit{if} \quad 3B_1^2\mu \leq 2(B_2-B_1+B_1^2), \\ \frac{B_1}{6} & \textit{if} \quad 2(B_2-B_1+B_1^2) \leq 3B_1^2\mu \leq 2(B_2+B_1+B_1^2), \\ \frac{(3\mu-2)B_1^2-2B_2}{12} & \textit{if} \quad 3B_1^2\mu \geq 2(B_2+B_1+B_1^2). \end{cases}$$

*Further, if $2(B_2-B_1+B_1^2) \leq 3B_1^2\mu \leq 2(B_2+B_1^2)$, then*

$$\left|a_3-\mu a_2^2\right| + \frac{3\mu B_1^2-2(B_2-B_1+B_1^2)}{3B_1^2}|a_3|^2 \leq \frac{B_1}{6}.$$

*If $2(B_2+B_1^2) \leq 3B_1^2\mu \leq 2(B_2+B_1+B_1^2)$, then*

$$\left|a_3-\mu a_2^2\right| + \frac{2(B_2+B_1+B_1^2)-3\mu B_1^2}{3B_1^2}|a_2|^2 \leq \frac{B_1}{6}.$$

*These results are sharp.*

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Chapter 2

(1) R. M. Ali, V. Ravichandran, N. Seenivasagan, On Bernardi's integral operator and the Briot-Bouquet differential subordination, J. Math. Anal. Appl. **334** (2006), 663–668.
(2) R. M. Ali, V. Ravichandran, N. Seenivasagan, Sufficient conditions for Janowski Starlikeness, Int. J. Math. Math. Sci. submitted.

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Chapter 5

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Chapter 6

(1) R. M. Ali, V. Ravichandran, and N. Seenivasagan, Coefficient bounds for $p$-valent functions, Appl. Math. Comput. **187** (2007), 35–46.

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