THE $\mathscr{U}$-RADIUS AND HANKEL DETERMINANT FOR ANALYTIC FUNCTIONS, AND PRODUCT OF LOGHARMONIC MAPPINGS

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by

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LIST OF SYMBOLS

		Page
$\mathcal{A}$	Class of analytic functions f of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{D})$	2
$\mathbb{C}$	Complex plane	1
$\mathcal{CV}$	Class of convex functions in $\mathcal{A}$	7
$\mathcal{CV}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$	8
$\mathcal{CV}(\varphi)$	$\left\{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z)\right\}$	11
$\mathcal{CCV}$	Class of close-to-convex functions in $\mathcal{A}$	8
$\mathbb{D}$	Open unit disk $\{z \in \mathbb{C} : \lvert z \rvert < 1\}$	2
$\mathcal{H}(\mathbb{D})$	Class of all analytic functions in $\mathbb{D}$	2
HG	Class of analytic functions φ in $\mathbb{D}$ of the form $\varphi(z) = zh(z)g(z)$, $h, g \in \mathcal{H}(\mathbb{D})$, $h(0) = g(0) = 1$	131
$H_q(n)$	Hankel determinants of functions $f \in \mathcal{A}$	70
Im	Imaginary part of a complex number	13
L_α	$\left\{f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} > 0\right\}$	9
$L(\alpha, \varphi)$	$\left\{f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z)\right\}$	99
M_α	$\left\{f \in \mathcal{A} : (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0\right\}$	9
$M(\alpha, \varphi)$	$\left\{f \in \mathcal{A} : (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z)\right\}$	105
$\mathcal{P}$	$\{p \in \mathcal{H}(\mathbb{D}) : p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,\ \mathrm{Re}\ p(z) > 0,\ z \in \mathbb{D}\}$	5
$\mathcal{P}(\alpha)$	$\{p \in \mathcal{H}(\mathbb{D}) : p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,\ \mathrm{Re}\ p(z) > \alpha,\ z \in \mathbb{D}\}$	5
$\mathcal{P}_{\mathbb{R}}$	Class of all functions $p \in \mathcal{P}$ with real coefficients	14
$\mathcal{P}_{Lh}$	Class of all logharmonic mappings f in $\mathbb{D}$ of the form $f(z) = h(z)\overline{g(z)}$, $h, g \in \mathcal{H}(\mathbb{D})$, $h(0) = g(0) = 1$ satisfying $p(z) = h(z)g(z) \in \mathcal{P}_{\mathbb{R}}$	134
$\mathbb{R}$	Set of all real numbers	3
Re	Real part of a complex number	5

$R_b(\varphi)$	$\left\{f \in \mathcal{A} : 1+\frac{1}{b}\left(f'(z)-1\right) \prec \varphi(z)\right\}$	88
$\mathcal{S}$	Class of all normalized univalent functions in $\mathcal{A}$	3
$\mathcal{S}_H$	Class of all normalized univalent and sense-preserving harmonic functions $f = h+\overline{g}$ in the unit disk $\mathbb{D}$	27
$\mathcal{S}_{Lh}$	Class of all normalized univalent logharmonic mappings	33
$\mathcal{ST}$	Class of starlike functions in $\mathcal{A}$	6
$\mathcal{S}_H^0$	The subclass of $\mathcal{S}_H$ consisting of functions $f = h+\overline{g}$ and $g'(0) = 0$	28
$\mathcal{ST}(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$	8
$\mathcal{ST}(\varphi)$	$\left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z)\right\}$	11
$\mathcal{ST}(\alpha,\varphi)$	$\left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} + \alpha\frac{z^2f''(z)}{f(z)} \prec \varphi(z)\right\}$	93
$\mathcal{ST}_\beta$	Class of strongly starlike functions of order β in $\mathcal{A}$	12
$\mathcal{ST}_L$	Class of *lemniscate of Bernoulli starlike* functions in $\mathcal{A}$	12
$\mathcal{ST}_p$	Class of parabolic starlike functions in $\mathcal{A}$	11
$\mathcal{ST}_{Lh}$	Class of all normalized univalent starlike logharmonic mappings	34
$\mathcal{SP}_{Lh}^{\alpha}$	Class of all univalent $\alpha-$spirallike logharmonic mappings	35
$\prec$	Subordinate to	10
T	Class of typically real functions in $\mathcal{A}$	13
TLh	Class of all logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ satisfying $\varphi(z) = zh(z)g(z) \in HG$ is typically real analytic in $\mathbb{D}$	131
$\mathcal{U}$	$\left\{f \in \mathcal{A} : \left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1, \quad z \in \mathbb{D}\right\}$	39

JEJARI $\mathscr{U}$ DAN PENENTU HANKEL UNTUK FUNGSI ANALISIS DAN HASIL DARAB PEMETAAN LOGHARMONIK

ABSTRAK

Tesis ini mengkaji tentang ciri-ciri geometrik dan analisis bagi fungsi analisis bernilai kompleks dan pemetaan log-harmonik tertakrif dalam cakera unit terbuka $\mathbb{D}$. Terdapat empat permasalahan penyelidikan yang dikaji. Sebagai permulaan, andaikan $\mathscr{U}$ sebagai kelas yang terdiri daripada fungsi analisis ternormal f yang memenuhi syarat $|(z/f(z))^2 f'(z)-1|<1$. Semua fungsi $f\in\mathscr{U}$ adalah univalen. Bagi permasalahan yang pertama, jejari-$\mathscr{U}$ ditentukan untuk beberapa kelas fungsi analisis termasuk kelas fungsi analisis yang memenuhi ketidaksamaan $\operatorname{Re} f(z)/g(z)>0$, atau $|f(z)/g(z)-1|<1$ dalam $\mathbb{D}$, untuk g yang terkandung dalam kelas fungsi analisis tertentu. Bagi kebanyakan kes, jejari-$\mathscr{U}$ yang tepat diperoleh. Konjektur oleh Obradovic dan Ponusamy berkenaan jejari univalen bagi hasil darab yang melibatkan fungsi univalen juga telah dibuktikan. Permasalahan kedua berkaitan dengan penentu Hankel bagi fungsi analisis. Bagi fungsi analisis ternormal f, andaikan $zf'(z)/f(z)$ atau $1+zf''(z)/f'(z)$ subordinat kepada suatu fungsi analisis φ dalam $\mathbb{D}$. Andaikan juga F sebagai jelmaan punca ke-k, iaitu, $F(z)=z\left[f\left(z^k\right)/z^k\right]^{\frac{1}{k}}$. Batas atas terbaik dalam bentuk pekali bagi fungsi φ yang diberi diperoleh bagi penentu Hankel kedua F, yang f terkandung dalam salah satu kelas di atas. Anggaran bagi penentu Hankel bagi penjelmaan ke-k untuk kelas fungsi α-cembung dan α-cembung secara logaritma juga diperoleh. Dua permasalahan terakhir adalah berkait dengan pemetaan logaritma dalam $\mathbb{D}$. Pertama, bagi pemetaan log-harmonik bak-bintang $f(z)=zh(z)\overline{g(z)}$, syarat cukup didapati bagi $F(z)=f(z)|f(z)|^{2\gamma}$ agar menjadi pemetaan α-bak lingkaren log-harmonik.

Syarat cukup juga diperoleh bagi dua pemetaan logharmonik f_1 dan f_2 yang memastikan hasil darab $F(z) = f_1^{\lambda}(z) f_2^{1-\lambda}(z),\ \ 0 \leq \lambda \leq 1$, adalah pemetaan log-harmonik bak bintang. Beberapa contoh telah dibangunkan daripada hasil darab tersebut. Permasalahan seterusnya melihat pada pemetaan log-harmonik ternormal $f(z) = zh(z)\overline{g(z)}$ dimana $\varphi(z) = zh(z)g(z)$ adalah fungsi analisis nyata biasa dalam $\mathbb{D}$. Pewakilan kamiran bagi pemetaan sedemikian diterbitkan, dan anggaran bagi jejari bak-bintangnya didapati. Anggaran atas terbaik pada lengkok juga ditentukan. Syarat-syarat geometri cukup dan perlu bagi $\varphi(z) = zh(z)g(z)$ untuk menjadi nyata biasa juga dikaji apabila $f(z) = zh(z)\overline{g(z)}$ mempunyai dilatasi dengan pekali nyata.

THE $\mathscr{U}$-RADIUS AND HANKEL DETERMINANT FOR ANALYTIC FUNCTIONS, AND PRODUCT OF LOGHARMONIC MAPPINGS

ABSTRACT

This thesis studies geometric and analytic properties of complex-valued analytic functions and logharmonic mappings in the open unit disk $\mathbb{D}$. It investigates four research problems. As a precursor to the first, let $\mathscr{U}$ be the class consisting of normalized analytic functions f satisfying $|(z/f(z))^2 f'(z) - 1| < 1$. All functions $f \in \mathscr{U}$ are univalent. In the first problem, the $\mathscr{U}$-radius is determined for several classes of analytic functions. These include the classes of functions f satisfying the inequality $\operatorname{Re} f(z)/g(z) > 0$, or $|f(z)/g(z) - 1| < 1$ in $\mathbb{D}$, for g belonging to a certain class of analytic functions. In most instances, the exact $\mathscr{U}$-radius are found. A recent conjecture by Obradović and Ponnusamy concerning the radius of univalence for a product involving univalent functions is also shown to hold true. The second problem deals with the Hankel determinant of analytic functions. For a normalized analytic function f, let $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ be subordinate to a given analytic function φ in $\mathbb{D}$. Further let F be its kth-root transform, that is, $F(z) = z\left[f\left(z^k\right)/z^k\right]^{\frac{1}{k}}$. A bound expressed in terms of the coefficients of the given function φ is obtained for the second Hankel determinant of F, where f belongs to either of the two classes above. Estimates for the Hankel determinant are also found for the kth-root transform of the class of α-convex functions and α-logarithmically convex functions. The final two studied problems studied relate to logharmonic mappings in $\mathbb{D}$. First, for a starlike logharmonic mapping $f(z) = zh(z)\overline{g(z)}$, sufficient conditions are obtained for $F(z) = f(z)|f(z)|^{2\gamma}$ to be $\alpha-$spirallike logharmonic mapping. In addition, sufficient

conditions are determined on two given logharmonic mappings f_1 and f_2 to ensure their product $F(z) = f_1^{\lambda}(z) f_2^{1-\lambda}(z),\ 0 \leq \lambda \leq 1$, is a univalent starlike logharmonic mapping. Several illustrative examples are constructed from this product. The latter problem looks at normalized logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ where $\varphi(z) = zh(z)g(z)$ is typically real analytic in $\mathbb{D}$. An integral representation for such mappings f is derived, and an estimate found on its radius of starlikeness. An upper estimate on arclength is also determined. Sufficient and necessary geometric conditions for $\varphi(z) = zh(z)g(z)$ to be typically real are also investigated when $f(z) = zh(z)\overline{g(z)}$ has a dilatation with real coefficients.

CHAPTER 1

INTRODUCTION

Geometric function theory is a branch of complex analysis with a long steeped history. It started in the early 20th century. Function theory studies geometric properties of complex-valued functions, and incorporate various tools from analysis.

This introductory chapter presents basic definitions and fundamental results important in the sequel. These are results on analytic functions, as well as on harmonic and log-harmonic mappings. It also serves to provide the motivations for the problems studied in the thesis.

1.1 Analytic Univalent Functions

In this thesis, the complex plane is denoted by $\mathbb{C}$. Further, let

$$\mathbb{D}(z_0, r) := \{z : z \in \mathbb{C}, |z - z_0| < r\}, \quad r > 0,$$

be the neighborhood of z_0. A set D of $\mathbb{C}$ is *open* if for every point $z_0 \in$ D, there is an $r > 0$ such that $\mathbb{D}(z_0, r) \subset$ D. An open set D is *connected* if there is a polygonal path in D joining any pair of points in D.

A *domain* D of $\mathbb{C}$ is an open connected set. A domain D is *simply connected* if the interior to every simple closed curve in D lies completely within D. Geometrically, a simply connected domain is a domain without any holes.

A complex-valued function f defined in D is *differentiable* at a point $z_0 \in$ D if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

exists. A function f defined in D is *analytic* at $z_0 \in$ D if it is differentiable in some neighbourhood of z_0. It is analytic in D if it is analytic at all points in D. It is known in [123, p. 167] that for $z \in \mathbb{D}(z_0, r) \subseteq$ D, an analytic function f in D has a Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n := \frac{f^{(n)}(z_0)}{n!}.$$

Denote by $\mathcal{H}(\mathbb{D})$ the class of all analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{A}$ denote the class of all normalized analytic functions f in $\mathcal{H}(\mathbb{D})$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A function f is *univalent* in D if it is one-to-one in D. Thus f is univalent if it takes different points in D to different values, that is, for any two distinct points z_1 and z_2 with $z_1 \neq z_2$ in D, $f(z_1) \neq f(z_2)$. A function f is called *locally univalent* at z_0 if it is one-to-one in some neighbourhood of z_0. It is known in [38, p. 5] that the condition $f'(z_0) \neq 0$ is necessary and sufficient for local univalence at z_0.

A function that preserves both the magnitude and orientation of angles is said to be *conformal*. For an analytic function f, the condition $f'(z_0) \neq 0$ is equivalent to it being conformal at z_0.

The Riemann mapping theorem is an important theorem in geometric function theory. It states that any simply connected domain which is not the entire complex plane, can be mapped conformally onto $\mathbb{D}$.

Theorem 1.1. (Riemann Mapping Theorem) [38, p. 11] *Let D be a simply connected domain which is a proper subset of the complex plane. Let ζ be a given point in D. Then there is a unique analytic and univalent function f which maps D onto the unit disk $\mathbb{D}$ satisfying $f(\zeta)=0$ and $f'(\zeta)>0$.*

Therefore, the study of conformal mappings on a simply connected domain can be confined to the study of functions that are analytic and univalent on the open unit disk $\mathbb{D}$.

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions. An example is the function k given by

$$k(z)=\frac{z}{(1-z)^2}=\frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2-1\right]=\sum_{n=1}^{\infty}nz^n, \qquad z\in\mathbb{D}. \tag{1.2}$$

This function is known as the Koebe function, and it maps $\mathbb{D}$ onto the entire complex plane except for a slit along the half-line $(-\infty,-1/4]$. The Koebe function and its rotations $e^{-i\beta}k(e^{i\beta}z)$, $\beta\in\mathbb{R}$, play an important role in the study of the class $\mathcal{S}$. These functions are extremal functions for various problems in the class $\mathcal{S}$.

In 1916, Bieberbach [30] conjectured the coefficients for $f(z)=\sum_{n=1}^{\infty}a_nz^n\in\mathcal{S}$ satisfy $|a_n|\leq n$. This conjecture is known as Bieberbach's conjecture. However, he only proved for the case when $n=2$, and this result is called the Bieberbach theorem.

Theorem 1.2. (Bieberbach theorem) [30] *Let $f\in\mathcal{S}$. Then*

$$|a_2|\leq 2.$$

Equality occurs if and only if f is a rotation of the Koebe function k.

In fact for many years, this conjecture has stood as a challenge to many mathematicians. The problem was resolved only for some initial values of n. Lowner [76] proved the Bieberbach conjecture for the case $n = 3$, followed by Garabedian and Schiffer [47] for $n = 4$. For $n = 6$, it was proved independently by Pederson [119] and Ozawa [116]. Pederson and Schiffer [118] proved the conjecture for $n = 5$. It was not until 1985 that de Branges [36] successfully proved the Bieberbach conjecture.

Theorem 1.3. (de Branges Theorem) [36] *The coefficients of each function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ *satisfy* $|a_n| \leq n$ *for* $n = 2, 3, \ldots$. *Equality occurs if and only if* f *is the Koebe function* k *or one of its rotations.*

Bieberbach theorem has significant implications in the theory of univalent functions. These include the well known covering theorem due to Koebe, which states the image of $\mathbb{D}$ under every $f \in \mathcal{S}$ must cover an open disk centered at the origin of radius $1/4$.

Theorem 1.4. (Koebe One-Quarter Theorem) [38, p. 31] *The range of every function of the class* $\mathcal{S}$ *contains the disk* $\{w : |w| < 1/4\}$.

One important consequence of the Bieberbach theorem is the distortion theorem which gives sharp bounds for $|f'(z)|$.

Theorem 1.5. (Distortion Theorem) [38, p. 32] *Let* $f \in \mathcal{S}$. *Then*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \qquad |z| = r < 1.$$

Equality occurs if and only if f *is a suitable rotation of the Koebe function* k.

The growth theorem which results from the distortion theorem provides sharp bounds for $|f(z)|$.

Theorem 1.6. (Growth Theorem) [38, p. 33] *Let* $f \in \mathcal{S}$. *Then*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \qquad |z| = r < 1.$$

Equality occurs if and only if f is a suitable rotation of the Koebe function k.

1.2 Subclasses of Analytic Univalent Functions

An important subclass of normalized analytic functions in the open unit disk $\mathbb{D}$ is the class of functions with positive real part.

Definition 1.1. (The class of functions with positive real part) [48, p. 78] *The class $\mathcal{P}$ consists of all analytic functions*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \qquad z \in \mathbb{D}, \tag{1.3}$$

with

$$\text{Re } p(z) > 0, \qquad z \in \mathbb{D}.$$

An important example of a function in $\mathcal{P}$ is the *Möbius function*

$$m(z) := \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n$$

which maps $\mathbb{D}$ onto the half-plane $\{w : \text{Re } w > 0\}$. The role of this Möbius function m is similar to that of the Koebe function in the class $\mathcal{S}$.

The sharp coefficient bound for functions in the class $\mathcal{P}$ is given in the following result.

Lemma 1.1. (Carathéodory's Lemma) [38, p. 41] *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$. Then the following sharp estimate holds:*

$$|c_n| \leq 2, \quad (n = 1, 2, 3, \ldots).$$

Equality occurs for the Möbius function m or its rotations.

More generally, for $0 \leq \alpha < 1$, let $\mathcal{P}(\alpha)$ denote the class of analytic functions p of the form (1.3) with

$$\operatorname{Re} p(z) > \alpha, \quad z \in \mathbb{D}.$$

The class $\mathcal{P}$ is closely related to several subclasses of univalent functions. These include the important classes of convex and starlike functions. Geometric and analytic relationships between them will soon be made evident.

A set D in $\mathbb{C}$ is called *starlike with respect to an interior point* w_0 in D if the line segment joining w_0 to every other point w in D lies entirely in D. Analytically, this condition is equivalent to

$$(1-t)w_0 + tw \in \mathrm{D}$$

for every $w \in \mathrm{D}$, and $0 \leq t \leq 1$. In the case $w_0 = 0$, the set D is called starlike with respect to the origin, or simply a starlike domain.

Definition 1.2. (Starlike function) [48, p. 108] *A function $f \in \mathcal{A}$ is called a starlike function with respect to w_0 if it maps $\mathbb{D}$ onto a domain that is starlike with respect to w_0. In the particular case that $w_0 = 0$, f is called a starlike function.*

Denote by $\mathcal{ST}$ the subclass of $\mathcal{S}$ consisting of all starlike functions in $\mathbb{D}$. The following theorem gives an analytic description of the class $\mathcal{ST}$.

Theorem 1.7. (Analytical characterization of starlike functions) [38, p. 41] *Let $f \in \mathcal{A}$. Then $f \in \mathcal{ST}$ if and only if*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad z \in \mathbb{D}. \tag{1.4}$$

Thus $f \in \mathcal{ST}$ if and only if $zf'/f \in \mathcal{P}$. The Koebe function in (1.2) is an example of starlike function in $\mathbb{D}$. The sharp coefficient bound for $f \in \mathcal{ST}$ is given by the following result.

Theorem 1.8. [89] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{ST}$. *Then*

$$|a_n| \leq n, \qquad (n = 2, 3, \ldots).$$

Equality occurs for all n when f is a rotation of the Koebe function k.

A set D in $\mathbb{C}$ is *convex* if it is starlike with respect to each of its points, that is, for every pair of points w_1 and w_2 lying in D, the line segment joining w_1 and w_2 also lies entirely in D. Analytically, this is equivalent to

$$tw_1 + (1-t)w_2 \in \mathrm{D}$$

for every pair $w_1, w_2 \in \mathrm{D}$, and $0 \leq t \leq 1$.

Definition 1.3. (Convex function) [48, p. 107] *A function* $f \in \mathcal{A}$ *is called a convex function if it maps* $\mathbb{D}$ *onto a convex domain.*

Denote by $\mathcal{CV}$ the subclass of $\mathcal{S}$ consisting of all convex functions in $\mathbb{D}$. The following is an analytic description of convex functions.

Theorem 1.9. (Analytical characterization of convex functions) [38, p. 42] *Let* $f \in \mathcal{A}$. *Then* $f \in \mathcal{CV}$ *if and only if*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0 \quad z \in \mathbb{D}. \tag{1.5}$$

The function

$$L(z) = \frac{z}{1-z} \tag{1.6}$$

which maps $\mathbb{D}$ onto the half-plane $\{w : \operatorname{Re}\, w > -1/2\}$ is a convex function and belongs to $\mathcal{CV}$. The following result gives sharp coefficient bound for the class $\mathcal{CV}$.

Theorem 1.10. [75] *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{CV}$. Then*

$$|a_n| \leq 1, \qquad (n = 2, 3, \ldots).$$

Equality occurs for all n when f is a rotation of the function L given by (1.6).

In 1915, Alexander [19] showed that there is a close connection between convex and starlike functions.

Theorem 1.11. [19] *Let $f \in \mathcal{A}$. Then f is convex in $\mathbb{D}$ if and only if $zf'(z)$ is starlike in $\mathbb{D}$.*

In 1936, Robertson [129] introduced the classes of *starlike and convex functions of order α*, $0 \leq \alpha < 1$. These are given by

$$\mathcal{ST}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \, , \, z \in \mathbb{D} \right\},$$

and

$$\mathcal{CV}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \, , \qquad z \in \mathbb{D} \right\},$$

respectively. In particular, $\mathcal{ST}(0) = \mathcal{ST}$ and $\mathcal{CV}(0) = \mathcal{CV}$. It is clear that

$$\mathcal{ST}(\alpha) \subseteq \mathcal{ST} \quad \text{and} \quad \mathcal{CV}(\alpha) \subseteq \mathcal{CV}.$$

A classical result of Strohhäcker [138] shows that $\mathcal{CV} \subset \mathcal{ST}(1/2)$.

A function $f \in \mathcal{A}$ is said to be *close-to-convex* in $\mathbb{D}$ if there is a convex function g and a real number θ, $-\pi/2 < \theta < \pi/2$, such that

$$\operatorname{Re} \left(e^{i\theta} \frac{f'(z)}{g'(z)} \right) > 0 \quad z \in \mathbb{D}.$$

This set of functions, denoted by $\mathcal{CCV}$, was introduced by Kaplan [67] in 1952. The subclasses of $\mathcal{S}$, namely convex, starlike and close-to-convex functions are related as follows:

$$\mathcal{CV} \subset \mathcal{ST} \subset \mathcal{CCV}.$$

Indeed, a significant result in the theory of univalent functions is the Noshiro-Warschawski theorem. This theorem states that a function $f \in \mathcal{A}$ whose derivative has positive real part in $\mathbb{D}$ is univalent.

Theorem 1.12. (Noshiro-Warschawski Theorem) [103] *If a function f is analytic in a convex domain D and*

$$\operatorname{Re}\left(e^{i\alpha} f'(z)\right) > 0$$

for some real α, then f is univalent in D.

Using the Noshiro-Warschawski theorem, Kaplan [67] proved that every close-to-convex function is univalent, and thus $\mathcal{CCV} \subset \mathcal{S}$.

For $\alpha \geqslant 0$, a function $f \in \mathcal{A}$ with $f'(z)f(z)/z \neq 0$ is said to be an α-*convex* function if and only if

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0, \quad z \in \mathbb{D}.$$

This class of functions, denoted by M_α, was introduced by Mocanu *et al.*[86]. In 1973, Miller *et al.*[84] proved that functions in the class M_α are univalent and starlike in $\mathbb{D}$. They also showed that all α-convex functions are convex for $\alpha \geq 1$. Evidently, M_0 reduces to the class $\mathcal{ST}$ and M_1 reduces to the class $\mathcal{CV}$.

An analytic function $f \in \mathcal{A}$ with $f'(z)f(z)/z \neq 0$ and $1 + zf''(z)f'(z) \neq 0$ is said to be an $\alpha-$*logarithmically convex* function in $\mathbb{D}$ if and only if

$$\operatorname{Re}\left(\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\right) > 0, \quad z \in \mathbb{D},$$

where $\alpha \in [0,1]$. This set of functions denoted by L_α was introduced by Lewandowski *et al.* [73]. Darus *et al* [35] proved that functions in this class are starlike. They also obtained bounds for $|a_2|$, $|a_3|$ and $|a_3 - \mu a_2|$, where μ is real. Some extreme coefficient problems are also solved. It is clear that L_0 reduces to the class $\mathcal{CV}$ and L_1 reduces to the class $\mathcal{ST}$.

An analytic function f is *subordinate* to g in $\mathbb{D}$, written $f(z) \prec g(z)$, if there exists an analytic function w in $\mathbb{D}$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{D}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. In terms of subordination, the analytic conditions (1.4) and (1.5) can be written respectively as

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}, \tag{1.7}$$

and

$$\frac{zf''(z)}{f'(z)} + 1 \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}. \tag{1.8}$$

This follows because the mapping $m(z) = (1+z)/(1-z)$ maps $\mathbb{D}$ onto the right-half plane, and thus $\operatorname{Re}(m(z)) > 0$.

Ma and Minda [77] gave a unified presentation of various subclasses of starlike and convex functions by replacing the superordinate function $m(z) = (1+z)/(1-z)$ in (1.7) and (1.8) by a more general analytic univalent function φ which has positive real part in $\mathbb{D}$ and normalized by the conditions $\varphi(0) = 1$, $\varphi'(0) > 0$. Furthermore, it is assumed that $\varphi(\mathbb{D})$ is starlike with respect to $\varphi(0) = 1$, and symmetric with respect to the real axis.

The class of *Ma-Minda starlike functions with respect to* φ, denoted by $\mathcal{ST}(\varphi)$, consists of functions $f \in \mathcal{A}$ satisfying the subordination $zf'(z)/f(z) \prec \varphi(z)$. This class can be written as

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z), \quad z \in \mathbb{D} \right\}. \tag{1.9}$$

Similarly the class of *Ma-Minda convex functions with respect to* φ, denoted by $\mathcal{CV}(\varphi)$, consists of functions $f \in \mathcal{A}$ satisfying the subordination $1 + zf''(z)/f'(z) \prec \varphi(z)$. This class is

$$\mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z), \quad z \in \mathbb{D} \right\}.$$

Note that $f \in \mathcal{CV}(\varphi)$ if and only if $zf' \in \mathcal{ST}(\varphi)$.

The class of Ma-Minda starlike functions with respect to φ envelops several well-known subclasses of univalent functions by appropriate choices of φ in (1.9). For instance, when φ is given by

$$\varphi_\alpha(z) := \frac{1 + (1 - 2\alpha)z}{1 - z} = 1 + 2(1 - \alpha)z + 2(1 - \alpha)z^2 + 2(1 - \alpha)z^3 + \cdots,$$

where $0 \leq \alpha < 1$, then $\varphi_\alpha(\mathbb{D}) = \{w : \operatorname{Re} w > \alpha\}$. Therefore, the class of starlike functions of order α which satisfies the analytical condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{D},$$

can be expressed in the form

$$\mathcal{ST}(\alpha) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi_\alpha(z), \quad z \in \mathbb{D} \right\}.$$

For the choice

$$\varphi_{PAR}(z) := 1 + \frac{2}{\pi^2}\left(\log \frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)^2 = 1 + \frac{8}{\pi^2}z + \frac{16}{3\pi^2}z^2 + \frac{184}{45\pi^2}z^3 + \cdots,$$

Rønning [131] showed that φ_{PAR} maps $\mathbb{D}$ onto the parabolic region $\{w = u + iv : v^2 < 2u - 1\} = \{w : \operatorname{Re} w > |w - 1|\}$. Consequently, the class $\mathcal{ST}_P$ of *parabolic starlike* functions which satisfies the analytical condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \mathbb{D},$$

can be expressed in the form

$$\mathcal{ST}_P := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi_{PAR}, \quad z \in \mathbb{D} \right\}.$$

With the choice

$$\varphi_\beta(z) := \left(\frac{1+z}{1-z}\right)^\beta = 1 + 2\beta z + 2\beta^2 z^2 + \frac{2}{3}\beta(1+2\beta^2)z^3 + \cdots, \quad 0 < \beta \le 1,$$

it is evident that $\left|\arg \varphi_\beta(z)\right| = \beta \left|\arg\left((1+z)/(1-z)\right)\right| < \beta\pi/2$. Thus the class $\mathcal{ST}_\beta$ of *strongly starlike functions of order* β which satisfies the analytical condition [31]

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D},$$

can be expressed in the form

$$\mathcal{ST}_\beta := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi_\beta, \quad z \in \mathbb{D} \right\}.$$

For the choice

$$\varphi_L := \sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 + \cdots,$$

it is clear that $\varphi_L(\mathbb{D}) = \{w : |w^2 - 1| < 1\}$. Therefore, the class $\mathcal{ST}_L$ of *lemniscate of Bernoulli starlike* functions which satisfies the analytical condition [136]

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1, \quad z \in \mathbb{D},$$

can be expressed in the form

$$\mathcal{ST}_L := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi_L, \quad z \in \mathbb{D} \right\}.$$

1.3 Analytic Typically Real Functions

An analytic function f is said to be *typically real* in $\mathbb{D}$ if it has real values on the real axis and nonreal values elsewhere. Therefore, typically real function maps the upper unit disk into either the upper half-plane or the lower half-plane, and similarly for the lower unit disk.

Denote by T the class consisting of typically real functions $f \in \mathcal{A}$. This class was introduced and investigated by Rogosinski [130].

For $f \in T$, by definition, f is real whenever z is real, that is, $f(z) = \overline{f(z)}$ for $z = x \in (-1,1)$. Thus $\sum_{n=2}^{\infty}(a_n - \overline{a_n})x^n = 0$, which yields $a_n = \overline{a_n}$. Hence f has real coefficients.

The converse does not hold, as illustrated by the function $f(z) = z + z^2 + 4z^3 \in \mathcal{A}$. It is clear that f has real coefficients. However, f is not typically real because $f(i/2) = -1/4$.

Furthermore, if $f \in T$, then $f'(0) > 0$, and thus near the origin f maps the upper unit disk into the upper half-plane, and the lower unit disk into the lower half-plane. Consequently,

$$(\operatorname{Im} z)(\operatorname{Im} f(z)) > 0, \quad z \in \mathbb{D} \backslash \mathbb{R},$$

when $f \in T$.

Proposition 1.1. *If $f \in \mathcal{S}$ has real coefficients, then $f \in T$.*

Proof. Since f has real coefficients, it follows that f is real whenever z is real. Suppose z is not real. Since f is univalent, it follows that $f(z) \neq f(\bar{z})$. Further f has real coefficients, that is, $f(\bar{z}) = \overline{f(z)}$. Therefore, $f(z) \neq \overline{f(z)}$, and thus $f(z)$ is not real. Hence $f(z)$ is real if and only if z is real, which yields the desired result. □

Note that a typically real function need not be univalent. For instance, let $f(z) = z+z^3$. Then f is not univalent in $\mathbb{D}$ since $f'(i/\sqrt{3}) = 0$. However,

$$\mathrm{Im}(f(z)) = \mathrm{Im}\left((x+iy)(1+x^2-y^2+2ixy)\right) = y(3x^2+1-y^2).$$

Thus

$$\mathrm{Im}(z)\mathrm{Im}(f(z)) = y^2\left(3x^2+(1-y^2)\right) > 0,$$

whenever Im $(z) \neq 0$, and hence f is typically real.

Let $\mathcal{P}_{\mathbb{R}}$ denote the class of all functions in $\mathcal{P}$ with real coefficients. The connection between functions in T and functions in $\mathcal{P}_{\mathbb{R}}$ was established by Rogosinski [130].

Theorem 1.13. [130] *A function $\varphi \in T$ if and only if there exists a function $p \in \mathcal{P}_{\mathbb{R}}$ such that $\varphi(z) = zp(z)/(1-z^2)$.*

1.4 The *k*th-root Transform

Let $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ with $f(z) \neq 0$ in $\mathbb{D}\backslash\{0\}$. Further, let $k \geq 2$ be a fixed integer. The *kth-root transform* of f is defined by

$$F(z) := \left(f\left(z^k\right)\right)^{\frac{1}{k}} = z\left(\frac{f\left(z^k\right)}{z^k}\right)^{\frac{1}{k}}.$$

The following lemma is required to prove the univalence of the kth-root transform whenever $f \in \mathcal{S}$.

Lemma 1.2. [123, p. 142] *If f is analytic in $\mathbb{D}$ with $0 \notin f(\mathbb{D})$, then there exist an analytic function h in $\mathbb{D}$ and an integer $k \geq 2$ such that $h^k = f$.*

Proof. Since $f(z) \neq 0$ in $\mathbb{D}$, it follows that $f'(z)/f(z)$ is analytic in $\mathbb{D}$. By Cauchy's integral theorem [123, p. 139] , there exists a function $g \in \mathcal{H}(\mathbb{D})$ such that

$$g'(z) = \frac{f'(z)}{f(z)}. \tag{1.10}$$

Let $s(z) = f(z)\exp\{-g(z)\}$. It follows from (1.10) that

$$s'(z) = \big(f'(z) - g'(z)f(z)\big)\exp\{-g(z)\} = 0.$$

Then for a fixed $z_0 \in \mathbb{D}$,

$$s(z) - s(z_0) = \int_{z_0}^{z} s'(\zeta)d\zeta = 0,$$

and thus $f(z)\exp\{-g(z)\} = s(z_0)$. As $s(z_0) \neq 0$ in $\mathbb{D}$, we can let $s(z_0) = \exp\{m\}$ for some m. Then $f(z)\exp\{-g(z)\} = \exp\{m\}$, that is, $f(z) = \exp\{g(z)+m\} = \exp\{G(z)\}$, where $G(z) = g(z)+m$. Hence the proof is completed by taking $h(z) = \exp\{G(z)/k\}$ for every $z \in \mathbb{D}$. $\square$

The following result shows that the kth-root transform preserves univalence.

Theorem 1.14. *Let $f \in \mathcal{S}$ and $g(z) = (f(z^k))^{1/k}$ be the kth-root transformation of f. Then $g \in \mathcal{S}$. The branch is chosen so that $\big(f(z^k)/z^k\big)^{1/k} = 1$ at $z = 0$.*

Proof. Since $f \in \mathcal{S}$, it follows that $f(z)/z$ is a nonvanishing analytic function. By applying Lemma 1.2, there exist an analytic function h and an integer $k \geq 2$ such that $h^k(z) = f(z)/z$. Let

$$g(z) = z\left(\frac{f(z^k)}{z^k}\right)^{\frac{1}{k}} = zh(z^k).$$

Since $f\left(z^k\right)/z^k = 1 + \sum_{n=2}^{\infty} a_n z^{k(n-1)} := 1 + x$, and

$$\begin{aligned}
(1+x)^{\frac{1}{k}} &= \sum_{n=0}^{\infty} \frac{(-1/k)_n}{n!}(-x)^n \\
&= 1 + \frac{1}{k}x + \frac{\left(\frac{-1}{k}\right)\left(\frac{-1}{k}+1\right)}{2!}x^2 - \frac{\left(\frac{-1}{k}\right)\left(\frac{-1}{k}+1\right)\left(\frac{-1}{k}+2\right)}{3!}x^3 + \cdots \\
&= 1 + \frac{1}{k}x - \frac{(k-1)}{2k^2}x^2 + \frac{(k-1)(2k-1)}{3!k^3}x^3 + \cdots,
\end{aligned}$$

it follows that

$$g(z) = z\left(\frac{f\left(z^k\right)}{z^k}\right)^{\frac{1}{k}} = z\left(1 + \frac{1}{k}\sum_{n=2}^{\infty} a_n z^{k(n-1)} - \frac{(k-1)}{2k^2}\left(\sum_{n=2}^{\infty} a_n z^{k(n-1)}\right)^2 + \cdots\right)$$

$$= z\left(1+\frac{1}{k}\sum_{n=1}^{\infty} a_{n+1}z^{nk} - \frac{(k-1)}{2k^2}\left(\sum_{n=1}^{\infty} a_{n+1}z^{nk}\right)^2 + \cdots\right).$$

Thus g is normalized with $g(0) = 0$ and $g'(0) = 1$.

Suppose $z_1, z_2 \in \mathbb{D}$ such that $g(z_1) = g(z_2)$. Then $g^k(z_1) = g^k(z_2)$, and thus $f(z_1^k) = f(z_2^k)$. The univalence of f in $\mathbb{D}$ implies that $z_1^k = z_2^k$, and hence, there exists $\beta \in \mathbb{C}$, $\beta^k = 1$, such that $z_2 = \beta z_1$. If $\beta = 1$, then $z_2 = z_1$. Assume that $\beta \neq 1$. It follows that

$$g(z_2) = g(\beta z_1) = \beta z_1 h(\beta^k z_1^k) = \beta z_1 h(z_1^k) = \beta g(z_1) = \beta g(z_2),$$

and thus $(1-\beta)g(z_2) = 0$. Since $\beta \neq 1$, it yields that $g(z_2) = 0$, that is, $z_2 = 0$. Furthermore, $g(z_1) = g(z_2)$ implies $z_1 = 0$, and hence $z_2 = z_1 = 0$. This completes the proof. $\square$

The next result shows that the kth-root transform preserves starlikeness.

Theorem 1.15. *Let $g(z) = (f(z^k))^{1/k}$ be the kth-root transformation of f. Then $g \in \mathcal{ST}$ if and only if $f \in \mathcal{ST}$.*

Proof. It is clear that for each $z \in \mathbb{D}$

$$\frac{zg'(z)}{g(z)} = z^k \frac{f'(z^k)}{f(z^k)}.$$

Thus

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) = \operatorname{Re}\left(z^k \frac{f'(z^k)}{f(z^k)}\right), \quad z \in \mathbb{D},$$

and hence g is starlike if and only if f is starlike. $\square$

The following result shows that the convexity of the kth-root transform of f implies convexity of f. However, the converse does not hold.

Theorem 1.16. *Let $g(z) = (f(z^k))^{1/k}$ be the kth-root transformation of f and $g \in \mathcal{CV}$. Then $f \in \mathcal{CV}$. However, the converse is false.*

Proof. Evidently,

$$g'(z) = g(z)\left(\frac{z^{k-1}f'(z^k)}{f(z^k)}\right). \tag{1.11}$$

Thus

$$1+\frac{zg''(z)}{g'(z)} = \frac{zg'(z)}{g(z)} + k\left(1+\frac{z^k f''(z^k)}{f'(z^k)}\right) - k\left(\frac{z^k f'(z^k)}{f(z^k)}\right).$$

From (1.11), the above equality is equivalent to

$$1+\frac{zg''(z)}{g'(z)} = \frac{zg'(z)}{g(z)} + k\left(1+\frac{z^k f''(z^k)}{f'(z^k)}\right) - k\frac{zg'(z)}{g(z)}. \tag{1.12}$$

It follows that

$$\operatorname{Re}\left(1+\frac{z^k f''(z^k)}{f'(z^k)}\right) = \frac{1}{k}\left(\operatorname{Re}\left(1+\frac{zg''(z)}{g'(z)}\right) + (k-1)\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right)\right),$$

and hence if g is convex, then f is convex.

On the other hand, $f(z) = z/(1-z)$ is a convex function such that for $z \in \mathbb{D}$,

$$\begin{aligned} k\ \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) - (k-1)\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= k\ \operatorname{Re}\left(\frac{1+z}{1-z}\right) - (k-1)\operatorname{Re}\left(\frac{1}{1-z}\right) \\ &< k\ \operatorname{Re}\left(\frac{1+z}{1-z}\right) - \frac{(k-1)}{2}. \end{aligned}$$

By taking $k = 2$ and $z_0 = \sqrt{3/5}i \in \mathbb{D}$, it is evident that $z_0^2 \in \mathbb{D}$, and

$$2\ \operatorname{Re}\left(1+\frac{z_0^2 f''(z_0^2)}{f'(z_0^2)}\right) - \operatorname{Re}\left(\frac{z_0^2 f'(z_0^2)}{f(z_0^2)}\right) < 2\ \operatorname{Re}\left(\frac{1+z_0^2}{1-z_0^2}\right) - \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0. \tag{1.13}$$

Equations (1.11) and (1.12) show that

$$\mathrm{Re}\left(1+\frac{zg''(z)}{g'(z)}\right) = k\ \mathrm{Re}\left(1+\frac{z^k f''(z^k)}{f'(z^k)}\right) - (k-1)\,\mathrm{Re}\left(\frac{z^k f'(z^k)}{f(z^k)}\right).$$

Thus, it follows from (1.13) that g is not convex. □

The kth-root transform has been widely used in a variety of ways in complex function theory. Bounds for the Fekete-Szegö coefficient functional associated with kth-root transform $(f(z^k))^{1/k}$ of normalized analytic functions f were derived in [20]. Annamalai *et al.* [25] obtained a bound of the Fekete Szegö coefficient functional for the Janowski α-Spirallike functions associated with the kth-root root transformation.

1.5 The Second Hankel Determinant

For positive integers q and n, the Hankel determinant $H_q(n)$ for an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is defined by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.14}$$

Hankel determinants play an important role in the study of singularities. For instance, Dienes [37, p.333] showed that if the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has at most p poles and no other singularities on the circumference of its circle of convergence, then $\lim_{n\to\infty}\left|\sqrt[n]{H_p(n)}\right| = 1$. Furthermore, Hankel determinants are useful in the study of a function of bounded characteristic. For example, Cantor [32] proved that if the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a ratio of two bounded analytic functions in $\mathbb{D}$, then

$\lim_{q\to\infty} H_{q+1}(n) = 0$.

The growth rate of Hankel determinant $H_q(n)$ as $n \to \infty$ was obtained by Pommerenke [121]. Various authors [92, 94] and [100] have investigated the growth rate of Hankel determinant $H_q(n)$ for a certain subclass of analytic functions by essentially following Pommerenke's method.

Pommerenke [122] proved that Hankel determinants of univalent functions satisfy

$$|H_q(n)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}} \quad (n = 1, 2, \ldots,\ q = 2, 3, \ldots),$$

where $\beta > 1/4000$ and K depends only on q.

Hankel determinants have also been discussed for several subclasses of analytic functions by many authors. For instance, in the works by Ehrenborg [42], Layman [71], Noor [95, 96, 97, 98, 99], Noor and Al-Bany [101] and Noor [102]. The Hankel determinant of meromorphic functions was obtained in [142]. Various properties of these determinants can be found in [141, Chapter 4].

It is evident that $H_2(1) = a_3 - a_2^2$ is the Fekete-Szegö coefficient functional for $f \in \mathcal{A}$. Interestingly the determinant also satisfies $H_2(1) = S_f(0)/6$, where S_f is the Schwarzian derivative of f defined in [33] by $S_f = (f''/f')' - (f''/f')^2/2$. Ali *et al.* [20] investigated the Fekete-Szegö coefficient functional for the kth-root transform of functions belonging to several classes defined via subordination.

In recent years, several authors have investigated bounds for the second Hankel determinant $H_2(2) = a_2a_4 - a_3^2$ of functions belonging to various subclasses of univalent and multivalent functions. For example, Elhosh obtained bounds for the second Hankel determinant of univalent functions and close-to-convex functions respectively in [43, 44]. In addition, Halim *et al.* [53, 63] and [64] obtained bounds for the second

Hankel determinant for certain subclasses of analytic functions. Singh [134] established a bound for the second Hankel determinant for analytic functions with respect to other points. Moreover, Lee *et al.* [72] investigated bounds for the second Hankel determinant for functions belonging to subclasses of Ma-Minda starlike and convex functions and two other related classes defined by subordination.

Hayami and Owa [55, 57] obtained a bound for the generalized functional $|a_n a_{n+2} - \mu a_{n+1}^2|$ by using the Hankel determinant $H_2(n)$ for all $n \geq 1$ and some real number μ for several subclasses of $\mathcal{A}$. These authors [54] also studied a bound for the functional $|a_{p+2} - \mu a_{p+1}^2|$ for p-valent analytic functions. They also obtained a bound for the functional $|a_{p+1}a_{p+3} - \mu a_{p+2}^2|$ for p-valent analytic functions in [56]. Similar study of finding bounds for other classes of p-valent analytic functions was discussed in [140].

1.6 Radius Problems

Another active topic of investigation in the theory of univalent functions is the radius problem. Although not all analytic functions $f \in \mathcal{A}$ are univalent in the unit disk, for z near to the origin, the behavior of a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is similar to the identity map. Therefore, f maps a sufficiently small disk $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ univalently onto some domain. The radius of the largest disk in $\mathbb{D}$ where f is univalent is called the radius of univalence for f. For instance, the function $f(z) = z + 3z^2 \in \mathcal{A}$ is not locally univalent at $z_0 = -1/6$ since $f'(-1/6) = 0$. However, the Noshiro-Warschawski result shows that the function f is univalent in the disk $|z| < 1/6$. Thus *the radius of univalence* for the function $f(z) = z + 3z^2$ is $r_0 = 1/6$.

Similarly, every univalent function $f \in \mathcal{A}$ is not necessarily starlike in the unit disk. However, we can find a sufficiently small disk $\mathbb{D}_r$ such that f maps $\mathbb{D}_r$ onto a starlike domain. The radius of the largest disk with this property is called the radius of starlikeness for f. Let $\mathcal{F} \subset \mathcal{A}$ be a set of analytic functions. The radius of the largest disk in $\mathbb{D}$ such that every function $f \in \mathcal{F}$ maps the disk onto a starlike domain is called *the radius of starlikeness* for the class $\mathcal{F}$. If $r_{\mathcal{ST}}$ is the radius of starlikeness for the class $\mathcal{F}$, then equivalently $r^{-1}f(rz) \in \mathcal{ST}$ for $r \leq r_{\mathcal{ST}}$, and $f \in \mathcal{F}$. In particular, if $\mathcal{F} = \mathcal{S}$, then the radius of starlikeness for the class $\mathcal{S}$ is $r_{\mathcal{ST}} = \tanh(\pi/4) \approx 0.65579$ [52].

Analogously, the radius of the largest disk in $\mathbb{D}$ such that every function $f \in \mathcal{F}$ maps the disk onto a convex domain is called *the radius of convexity* for the class $\mathcal{F}$. If $r_{\mathcal{CV}}$ is the radius of convexity for the class $\mathcal{F}$, then equivalently $r^{-1}f(rz) \in \mathcal{CV}$ for $r \leq r_{\mathcal{CV}}$, and $f \in \mathcal{F}$. It is known [90] that the *radius of convexity* for the class $\mathcal{S}$ is $r_{\mathcal{CV}} = 2-\sqrt{3} \approx 0.26795$.

Let $\mathcal{F} \subset \mathcal{A}$ be a set of analytic functions, and let $\mathcal{U}$ be the class of functions $f \in \mathcal{A}$ satisfying $|(z/f(z))^2 f'(z)-1| < 1$ for $z \in \mathbb{D}$. Then every analytic function $f \in \mathcal{F}$ is not necessarily in the class $\mathcal{U}$. However, we can find a sufficiently small disk $\mathbb{D}_r$ such that f satisfies $|(z/f(z))^2 f'(z)-1| < 1$ in the disk $\mathbb{D}_r$. The radius of the largest disk in $\mathbb{D}$ such that every function $f \in \mathcal{F}$ satisfies $r^{-1}f(rz) \in \mathcal{U}$ is called the $\mathcal{U}$-radius for the class $\mathcal{F}$ and denoted by $r_{\mathcal{U}}$.

In general, for two families $\mathcal{G}$ and $\mathcal{F}$ of $\mathcal{A}$, the $\mathcal{G}$*-radius* for the class $\mathcal{F}$, denoted by $R_{\mathcal{G}(\mathcal{F})}$, is the largest number R such that $r^{-1}f(rz) \in \mathcal{G}$ for $0 < r \leq R$, and $f \in \mathcal{F}$.

The radius of close-to-convexity for the class $\mathcal{S}$ was determined by Krzyż [69]. Several authors have investigated the problem of finding the radius constants for subclasses of $\mathcal{A}$. For instance, Ali *et al.* [23] obtained radius constants for several classes of analytic functions on the unit disk $\mathbb{D}$ which includes the radius of starlikeness of positive order, radius of parabolic starlikeness, radius of Bernoulli lemniscate starlikeness, and radius of uniform convexity. Some results of radius problems have also been derived by Goodman [48, Chapter 13].

1.7 Harmonic Mappings

Let D be a domain in $\mathbb{R}^2$. A real-valued function $u : \mathrm{D} \longrightarrow \mathbb{R}$ is called *harmonic* if all its second partial derivatives exist and are continuous in D, and satisfies the Laplacian equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A complex-valued function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ in a domain D is harmonic if the two coordinate functions u and v are real harmonic in D. Thus a complex-valued harmonic function f satisfies Laplacian equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Since $z = x + iy$, it follows that

$$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z-\bar{z}}{2i}.$$

By using the chain rule, it is evident that

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{1.15}$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{1.16}$$

Consequently, for a complex valued function $w = u + iv$ with continuous partial derivatives, it is clear that

$$\begin{aligned}\frac{\partial \overline{w}}{\partial z} &= \frac{1}{2}\big((u_x - iv_x) - i(u_y - iv_y)\big) = \frac{1}{2}\big((u_x - v_y) - i(u_y + v_x)\big) \\ &= \overline{\frac{1}{2}\big((u_x - v_y) + i(u_y + v_x)\big)} = \overline{\left(\frac{\partial w}{\partial \overline{z}}\right)},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \overline{w}}{\partial \overline{z}} &= \frac{1}{2}\big((u_x - iv_x) + i(u_y - iv_y)\big) = \frac{1}{2}\big((u_x + v_y) + i(u_y - v_x)\big) \\ &= \overline{\frac{1}{2}\big((u_x + v_y) - i(u_y - v_x)\big)} = \overline{\left(\frac{\partial w}{\partial z}\right)}.\end{aligned}$$

Since

$$f_{\overline{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}\big((u_x + iv_x) + i(u_y + iv_y)\big) = \frac{1}{2}\big((u_x - v_y) + i(u_y + v_x)\big),$$

it follows from the Cauchy Riemann equations $u_x = v_y$ and $u_y = -v_x$, that a function f is analytic in a domain D if and only if $f_{\overline{z}} = 0$, that is, f is independent of $\overline{z}$.

A direct calculation shows that the Laplacian of f becomes

$$\begin{aligned}\Delta f = f_{xx} + f_{yy} = f_{xx} - if_{yx} + if_{xy} + f_{yy} &= (f_x - if_y)_x + i(f_x - if_y)_y \\ &= 4\left(\frac{f_x - if_y}{2}\right)_{\overline{z}} = 4f_{z\overline{z}}.\end{aligned}$$

Thus f is harmonic if and only if f_z is analytic.

Proposition 1.2. *Let f be a harmonic function in a domain* D. *Then the composition $f \circ \psi$ is harmonic in Ω for any analytic function $\psi : \Omega \longrightarrow$* D.

Proof. Setting

$$F(z) = (f \circ \psi)(z) = f(\psi(z)) = f(w),$$

then

$$F_z(z) = \frac{\partial f(w)}{\partial w}\frac{\partial w(z)}{\partial z} + \frac{\partial f(w)}{\partial \overline{w}}\frac{\partial \overline{w(z)}}{\partial z}.$$

Since w is analytic, it follows that $\partial \overline{w(z)}/\partial z = \overline{(\partial w(z)/\partial \overline{z})} = 0$. Thus

$$F_z(z) = \frac{\partial f(w)}{\partial w}\frac{\partial w(z)}{\partial z}.$$

Also, f is harmonic. Then

$$\begin{aligned} F_{z\overline{z}}(z) &= \frac{\partial}{\partial \overline{z}}\left(\frac{\partial f(w)}{\partial w}\frac{\partial w(z)}{\partial z}\right) = \left(\frac{\partial^2 f(w)}{\partial w^2}\frac{\partial w(z)}{\partial \overline{z}} + \frac{\partial^2 f(w)}{\partial \overline{w}\partial w}\frac{\partial \overline{w(z)}}{\partial \overline{z}}\right)\frac{\partial w}{\partial z} + \frac{\partial f(w)}{\partial w}\left(\frac{\partial^2 w(z)}{\partial \overline{z}\partial z}\right) \\ &= \frac{\partial^2 f(w)}{\partial \overline{w}\partial w}\frac{\partial w(z)}{\partial z}\frac{\partial \overline{w(z)}}{\partial \overline{z}} = f_{w\overline{w}}(w)\frac{\partial w(z)}{\partial z}\overline{\left(\frac{\partial w(z)}{\partial z}\right)} = f_{w\overline{w}}(w)\left|\frac{\partial w(z)}{\partial z}\right|^2 = 0. \end{aligned}$$

Hence the composition $f \circ \psi$ is harmonic. $\square$

However, if f is a harmonic function and ψ is analytic, then $\psi \circ f$ need not be harmonic. For instance, $F(z) = \psi \circ f = (z + \overline{z}/2)^2$, where $f(z) = z + \overline{z}/2$ and $\psi = z^2$. It is evident that $f_{z\overline{z}} = 0$, but $F_{z\overline{z}} = 1$. Thus $\psi \circ f$ is not harmonic.

A mapping is said to be sense-preserving if it preserves the orientation, or sense of the angle between two curves. A sense-preserving mapping does not necessarily preserve the magnitude of the angle between the intersecting curves.

The Jacobian of a function $f(z) = u(x,y) + iv(x,y)$ at a point z is given by

$$J_f(z) := \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

If f is an analytic function, then f satisfies the Cauchy Riemann equations, and thus its Jacobian has the following form

$$J_f(z) = (u_x)^2 + (v_x)^2 = |f'(z)|^2.$$

Hence an analytic function f is locally univalent in $\mathbb{D}$ if and only if $J_f(z) \neq 0$. In 1936, Lewy [74] showed that this property remains true for harmonic functions. In view of Lewy's theorem, harmonic mappings are sense-preserving (or orientation-preserving) if $J_f(z) > 0$, and sense-reversing if $J_f(z) < 0$ throughout the domain D.

The Jacobian may be expressed equivalently in terms of z derivative and $\bar{z}$ derivative:

$$\begin{aligned}
J_f(z) = u_x v_y - u_y v_x &= \frac{1}{4}\left((u_x+v_y)^2 + (u_y-v_x)^2 - (u_x-v_y)^2 - (u_y+v_x)^2\right) \\
&= \frac{1}{4}\left(|(u_x+v_y) - i(u_y-v_x)|^2 - |(u_x-v_y) + i(u_y+v_x)|^2\right) \\
&= \frac{1}{4}\left(|u_x+iv_x - i(u_y+iv_y)|^2 - |u_x+iv_x + i(u_y+iv_y)|^2\right) \\
&= \frac{1}{4}\left(|f_x - if_y|^2 - |f_x+if_y|^2\right) \\
&= |f_z(z)|^2 - |f_{\bar{z}}(z)|^2.
\end{aligned}$$

It is evident that if $u = u(x,y)$ is harmonic, then $\phi = u_x - iu_y$ is analytic. Also, it is known that for any analytic function g in a simply connected domain D, there exists an antiderivative G in D, that is, $G'(z) = g(z)$ [123, p. 139]. Thus it follows that if u is a real-valued harmonic function defined in a simply connected domain D, then there is an analytic function Φ such that $\operatorname{Re}\Phi(z) = u$.

Let $f = u + iv$ be a harmonic function in a simply connected domain D. Then there exist analytic functions F and G in D such that

$$u(z) = \operatorname{Re}F(z) = \frac{F(z)+\overline{F(z)}}{2}, \quad v(z) = \operatorname{Re}G(z) = \frac{G(z)+\overline{G(z)}}{2}.$$

Thus

$$\begin{aligned}
f(z) = u(z) + iv(z) &= \frac{F(z)+\overline{F(z)}}{2} + i\frac{G(z)+\overline{G(z)}}{2} \\
&= \left(\frac{F(z)+iG(z)}{2}\right) + \overline{\left(\frac{F(z)-iG(z)}{2}\right)}.
\end{aligned}$$

Write $h = (F+iG)/2$, $g = (F-iG)/2$. Then f has the canonical representation $f(z) = h(z) + \overline{g(z)}$, where h and g are analytic in $\mathbb{D}$. The functions h and g are respectively called the *analytic* and *co-analytic* parts of f.

Now if $f(z) = h(z) + \overline{g(z)}$ is a harmonic function, then its Jacobian has the form

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |h'(z)|^2 - |\overline{g'(z)}|^2 = |h'(z)|^2 - |g'(z)|^2.$$

Thus a harmonic function $f(z) = h(z) + \overline{g(z)}$ is locally univalent and sense-preserving in D if $|h'(z)| > |g'(z)|$. On the other hand, f is locally univalent and sense-reversing if $|h'(z)| < |g'(z)|$.

Let $B(\mathrm{D})$ denote the set of bounded analytic functions $a \in \mathcal{H}(\mathrm{D})$ satisfying $|a(z)| < 1$ for $z \in \mathrm{D}$.

Necessary and sufficient conditions for a function f to be harmonic is obtained in the following result.

Theorem 1.17. [39, p. 6] *Let f be a complex-valued function defined in a domain* D *having continuous second partial derivatives. Then f is a harmonic and sense-preserving mapping in* D *if and only if f is a solution of the elliptic partial differential equation*

$$\overline{f_{\bar{z}}} = a f_z$$

for some $a \in B(\mathrm{D})$.

Proof. Suppose that $f = h + \bar{g}$ is a harmonic function and sense-preserving in D. Then $f_z(z) = h'(z) \neq 0$ and $\overline{f_{\bar{z}}} = g'(z)$. Define a function a by

$$a(z) := \frac{g'(z)}{h'(z)}.$$

Then a is analytic in D and $|a(z)| < 1$. The last relation yields the desired result.

Conversely, suppose that f is a solution of the partial differential equation $\overline{f_{\overline{z}}} = af_z$. Computations from (1.15) and (1.16) show that

$$\overline{(f_z)} = \left(\overline{f}\right)_{\overline{z}}.$$

Replacing f by $f_{\overline{z}}$ in the above equality gives

$$\overline{(f_{\overline{z}z})} = \left(\overline{f_{\overline{z}}}\right)_{\overline{z}}.$$

Differentiating the equation $\overline{f_{\overline{z}}} = af_z$ with respect to $\overline{z}$, leads to

$$\left(\overline{f_{\overline{z}}}\right)_{\overline{z}} = \overline{(f_{\overline{z}z})} = af_{z\overline{z}} + a_{\overline{z}}f_z. \tag{1.17}$$

Since a is analytic, it implies that $a_{\overline{z}} = 0$, and thus (1.17) yields $\overline{f_{\overline{z}z}} = af_{z\overline{z}}$. Further, as $|a(z)| < 1$, then $f_{z\overline{z}} = 0$. Therefore, f is harmonic. Also, since $|f_{\overline{z}}| < |f_z|$, it follows that $J_f = |f_z|^2 - |f_{\overline{z}}|^2 > 0$. Hence f is sense-preserving. $\square$

Definition 1.4. *The equation $\overline{f_{\overline{z}}} = af_z$, where $|a(z)| < 1$ for $z \in$ D is called the Beltrami equation. The complex-valued function a is called the second dilatation of f.*

Denote by $\mathcal{S}_H$ the class of all univalent and sense-preserving harmonic functions $f = h + \overline{g}$ in the unit disk $\mathbb{D}$, normalized by $h(0) = g(0) = 0$ and $h'(0) = 1$. Thus the power series expansions of the analytic functions h and g are given by

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k.$$

Since $J_f > 0$, it follows that $|g'(0)| < |h'(0)| = 1$.

This class was introduced and investigated by Clunie and Sheil-Small [34]. The class $\mathcal{S}_H$ contains the standard class S of analytic univalent functions.

Duren [38, p. 9] proved that the class $\mathcal{S}$ is *normal* in $\mathbb{D}$, that is, every sequence of functions f_n in $\mathcal{S}$ has a subsequence which converges locally uniformly in $\mathbb{D}$. Note that the definition of a normal family does not require that the limit function of a convergent subsequence be necessarily in $\mathcal{S}$. He also showed in [38, p. 9] that $\mathcal{S}$ is *compact* in $\mathbb{D}$, that is, the limits of all converging sequence of functions f_n in $\mathcal{S}$ are functions belonging to $\mathcal{S}$.

Clunie and Sheil-Small [34] showed that $\mathcal{S}_H$ is normal, but not compact. For instance, consider the sequence of functions $f_n \in \mathcal{S}_H$ given by

$$f_n(z) = z + \frac{n}{n+1}\bar{z}.$$

Then $\lim_{n\to\infty} f_n(z) = f(z) = 2\operatorname{Re} z$. For a given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| = \left| z + \frac{n}{n+1}\bar{z} - (z+\bar{z}) \right| = \left| \frac{\bar{z}}{n+1} \right| < \frac{1}{n+1} < \varepsilon$$

for any $z \in \mathbb{D}(z_0, r)$ and $n > N$. This shows that f_n converges locally uniformly to the function f in $\mathbb{D}$. It is evident that the limit function is harmonic in $\mathbb{D}$ since $f_{z\bar{z}} = 0$. However, f is not univalent, and thus $f \notin \mathcal{S}_H$. Hence $\mathcal{S}_H$ is not compact.

Clunie and Sheil-Small [34] also investigated the subclass $\mathcal{S}_H^0$ consisting of functions $f = h + \overline{g} \in \mathcal{S}_H$ with $g'(0) = 0$. The series expansions of h and g for the subclass $\mathcal{S}_H^0$ are given by

$$h(z) = z + \sum_{n=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_k z^k.$$

They also proved that $\mathcal{S}_H^0$ is normal and compact.

Univalent harmonic functions have been extensively studied in [16, 34, 38, 40, 41, 59, 60, 61, 62] and [66].

1.8 Logharmonic Mappings

Recall that $B(\mathrm{D})$ is the set of functions $a \in \mathcal{H}(\mathrm{D})$ satisfying $|a(z)| < 1$ for $z \in \mathrm{D}$. Let B_0 denote its subclass consisting of $a \in B(\mathrm{D})$ with $a(0) = 0$.

Definition 1.5. *A nonconstant function f in* D *is called logharmonic with respect to a if f is a solution of the nonlinear elliptic partial differential equation*

$$\overline{\left(\frac{f_{\bar{z}}(z)}{f(z)}\right)} = a(z)\frac{f_z(z)}{f(z)}. \tag{1.18}$$

Suppose $0 \notin f(\mathrm{D})$. Then the equation (1.18) is equivalent to

$$\overline{\frac{\partial}{\partial \bar{z}} \log(f(z))} = a(z)\frac{\partial}{\partial z}\log(f(z)).$$

Hence $\log(f(z))$ is harmonic and sense-preserving in D, and the function a is called the second dilatation of $\log f$.

The Jacobian of a logharmonic function f with respect to a satisfies

$$\begin{aligned} J_f = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 &= |f_z(z)|^2\left(1 - \left|\frac{\overline{f_{\bar{z}}(z)}f(z)}{\overline{f(z)}f_z(z)}\right|^2\right) \\ &= |f_z(z)|^2\,(1 - |a(z)|^2). \end{aligned}$$

Thus J_f is positive. Therefore, all nonconstant logharmonic mappings are sense-preserving and locally univalent in D. It is evident that if $a = 0$, then $f \in \mathcal{H}(\mathrm{D})$.

Proposition 1.3. *Let f and g be logharmonic mappings with respect to $a \in B(\mathrm{D})$. Then fg is logharmonic with respect to a.*

Proof. Since

$$\overline{\left(\frac{(fg)_{\overline{z}}}{fg}\right)} = \overline{\left(\frac{f_{\overline{z}}}{f}\right)} + \overline{\left(\frac{g_{\overline{z}}}{g}\right)} = a(z)\left(\frac{f_z}{f} + \frac{g_z}{g}\right) = a(z)\frac{(fg)_z}{fg},$$

it follows that fg is a solution of the equation (1.8). Therefore, the function fg is logharmonic with respect to a. □

Proposition 1.4. *Let f and g be logharmonic mappings with respect to $a \in B(\mathrm{D})$ and $0 \notin g(\mathrm{D})$. Then f/g is logharmonic with respect to a.*

Proof. Since

$$\overline{\left(\frac{\left(\frac{f}{g}\right)_{\overline{z}}}{\frac{f}{g}}\right)} = \overline{\left(\frac{f_{\overline{z}}}{f}\right)} - \overline{\left(\frac{g_{\overline{z}}}{g}\right)} = a(z)\left(\frac{f_z}{f} - \frac{g_z}{g}\right) = a(z)\frac{\left(\frac{f}{g}\right)_z}{\frac{f}{g}},$$

it follows that f/g is a solution of the equation (1.8). Therefore, the function f/g is logharmonic with respect to a. □

Remark 1.1. Let f be a logharmonic mapping with respect to $a \in B(\mathrm{D})$. Then the translations in the image do not preserve logharmonicity.

To see this, consider $f(z) = z|z|^2 = z^2\overline{z}$, $w_0 = -1$, and $F(z) = f(z) - w_0 = z|z|^2 + 1$. Then

$$\frac{f_z(z)}{f(z)} = \frac{2}{z}, \quad \text{and} \quad \overline{\left(\frac{f_{\overline{z}}(z)}{f(z)}\right)} = \frac{1}{z},$$

and thus

$$a(z) = \frac{\overline{\left(\frac{f_{\overline{z}}(z)}{f(z)}\right)}}{\frac{f_z(z)}{f(z)}} = \frac{1}{2}.$$

Hence f is a solution of the equation (1.8). Therefore, f is logharmonic with respect to a.

Since

$$\frac{F_z(z)}{F(z)} = \frac{2z\overline{z}}{z^2\overline{z}+1} \quad \text{and} \quad \overline{\left(\frac{F_{\overline{z}}(z)}{F(z)}\right)} = \overline{\left(\frac{z^2}{z^2\overline{z}+1}\right)},$$

it follows that

$$a(z) = \frac{\overline{\left(\frac{F_{\overline{z}}(z)}{F(z)}\right)}}{\frac{F_z(z)}{F(z)}} = \frac{\overline{z}^2}{z\overline{z}^2+1}\,\frac{z^2\overline{z}+1}{2z\overline{z}} = \frac{\overline{z}(z^2\overline{z}+1)}{2z(z\overline{z}^2+1)}.$$

It is evident that a is dependent on $\overline{z}$, and thus a is not analytic. Hence, F is not logharmonic.

Remark 1.2. Let f be a logharmonic mapping with respect to $a \in B(\mathrm{D})$. Then the inverse of a logharmonic mapping is not necessarily logharmonic.

To see this, consider $f(z) = (z-1)|z-1|^2 = (z-1)^2\overline{(z-1)}$. Then

$$\frac{f_z(z)}{f(z)} = \frac{2}{z-1}, \quad \text{and} \quad \overline{\left(\frac{f_{\overline{z}}(z)}{f(z)}\right)} = \overline{\left(\frac{1}{\overline{z-1}}\right)} = \frac{1}{z-1},$$

and thus

$$a(z) = \frac{\overline{\left(\frac{f_{\overline{z}}(z)}{f(z)}\right)}}{\frac{f_z(z)}{f(z)}} = \frac{1}{2}.$$

Hence f is a solution of the equation (1.8). Therefore, f is logharmonic with respect to a. Further, it is known in [7, Theorem 3.4] that f is univalent.

Now, let $g(z) = z^{2/3}\overline{z}^{-1/3}+1$. Then

$$(f\circ g)(z) = f(z^{2/3}\overline{z}^{-1/3}+1) = (z^{2/3}\overline{z}^{-1/3})^2(\overline{z}^{2/3}z^{-1/3}) = z,$$

and

$$(g\circ f)(z) = g\left((z-1)^2\overline{(z-1)}\right) = \left((z-1)^2\overline{(z-1)}\right)^{2/3}\left(\overline{(z-1)^2\overline{(z-1)}}\right)^{-1/3}+1$$
$$= (z-1)+1 = z.$$

Thus g is the inverse of the function f.

Since

$$\frac{g_z(z)}{g(z)} = \frac{2\overline{z}^{-1/3}}{3z^{1/3}(z^{2/3}\overline{z}^{-1/3}+1)},$$

and

$$\overline{\left(\frac{g_{\overline{z}}(z)}{g(z)}\right)} = \overline{\left(\frac{-z^{2/3}}{3\overline{z}^{4/3}(z^{2/3}\overline{z}^{-1/3}+1)}\right)},$$

it follows that

$$a(z) = \frac{\overline{\left(\frac{g_{\overline{z}}(z)}{g(z)}\right)}}{\frac{g_z(z)}{g(z)}} = \frac{-\overline{z}^{2/3}}{3z^{4/3}(\overline{z}^{2/3}z^{-1/3}+1)} \frac{3z^{1/3}(z^{2/3}\overline{z}^{-1/3}+1)}{2\overline{z}^{-1/3}} = \frac{-\overline{z}(z^{2/3}\overline{z}^{-1/3}+1)}{2z(\overline{z}^{2/3}z^{-1/3}+1)},$$

which shows that a is dependent on $\overline{z}$, and thus a is not analytic. Hence, g is not logharmonic.

The study of logharmonic mappings was initiated mainly by Abdulhadi and Bshouty [7]. The basic theory of logharmonic mappings was developed by Abdulhadi and Hengartner in [9, 8, 10, 11, 12, 13] and [14].

A local representation for logharmonic mappings was given by Abdulhadi and Bshouty in [7]. In particular, the local representation for a logharmonic function f at a point $z_0 \in D$ where f vanishes is given in the following result.

Theorem 1.18. [7] *Let f be a logharmonic mapping in D with respect to $a \in B(D)$. Suppose that $f(z_0) = 0$ and $\mathbb{B}(z_0,\rho)\backslash\{z_0\} \subset D\backslash Z(f)$, where $\mathbb{B}(z_0,\rho) = \{z : |z-z_0| < \rho\}$ and $Z(f)$ the zero set of f. Then f admits the representation*

$$f(z) = (z-z_0)^m |z-z_0|^{2\beta m} h(z)\overline{g(z)}, \quad z \in \mathbb{B}(z_0,\rho), \tag{1.19}$$

where $m \in \mathbb{N}$, and

$$\beta = \frac{\overline{a(z_0)}(1+a(z_0))}{1-|a(z_0)|^2}.$$

The functions h and g are in $\mathcal{H}(\mathbb{B}(z_0,\rho))$, with $h(z_0)\neq 0$ and $g(z_0)=1$.

Note that

$$\begin{aligned}
\operatorname{Re}\beta &= \operatorname{Re}\left(\frac{\overline{a(z_0)}(1+a(z_0))}{1-|a(z_0)|^2}\right)\\
&= \frac{\operatorname{Re} a(z_0)+|a(z_0)|^2}{1-|a(z_0)|^2}\\
&\geq \frac{|a(z_0)|^2-|a(z_0)|}{1-|a(z_0)|^2}\\
&= \frac{-|a(z_0)|}{1+|a(z_0)|} > -\frac{1}{2}.
\end{aligned}$$

Let f be a nonconstant logharmonic mapping defined in $\mathbb{D}$ that vanishes only at $z=0$. Then the representation (1.19) of f becomes

$$f(z)=z^m|z|^{2\beta m}h(z)\overline{g(z)}, \quad z\in\mathbb{D}, \tag{1.20}$$

where m is a nonnegative integer, $\operatorname{Re}\beta > -1/2$. Further, the functions h and g are analytic functions in $\mathbb{D}$ satisfying $h(0)\neq 0$ and $g(0)=1$. The exponent β in (1.20) depends only on $a(0)$ and is given by

$$\beta=\overline{a(0)}\frac{1+a(0)}{1-|a(0)|^2}.$$

Note that $f(0)\neq 0$ if and only if $m=0$, and that a univalent logharmonic mapping in $\mathbb{D}$ vanishes at the origin if and only if $m=1$, that is, f has the form

$$f(z)=z|z|^{2\beta}h(z)\overline{g(z)}, \quad z\in\mathbb{D},$$

where $\operatorname{Re}\beta > -1/2$, $0\notin (hg)(\mathbb{D})$ and $g(0)=1$.

Denote by $\mathcal{S}_{Lh}$ the class consisting of univalent logharmonic mappings f in $\mathbb{D}$ with respect to some $a\in B_0$ of the form

$$f(z) = zh(z)\overline{g(z)},$$

normalized by $h(0) = g(0) = 1$ and $0 \notin (hg)(\mathbb{D})$. This class has been studied extensively in recent years, for instance, in the works of [1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 13, 14, 15] and [82].

Let $f = zh(z)\overline{g(z)}$ be a univalent logharmonic mapping. Then f is a *starlike logharmonic* of order α if

$$\frac{\partial}{\partial\theta}\arg f(re^{i\theta}) > \alpha, \quad 0 \le \alpha < 1.$$

Since

$$\begin{aligned}
\frac{\partial}{\partial\theta}\arg f(re^{i\theta}) =& \frac{\partial}{\partial\theta}\operatorname{Im}\log f(re^{i\theta}) \\
&= \operatorname{Im}\left(\frac{\partial}{\partial\theta}\log f(re^{i\theta})\right) \\
&= \operatorname{Im}\left(\frac{\partial \log f(z)}{\partial z}\frac{\partial z}{\partial\theta} + \frac{\partial \log f(z)}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial\theta}\right) \\
&= \operatorname{Im} i\left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f}\right) \\
&= \operatorname{Re}\frac{zf_z - \bar{z}f_{\bar{z}}}{f},
\end{aligned}$$

it follows that f is starlike logharmonic of order α if f satisfies the condition

$$\operatorname{Re}\frac{zf_z - \bar{z}f_{\bar{z}}}{f} > \alpha, \quad 0 \le \alpha < 1,$$

for all $z \in \mathbb{D}$. Denote by $\mathcal{ST}_{Lh}(\alpha)$ the subclass of $\mathcal{S}_{Lh}$ consisting of all starlike logharmonic mappings of order α. If $\alpha = 0$, then $\mathcal{ST}_{Lh}(0) := \mathcal{ST}_{Lh}$ is the class of starlike logharmonic mappings.

Two representation theorems for functions in $\mathcal{ST}_{Lh}$ were obtained by Abdulhadi and Abu Muhanna [4]. First, they established a connection between the class of starlike logharmonic mappings of order α and the class of starlike analytic functions of order

α.

Theorem 1.19. [4] *Let $f(z) = zh(z)\overline{g(z)}$ be logharmonic in $\mathbb{D}$ with $0 \notin hg(\mathbb{D})$. Then $f \in \mathcal{ST}_{Lh}(\alpha)$ if and only if $\varphi(z) = zh(z)/g(z) \in \mathcal{ST}(\alpha)$ for $0 \le \alpha < 1$.*

The second is an integral representation theorem. This result can be found in [4]. Starlike logharmonic mapping is an active subject of investigation, and several recent works include those of [27, 28] and [120].

Let D be a simply connected domain in $\mathbb{C}$ containing the origin. Then D is said to be $\alpha-$*spirallike,* $|\alpha| < \pi/2$, if $w\exp(-te^{i\alpha}) \in D$ for all $t \ge 0$ whenever $w \in$ D. The class of $\alpha-$spirallike was defined by Spacek [137]. Evidently, if $\alpha = 0$, then D is starlike with respect to the origin (see page 6).

Definition 1.6. ($\alpha-$spirallike function) [38, p. 52] *A function $f \in \mathcal{A}$ is called a $\alpha-$spirallike function if it maps $\mathbb{D}$ onto a $\alpha-$spirallike domain.*

The following theorem gives a sufficient and necessary conditions for analytic functions to be $\alpha-$spirallike.

Theorem 1.20. [38, p. 52] *Let $f \in \mathcal{A}$, and $|\alpha| < \pi/2$. Then f is $\alpha-$spirallike in $\mathbb{D}$ if and only if*

$$\operatorname{Re}\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > 0, \qquad z \in \mathbb{D}.$$

Denote by $\mathcal{SP}^{\alpha}_{Lh}$ the subclass of $\mathcal{S}_{Lh}$ consisting of all $\alpha-$spirallike logharmonic mappings. Also, denote by $\mathcal{SP}^{\alpha}$ the subclass of $\mathcal{SP}^{\alpha}_{Lh}$ such that $f \in \mathcal{H}(\mathbb{D})$.

Abdulhadi and Hengartner [9] gave a representation theorem for mappings in the class $\mathcal{SP}^{\alpha}_{Lh}$. They also established a connection between the class of $\alpha-$spirallike logharmonic mappings $\mathcal{SP}^{\alpha}_{Lh}$ and the class of $\alpha-$spirallike analytic functions $\mathcal{SP}^{\alpha}$. This result is stated in the following theorem.

Theorem 1.21. [9, Theorem 2.1] *Let* $f(z) = z|z|^{2\beta} h(z)\overline{g(z)}$ *be logharmonic with respect to a in* $\mathbb{D}$ *with* $0 \notin hg(\mathbb{D})$, *where* $\beta = \overline{a(0)}(1+a(0))/(1-|a(0)|^2)$. *Then* $f \in \mathcal{SP}^{\alpha}_{Lh}$ *if and only if* $\psi(z) = zh(z)/(g(z))^{e^{2i\alpha}} \in \mathcal{SP}^{\alpha}$.

1.9 Scope of The Thesis

This thesis is composed of six chapters including four research problems followed by references.

In Chapter 1, some fundamental concepts regarding univalent functions, harmonic, and logharmonic mappings are presented. The basic notations, definitions and known results required in this thesis are given.

In Chapter 2, the $\mathcal{U}$-radius is obtained for several classes of functions. These include the class of normalized analytic functions f satisfying the inequality $\operatorname{Re} f(z)/g(z) > 0$ or $|f(z)/g(z)-1| < 1$ in $\mathbb{D}$, where g belongs to a certain class of analytic functions. The estimation for the $\mathcal{U}$-radius of the class of functions f satisfying the inequality $|f'(z)-1| < 1$ or $\operatorname{Re} f(z)/z > \alpha$, $0 \leq \alpha < 1$, in $\mathbb{D}$ is also determined. A conjecture by Obradović and Ponnusamy concerning the radius of univalence for a product involving univalent functions is validated.

In Chapter 3, bounds for the second Hankel determinant of the kth-root transform are obtained for several classes of functions defined via subordination. These classes can be seen as belonging to the genre of Ma-Minda starlike and convex functions. Bounds for the second Hankel determinant are also derived for the kth-root transform of various other classes, which include the class of α-convex functions and

α-logarithmically convex functions. Connections are made with earlier known results. In particular, the bounds obtained by Bansal [29], Janteng *et al.*[63] and Lee *et al.* [72] are shown to be special cases of the results obtained in this chapter.

In Chapter 4, for a starlike logharmonic mapping $f(z) = zh(z)\overline{g(z)}$, sufficient conditions for a function $F(z) = f(z)|f(z)|^{2\gamma}$ to be $\alpha-$spirallike logharmonic mapping are obtained. A new logharmonic mapping with a specified property is constructed by taking product combination of two mappings possessing a given property. Specifically, if $f_1(z) = zh_1(z)\overline{g_1(z)}$, and $f_2(z) = zh_2(z)\overline{g_2(z)}$ are univalent starlike logharmonic with respect to the same $a \in B_0$. Then a new univalent starlike logharmonic mapping $F(z) = f_1^{\lambda}(z)f_2^{1-\lambda}(z),\ 0 \leq \lambda \leq 1$, with respect to the same a is established. In addition, if $f_1(z) = zh_1(z)\overline{g_1(z)}$ is logharmonic with respect to $a_1 \in B_0$, and $f_2(z) = zh_2(z)\overline{g_2(z)}$ is logharmonic with respect to $a_2 \in B_0$, then sufficient conditions are obtained to ensure their product $F(z) = f_1^{\lambda}(z)f_2^{1-\lambda}(z),\ 0 \leq \lambda \leq 1$ is a univalent starlike logharmonic mapping with respect to some $\mu \in B_0$. The work concludes with several examples of univalent starlike logharmonic mappings constructed from this product.

In Chapter 5, the class of normalized logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ in the unit disk satisfying $\varphi(z) = zh(z)g(z)$ is a typically real analytic function is considered. An integral representation for such a mapping is given. Moreover, the connection between this class and the class of logharmonic mappings with positive real part is established. The radius of starlikeness for this class, as well as an upper estimate for its arclength are determined. Sufficient and necessary geometric conditions for $\varphi(z) = zh(z)g(z)$ to be typically real are also derived when $f(z) = zh(z)\overline{g(z)}$ has a dilatation with real coefficients. In the second part of this chapter, we explore an integral representation and the radius of starlikeness for a subclass of this class.

In the final chapter, a summary of the work done in this thesis is presented. Some open problems are suggested for further research.

CHAPTER 2

THE $\mathscr{U}$-RADIUS FOR CLASSES OF ANALYTIC FUNCTIONS

2.1 Introduction

Let $\mathscr{U}$ denote the class of functions $f \in \mathcal{A}$ satisfying $|\mathscr{U}_f(z)| < 1$ for $z \in \mathbb{D}$, where

$$\mathscr{U}_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1.$$

Since $f \in \mathscr{U}$ satisfies the condition $|(z/f(z))^2 f'(z)| < 2$, it follows that $(z/f(z))^2 f'(z)$ is bounded, and $f(z) \neq 0$ for $z \in \mathbb{D}\backslash\{0\}$. Furthermore, if $f'(z) = 0$, then $|\mathscr{U}_f(z)| = 1$, which contradicts the assumption that $|\mathscr{U}_f(z)| < 1$. Thus $f'(z) \neq 0$, and f is locally univalent. We shall in fact prove in Theorem 2.2 that f is not only locally univalent but also univalent.

Functions in the class $\mathscr{U}$ have the following characterization.

Theorem 2.1. (Characterization for $\mathscr{U}$) [114] *If $f \in \mathscr{U}$, then*

$$\frac{z}{f(z)} = 1 - a_2 z - z\int_0^z \frac{w(t)}{t^2}\,dt,$$

where $a_2 = f''(0)/2$ and w is an analytic function in the unit disk $\mathbb{D}$ such that $w(0) = 0 = w'(0)$, and $|w(z)| < 1$ for $z \in \mathbb{D}$.

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in $\mathscr{U}$. Then $f(z)/z \neq 0$ and

$$\begin{aligned}\left(\frac{z}{f(z)}\right)^2 f'(z) &= 1 + (a_3 - a_2^2)z^2 + \dots\\ &= 1 + w(z),\end{aligned}$$

where $w(0) = 0 = w'(0)$, and $|w(z)| = |\,(z/f(z))^2 f'(z) - 1| < 1$. By Schwarz's lemma [123, p.240], $|w(z)| \leq |z|^2$, $z \in \mathbb{D}$. Note that

$$w(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 = -z^2 \frac{d}{dz}\left(\frac{1}{f(z)} - \frac{1}{z}\right).$$

It is evident that

$$\left(\frac{1}{f(z)} - \frac{1}{z}\right)\Big|_{z=0} = -\frac{f''(0)}{2} = -a_2,$$

and thus

$$\begin{aligned}\int_0^z \frac{w(t)}{t^2} dt &= -\int_0^z \left(\frac{1}{f(t)} - \frac{1}{t}\right)' dt \\ &= -\left(\frac{1}{f(z)} - \frac{1}{z}\right) - a_2,\end{aligned}$$

which yields the desired result. □

Following the idea of Aksentév [17], functions in the class $\mathscr{U}$ can readily be shown to be univalent in $\mathbb{D}$.

Theorem 2.2. *Every $f \in \mathscr{U}$ is univalent in $\mathbb{D}$, that is, f belongs to $\mathcal{S}$.*

Proof. Since $f \in \mathscr{U}$, it follows from Theorem 2.1 that

$$F(z) = \frac{1}{f(z)} = \frac{1}{z} - a_2 + \int_0^z a(t)dt,$$

where $a(t) = -w(t)/t^2$, and w is given by Theorem 2.1. Then for $z_1, z_2 \in \mathbb{D}$, and $z_1 \neq z_2$,

$$F(z_1) - F(z_2) = \frac{1}{f(z_1)} - \frac{1}{f(z_2)} = \left(\frac{1}{z_1} - \frac{1}{z_2}\right) - \int_{z_1}^{z_2} a(t)dt.$$

Setting

$$t = z_1 + s(z_2 - z_1), \qquad 0 \leq s \leq 1,$$

the integral can be written in the form

$$F(z_1) - F(z_2) = (z_2 - z_1)\left(\frac{1}{z_1 z_2} - \int_0^1 a(z_1 + s(z_2 - z_1))ds\right).$$

It follows that

$$\begin{aligned}|F(z_1) - F(z_2)| &\geq |z_1 - z_2|\left(\left|\frac{1}{z_1 z_2}\right| - \int_0^1 |a(z_1 + s(z_2 - z_1))|ds\right)\\ &> |z_1 - z_2|\left(1 - \int_0^1 ds\right) = 0.\end{aligned}$$

This shows that $F(z_1) \neq F(z_2)$ for $z_1 \neq z_2$. Therefore, f is univalent in $\mathbb{D}$. $\square$

The converse of the result in Theorem 2.2 does not hold, as illustrated by the function $f(z) = -\log(1-z)$. It is evident that

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) = \operatorname{Re}\left(\frac{1}{1-z}\right) > \frac{1}{2} > 0.$$

Thus f is a univalent convex function in $\mathcal{S}$. However,

$$\left|\mathscr{U}_f(z)\right| = \left|\frac{z^2}{(1-z)\log^2(1-z)} - 1\right|,$$

which at $z_0 = 0.95 \in \mathbb{D}$ gives $\left|\mathscr{U}_f(0.95)\right| \approx 1.01127 > 1$. Hence $f \notin \mathscr{U}$.

Functions in $\mathscr{U}$ need not be starlike [45, 107]. For example, the function $f(z) = z/(1 + z/2 + z^3/2)$, satisfies

$$\left|\mathscr{U}_f(z)\right| = \left|\left(\frac{(1-z^3)(1 + z/2 + z^3/2)}{1 + z/2 + z^3/2}\right) - 1\right| = |z^3| < 1,$$

and thus $f \in \mathscr{U}$. However,

$$\frac{zf'(z)}{f(z)} = \frac{1 - z^3}{1 + z/2 + z^3/2},$$

and at $z_0 = (-1+i)/\sqrt{2} \in \partial\mathbb{D}$,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) = \mathrm{Re}\left(\frac{1-\sqrt{2}}{3} + \frac{(1-2\sqrt{2})i}{3}\right) = \frac{1-\sqrt{2}}{3} < 0.$$

Thus $\mathrm{Re}(zf'(z)/f(z)) < 0$ for $z \in \mathbb{D}$ near z_0, and hence $f \notin \mathcal{ST}$.

The Koebe function $k(z) = z/(1-z)^2$ is an important example of a function which belongs to $\mathcal{U} \cap \mathcal{ST}$. It is interesting to note that each function in the set

$$\mathcal{S}_{\mathbb{Z}} = \left\{ z,\ \frac{z}{(1 \pm z)^2},\ \frac{z}{1 \pm z},\ \frac{z}{1 \pm z^2},\ \frac{z}{1 \pm z + z^2} \right\}$$

belongs to $\mathcal{U}$. For instance, the function $f(z) = z/(1+z+z^2)$ satisfies

$$\left|\mathcal{U}_f(z)\right| = \left| \frac{(1-z^2)(1+z+z^2)}{1+z+z^2} - 1 \right| = |z^2| < 1.$$

Furthermore, functions in $\mathcal{S}_{\mathbb{Z}}$ are known [46] to be the only functions in $\mathcal{S}$ with integer coefficients in their series expansions. Thus $\mathcal{S}_{\mathbb{Z}} \subset \mathcal{U} \subset \mathcal{S}$.

The class $\mathcal{U}$ has been widely studied in recent years, for example in the works of [105, 106, 107, 108, 109, 110, 111, 112, 113, 114] and [125]. Several interesting properties of the class $\mathcal{U}$ are shaped by the coefficients of its mappings. If $f \in \mathcal{S}$, then $z/f(z)$ is nonvanishing in $\mathbb{D}$ and has a series representation of the form

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n. \tag{2.1}$$

It follows from the area theorem [49, Theorem 11, p.193] that

$$\sum_{n=2}^{\infty} (n-1)|b_n|^2 \leq 1. \tag{2.2}$$

Obradović and Ponnusamy [114] showed that every $f \in \mathcal{A}$ of the form (2.1) belongs to the class $\mathcal{U}$ whenever $\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1$. They also showed in [110] that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying $\sum_{n=2}^{\infty} n|a_n| \leq 1$ belongs to $\mathcal{U} \cap \mathcal{ST}$. On the other hand, it was shown in [24] that functions $f \in \mathcal{U}$ of the form (2.1) necessarily satisfy $\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq 1$.

In [24], Ali *et al.* showed that condition (2.2) does not ensure univalence, and they obtained the sharp radius of univalence $r_0 = 1/\sqrt{2}$ for functions $f \in \mathcal{A}$ satisfying (2.2). In [106], the $\mathcal{U}$-radius for $\mathcal{S}$ was determined to be $1/\sqrt{2}$. Evidently, radius problems have continued to be an important area of study.

In [79, 80], MacGregor obtained the radius of starlikeness for the class of functions $f \in \mathcal{A}$ satisfying either

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0 \quad (z \in \mathbb{D}) \quad \text{or} \quad \left|\frac{f(z)}{g(z)} - 1\right| < 1 \quad (z \in \mathbb{D}) \tag{2.3}$$

for some $g \in \mathcal{CV}$. Ratti [128] determined the radius of starlikeness for the class (2.3) when g belongs to certain classes of analytic functions. MacGregor in [81] also found the radius of convexity for univalent functions satisfying $|f'(z) - 1| < 1$.

This chapter finds the $\mathcal{U}$-radius for three classes of functions:

(a) the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}, \tag{2.4}$$

for some $g \in \mathcal{A}$ with

$$\operatorname{Re}\left(\frac{g(z)}{z}\right) > 0, \quad z \in \mathbb{D};$$

(b) the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}, \tag{2.5}$$

for some $g \in \mathcal{A}$ with

$$\operatorname{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2}, \quad z \in \mathbb{D};$$

(c) and the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1, \quad z \in \mathbb{D},$$

for some $g \in \mathcal{A}$ with

$$\operatorname{Re}\left(\frac{g(z)}{z}\right) > 0, \quad z \in \mathbb{D}.$$

Additionally, the radius r_0 is also investigated in this chapter so that

$$\left|\mathcal{U}_f(z)\right| = \left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1$$

in the disk $|z| < r_0$ for the following two classes of functions:

(a) the subclass of close-to-convex functions $f \in \mathcal{A}$ satisfying

$$\left|f'(z) - 1\right| < 1, \quad z \in \mathbb{D}; \tag{2.6}$$

(b) and the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\frac{f(z)}{z} > \alpha, \quad 0 \leq \alpha < 1, \quad z \in \mathbb{D}. \tag{2.7}$$

It is known [38, p. 251] that each convex function in $\mathcal{CV}$ belongs to the class (2.7) for $\alpha = 1/2$. Also, Obradović and Ponnusamy in [110] proved that the class defined by (2.7) contains $f \in \mathcal{U}$ satisfying $f''(0) = 0$.

Ratti [128] showed that the radius of starlikeness for the class defined by (2.4) is $\sqrt{5}-2$, and that the radius can be improved to 1/3 for the class given by (2.5). The radius of convexity for the class given by (2.6) was obtained by MacGregor [81]. Several radius constants, which include the radius of starlikeness of a positive order, radius of parabolic starlikeness, radius of Bernoulli lemniscate starlikeness, and radius of uniform convexity, have been obtained for the classes defined by (2.4) and (2.5) in [23].

Obradović and Ponnusamy in [113] also considered the product of functions f and g belonging to certain subsets of $\mathcal{S}$. They showed that whenever $f,g \in \mathcal{ST}$, the product $F(z) = f(z)g(z)/z$ is starlike in the disk $|z| < 1/3$, and that this radius is sharp. They also conjectured that F is univalent in the disk $|z| < 1/3$ when $f,g \in \mathcal{S}$, and that the radius $1/3$ is best possible. In Section 2.3, this conjecture is shown in the affirmative.

To prove the results in this chapter, the following lemmas are required.

Lemma 2.1. [51] *For each $f \in \mathcal{S}$,*

$$\left|\log \frac{zf'(z)}{f(z)}\right| \leq \log \frac{1+r}{1-r}, \quad |z| = r < 1.$$

Lemma 2.2. [83, Theorem 3.1b, p. 71] *Let $p(z) = 1 + p_1 z + \cdots$ be analytic in $\mathbb{D}$, and h be convex. If*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z), \tag{2.8}$$

where $\gamma \neq 0$ and $Re\,\gamma \geq 0$, then

$$p(z) \prec \frac{\gamma}{z^\gamma} \int_0^z h(t) t^{\gamma-1} dt.$$

Lemma 2.3. [109] *Let f be analytic in $\mathbb{D}$ of the form*

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots,$$

with $b_n \geq 0$ for all $n \geq 2$. Then the following are equivalent:

(a) $f \in \mathcal{S}$,

(b) $\frac{f(z)f'(z)}{z} \neq 0, \quad z \in \mathbb{D}$,

(c) $\sum_{n=2}^{\infty}(n-1)b_n \leq 1$,

(d) $f \in \mathcal{U}$.

Lemma 2.4. [128] *An analytic function f in $\mathbb{D}$ satisfies $f(0) = 1$ and $\operatorname{Re}(f(z)) > \alpha$, $0 \leq \alpha < 1$ for $z \in \mathbb{D}$, if and only if $f(z) = (1 + (2\alpha - 1)z\phi(z))/(1 + z\phi(z))$, where ϕ is analytic satisfying $|\phi(z)| \leq 1$ in $\mathbb{D}$.*

2.2 The $\mathcal{U}$-radius for Classes of Analytic Functions

The following result determines the $\mathcal{U}-$radius for the class of functions satisfying (2.4).

Theorem 2.3. *The $\mathcal{U}$-radius for the class of functions $f \in \mathcal{A}$ satisfying the inequality*

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D},$$

for some $g \in \mathcal{A}$ with

$$\operatorname{Re}\left(\frac{g(z)}{z}\right) > 0, \quad z \in \mathbb{D},$$

is $r_{\mathcal{U}} = \sqrt{5} - 2 \approx 0.23607$.

Proof. Writing $p(z) = g(z)/z$ and $h(z) = f(z)/g(z)$, it follows that $p, h \in \mathcal{P}$ and $f(z) = zp(z)h(z)$. A computation shows that

$$\mathcal{U}_f(z) = z^2 \frac{f'(z)}{f^2(z)} - 1 = -z^2\left(\frac{1}{f(z)} - \frac{1}{z}\right)' = -z^2\left(\frac{1}{z}\left(\frac{1}{p(z)h(z)} - 1\right)\right)'$$

$$
\begin{aligned}
&= -z^2\left(-\frac{1}{z^2}\left(\frac{1}{p(z)h(z)}-1\right)+\frac{1}{z}\left(\frac{1}{h(z)}\left(\frac{1}{p(z)}\right)'+\frac{1}{p(z)}\left(\frac{1}{h(z)}\right)'\right)\right)\\
&= \frac{1}{p(z)h(z)}-1-\frac{z}{h(z)}\left(\frac{1}{p(z)}\right)'-\frac{z}{p(z)}\left(\frac{1}{h(z)}\right)'\\
&= \frac{1}{h(z)}\left(\frac{1}{p(z)}-z\left(\frac{1}{p(z)}\right)'-1\right)+\frac{1}{p(z)}\left(\frac{1}{h(z)}-z\left(\frac{1}{h(z)}\right)'-1\right)\\
&\quad -\left(\frac{1}{p(z)}-1\right)\left(\frac{1}{h(z)}-1\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\mathscr{U}_f(z)| \le{}& \left|\frac{1}{h(z)}\right|\left|\frac{1}{p(z)}-z\left(\frac{1}{p(z)}\right)'-1\right|+\left|\frac{1}{p(z)}\right|\left|\frac{1}{h(z)}-z\left(\frac{1}{h(z)}\right)'-1\right|\\
&+\left|\frac{1}{h(z)}-1\right|\left|\frac{1}{p(z)}-1\right|. \qquad (2.9)
\end{aligned}
$$

Since $1/p(z) = 1+\sum_{n=1}^{\infty} c_n z^n$ and $1/h(z) = 1+\sum_{n=1}^{\infty} d_n z^n$ are in the class $\mathcal{P}$, it follows that $|c_n| \le 2$ and $|d_n| \le 2$ for $n \ge 1$. Thus for $|z| = r$,

$$
\left|\frac{1}{p(z)}-1\right| \le \sum_{n=1}^{\infty} |c_n||z|^n \le 2\sum_{n=1}^{\infty} r^n = \frac{2r}{1-r}, \qquad (2.10)
$$

$$
\left|\frac{1}{p(z)}\right| \le \frac{2r}{1-r}+1 = \frac{1+r}{1-r}, \qquad (2.11)
$$

and

$$
\begin{aligned}
\left|\frac{1}{p(z)}-z\left(\frac{1}{p(z)}\right)'-1\right| &\le \sum_{n=2}^{\infty}(n-1)|c_n||z|^n \le 2\sum_{n=2}^{\infty}(n-1)r^n\\
&= 2r^2\sum_{n=1}^{\infty} n r^{n-1} = \frac{2r^2}{(1-r)^2}. \qquad (2.12)
\end{aligned}
$$

Similar estimates are obtained for the function $1/h$

$$
\left|\frac{1}{h(z)}-1\right| \le \frac{2r}{1-r}, \quad \left|\frac{1}{h(z)}\right| \le \frac{1+r}{1-r}, \quad \text{and} \quad \left|\frac{1}{h(z)}-z\left(\frac{1}{h(z)}\right)'-1\right| \le \frac{2r^2}{(1-r)^2}. \quad (2.13)
$$

Substituting (2.10), (2.11), (2.12) and (2.13) into (2.9) yields

$$|\mathscr{U}_f(z)| \leq 2\left(\frac{1+r}{1-r}\frac{2r^2}{(1-r)^2}\right) + \frac{4r^2}{(1-r)^2} = \frac{8r^2}{(1-r)^3}.$$

Hence $|\mathscr{U}_f(z)| < 1$ if $|z| < \sqrt{5}-2$, where $r_{\mathscr{U}} = \sqrt{5}-2$ is the root of the equation $r^2+4r-1=0$.

To demonstrate the sharpness, let $f_0(z) = z\big((1-z)/(1+z)\big)^2$, and $g_0(z) = z(1-z)/(1+z)$. Evidently,

$$\begin{aligned} |\mathscr{U}_{f_0}(r)| &= \left|\left(\frac{r}{f_0(r)}\right)^2 f_0{}'(r) - 1\right| \\ &= \left|\frac{(1+r)(1-4r-r^2)}{(1-r)^3} - 1\right| = \frac{8r^2}{(1-r)^3}. \end{aligned}$$

Since $r^2/(1-r)^3$ is increasing, it follows that $|\mathscr{U}_{f_0}(r)| > 1$ for $\sqrt{5}-2 < r < 1$. $\square$

The $\mathscr{U}$ −radius for the second class is derived in the following result.

Theorem 2.4. *The $\mathscr{U}$-radius for the class of functions $f \in \mathcal{A}$ satisfying the inequality*

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D},$$

for some $g \in \mathcal{A}$ with

$$\operatorname{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2}, \quad z \in \mathbb{D},$$

is $r_{\mathscr{U}} = 1/3$.

Proof. Let $p(z) = g(z)/z$, $h(z) = f(z)/g(z)$, and $f(z) = zp(z)h(z)$. Since $p \in \mathcal{P}(1/2)$, it follows from Lemma 2.4 that $p(z) = 1/(1+z\phi(z))$, where $|\phi(z)| \leq 1$. For $|z| = r$, and $|\phi(z)| = x$, $0 \leq x \leq 1$, it is evident that

$$\left|\frac{1}{p(z)} - 1\right| = |z||\phi(z)| = rx,$$

and the Schwarz-Pick inequality [123, p.243] gives

$$\left|\frac{1}{p(z)} - z\left(\frac{1}{p(z)}\right)' - 1\right| = |z|^2|\phi'(z)| \leq \frac{|z|^2(1-|\phi(z)|^2)}{1-|z|^2} = \frac{r^2(1-x^2)}{1-r^2}.$$

The function $h \in \mathcal{P}$ satisfies the estimates (2.10) and (2.12). It follows from (2.9) that

$$|\mathcal{U}_f(z)| \leq \frac{1+r}{1-r}\frac{r^2(1-x^2)}{1-r^2} + \frac{2r^2(1+rx)}{(1-r)^2} + \frac{2r^2x}{1-r} = \frac{r^2(3+2x-x^2)}{(1-r)^2}.$$

Since $\lambda(x) = 3+2x-x^2$ is increasing over $0 \leq x \leq 1$, this leads to

$$|\mathcal{U}_f(z)| \leq \frac{4r^2}{(1-r)^2} < 1$$

if $r < r_{\mathcal{U}}$, where $r_{\mathcal{U}} = 1/3$ is the root of the equation $3r^2+2r-1=0$.

For the sharpness, consider $f_0(z) = z(1-z)/(1+z)^2$, and $g_0(z) = z/(1+z)$. Then

$$\frac{z}{f_0(z)} = \frac{(1+z)^2}{1-z} = 1+3z+4\sum_{n=2}^{\infty} z^n.$$

It follows from Lemma 2.3 that $r^{-1}f_0(rz) \in \mathcal{U}$ provided $0 < r \leq 1$ satisfies

$$4\sum_{n=2}^{\infty}(n-1)r^n = 4r\sum_{n=1}^{\infty} nr^n = \left(\frac{2r}{1-r}\right)^2 \leq 1,$$

that is, if $r \leq r_{\mathcal{U}}$, where $r_{\mathcal{U}} = 1/3$ is the root of the equation $3r^2+2r-1=0$. $\square$

Next, the $\mathcal{U}$-radius for the third class is obtained.

Theorem 2.5. *The $\mathcal{U}$-radius for the class of functions $f \in \mathcal{A}$ satisfying the inequality*

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1, \quad z \in \mathbb{D},$$

for some $g \in \mathcal{A}$ with

$$\operatorname{Re}\left(\frac{g(z)}{z}\right) > 0, \quad z \in \mathbb{D},$$

is $r_{\mathscr{U}} = (\sqrt{17}-3)/4 \approx 0.28078$.

Proof. Let $p(z) = g(z)/z$, $h(z) = f(z)/g(z)$, and $f(z) = zp(z)h(z)$. Since $|h(z)-1| < 1$, it follows that $|1-1/h(z)|^2 < 1/|h(z)|^2$, that is, $1 - 2\operatorname{Re}(1/h(z)) < 0$. Thus $1/h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$ is in $\mathcal{P}(1/2)$, and hence $|d_n| \leq 1$ for $n \geq 1$. Now, for $|z| = r$,

$$\left|\frac{1}{h(z)} - 1\right| \leq \sum_{n=1}^{\infty} |d_n||z|^n \leq \sum_{n=1}^{\infty} r^n = \frac{r}{1-r},$$
$$\left|\frac{1}{h(z)}\right| \leq \frac{r}{1-r} + 1 = \frac{1}{1-r},$$

and

$$\left|\frac{1}{h(z)} - z\left(\frac{1}{h(z)}\right)' - 1\right| \leq \sum_{n=1}^{\infty} (n-1)|d_n||z|^n \leq \sum_{n=1}^{\infty} (n-1)r^n = r^2 \sum_{n=2}^{\infty} (n-1)r^{n-2}$$
$$= r^2 \sum_{n=1}^{\infty} nr^{n-1} = \frac{r^2}{(1-r)^2}.$$

Further, the function p satisfies the estimates (2.10) and (2.12). It follows from (2.9) that

$$|\mathscr{U}_f(z)| \leq \frac{1}{1-r}\frac{2r^2}{(1-r)^2} + \frac{1+r}{1-r}\frac{r^2}{(1-r)^2} + \frac{r}{1-r}\frac{2r}{1-r} = \frac{5r^2-r^3}{(1-r)^3}.$$

Hence $|\mathscr{U}_f(z)| < 1$ if $r < r_{\mathscr{U}}$, where $r_{\mathscr{U}} = (\sqrt{17}-3)/4$ is the root of the equation $2r^2 + 3r - 1 = 0$.

To demonstrate the sharpness, let $f_0(z) = z(1-z)^2/(1+z)$, and $g_0(z) = z(1-z)/(1+z)$. Then

$$\frac{z}{f_0(z)} = \frac{1+z}{(1-z)^2} = 1 + \sum_{n=1}^{\infty} (2n+1)z^n.$$

Lemma 2.3 will be used to show that $r^{-1}f_0(rz) \in \mathscr{U}$. For $0 < r \leq 1$,

$$\frac{rz}{f_0(rz)} = 1 + \sum_{n=1}^{\infty} (2n+1)r^n z^n,$$

and

$$\begin{aligned}\sum_{n=2}^{\infty}(n-1)(2n+1)r^n &= 2\sum_{n=2}^{\infty}n(n-1)r^n+\sum_{n=2}^{\infty}(n-1)r^n\\ &= 2r^2\sum_{n=2}^{\infty}n(n-1)r^{n-2}+r^2\sum_{n=1}^{\infty}nr^{n-1}\\ &= \frac{4r^2}{(1-r)^3}+\frac{r^2}{(1-r)^2}=\frac{5r^2-r^3}{(1-r)^3}\le 1\end{aligned}$$

if and only if $r\le(\sqrt{17}-3)/4$, where $r_{\mathscr{U}}=(\sqrt{17}-3)/4$ is the root of the equation $2r^2+3r-1=0$. $\square$

The following result estimates the $\mathscr{U}$-radius for functions $f\in\mathcal{A}$ satisfying inequality $|f'(z)-1|<1$.

Theorem 2.6. *Let $f\in\mathcal{A}$ satisfying*

$$|f'(z)-1|<1, \quad z\in\mathbb{D}.$$

Then $f\in\mathscr{U}$ in the disk $|z|<r_0$, where $r_0=\sqrt{(\sqrt{5}-1)/2}\approx 0.78615$.

Proof. Evidently the subordination (2.8) translates to $f'(z)\prec 1+z$ by choosing $\gamma=1$, $p(z)=f(z)/z$, and $h(z)=1+z$. It follows that

$$\frac{f(z)}{z}\prec 1+\frac{z}{2}.$$

So there exists an analytic self-map w of $\mathbb{D}$ with $w(0)=0$ and $f(z)/z=1+w(z)/2$.

Direct computations show that

$$\begin{aligned}|\mathscr{U}_f(z)| &= \left|\left(\frac{1}{1+\frac{w(z)}{2}}\right)^2\left(1+\frac{w(z)}{2}+\frac{zw'(z)}{2}\right)-1\right|\\ &= \frac{1}{\left|1+\frac{w(z)}{2}\right|^2}\left|\frac{w(z)}{2}+\frac{zw'(z)}{2}-w(z)-\frac{w^2(z)}{4}\right|\\ &= \frac{1}{4\left|1+\frac{w(z)}{2}\right|^2}\left|2\left(zw'(z)-w(z)\right)-w^2(z)\right|\end{aligned}$$

$$\leqslant \frac{1}{4\left(1-\frac{|w(z)|}{2}\right)^2}\left(2\left|zw'(z)-w(z)\right|+|w^2(z)|\right).$$

Thus

$$|\mathscr{U}_f(z)| \leq \frac{2\,|zw'(z)-w(z)|+|w^2(z)|}{4\left(1-\frac{|w(z)|}{2}\right)^2}. \tag{2.14}$$

The Schwarz-Pick inequality [123, p.243] applied to $w(z)/z$ yields

$$|zw'(z)-w(z)| \leq \frac{|z|^2-|w(z)|^2}{1-|z|^2}. \tag{2.15}$$

Substituting (2.15) into (2.14), and writing $|w(z)|=t$, $|z|=r$, $0\leq t\leq r$, leads to

$$\begin{aligned}
|\mathscr{U}_f(z)| &\leqslant \frac{1}{4\left(1-\frac{t}{2}\right)^2}\left(2\left(\frac{r^2-t^2}{1-r^2}\right)+t^2\right)\\
&= \frac{1}{(2-t)^2}\left(\frac{2r^2-2t^2+t^2-r^2t^2}{1-r^2}\right)\\
&= \frac{1}{(1-r^2)}\left(\frac{-\left(1+r^2\right)t^2+2r^2}{(2-t)^2}\right)\\
&:= \frac{1}{(1-r^2)}\Phi(t,r).
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial\Phi(t,r)}{\partial t} &= \frac{-2(1+r^2)(2-t)^2t+2(2-t)(-(1+r^2)t^2+2r^2)}{(2-t)^4}\\
&= \frac{-2(1+r^2)(2-t+t)t+4r^2}{(2-t)^3}\\
&= \frac{4\left(r^2-(1+r^2)t\right)}{(2-t)^3},
\end{aligned}$$

the function $\Phi(t,r)$ attains its maximum at the point $t_0=r^2/(1+r^2)$, that is,

$$\begin{aligned}
\Phi(t,r)\leq\Phi(t_0,r)=\Phi\left(\frac{r^2}{1+r^2},r\right) &= \frac{\frac{-r^4(1+r^2)}{(1+r^2)^2}+2r^2}{\left(2-\frac{r^2}{1+r^2}\right)^2}\\
&= \frac{r^2\left(r^2+2\right)\left(r^2+1\right)}{\left(r^2+2\right)^2}=\frac{r^2\left(r^2+1\right)}{r^2+2}.
\end{aligned}$$

Thus

$$|\mathscr{U}_f(z)| \leq \frac{r^2\left(r^2+1\right)}{(1-r^2)\left(r^2+2\right)},$$

and hence $|\mathscr{U}_f(z)| < 1$ if $|z| < r_0$, where $r_0 = \sqrt{(\sqrt{5}-1)/2}$ is the root of the equation $r^4 + r^2 - 1 = 0$. $\square$

Remark 2.1. Ozaki [115] introduced the class $\mathcal{G}$ consisting of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2},$$

and proved that these functions are necessarily univalent in $\mathbb{D}$. Umezawa [139] showed that these functions are convex in one direction. Sakaguchi [132] proved that $|\arg f'(z)| < \pi/2$ whenever $f \in \mathcal{G}$, and indeed, $\mathcal{G} \subset \mathcal{ST}$, see [104, 135]. There has been a continued interest in recent years over the class $\mathcal{G}$, see for example, the works in [124, 126]. It follows from [65, Theorem 2] that $|f'(z) - 1| < 1$ whenever $f \in \mathcal{G}$. Thus Theorem 2.6 shows that the $\mathscr{U}$-radius for $f \in \mathcal{G}$ is $\sqrt{(\sqrt{5}-1)/2}$.

The last result in this section estimates the $\mathscr{U}$-radius for functions $f \in \mathcal{A}$ satisfying inequality $\operatorname{Re}(f(z)/z) > \alpha$, $0 \leq \alpha < 1$.

Theorem 2.7. *Let $f \in \mathcal{A}$ satisfy*

$$\operatorname{Re}\frac{f(z)}{z} > \alpha, \quad 0 \leq \alpha < 1, \quad z \in \mathbb{D}.$$

Then $f \in \mathscr{U}$ in the disk $|z| < r(\alpha)$, where

$$r(\alpha)=\begin{cases}\sqrt{\frac{2(1-\alpha)}{1-2\alpha}}-1, & 0\le\alpha\le\frac{1}{10},\\ \sqrt{\frac{\sqrt{\alpha(1-\alpha)}-\alpha}{1-2\alpha}}, & \frac{1}{10}\le\alpha\le\frac{1}{2},\\ \sqrt{\frac{\sqrt{2-4\alpha(1-\alpha)}-2(1-\alpha)}{2(2\alpha-1)}}, & \frac{1}{2}\le\alpha\le\tau_\alpha,\\ \frac{1}{4\alpha-3}\left(1-\sqrt{\frac{2(1-\alpha)}{2\alpha-1}}\right), & \tau_\alpha\le\alpha<1,\end{cases}$$

and $\tau_\alpha=(8-11/\sqrt[3]{71+6\sqrt{177}}+\sqrt[3]{71+6\sqrt{177}})/12\approx 0.93804$ *is the root of the equation*

$$4\alpha-2-\left((2\alpha-1)+\sqrt{(2\alpha-1)(10\alpha-9)}\right)\left((2\alpha-1)+\sqrt{2(1-\alpha)(2\alpha-1)}\right)=0$$

in the interval $[9/10,1)$. *The result is sharp for the case* $\alpha\in[0,1/10]$.

Proof. Since $\operatorname{Re} f(z)/z>\alpha$, it follows that

$$\frac{f(z)}{z}\prec\frac{1+(1-2\alpha)z}{1-z},$$

there exists an analytic self-map w of $\mathbb{D}$ satisfying $w(0)=0$ and

$$\frac{f(z)}{z}=\frac{1+(1-2\alpha)w(z)}{1-w(z)}.$$

A further computation shows that

$$\begin{aligned}\left(\frac{z}{f(z)}\right)^2 f'(z)&=\frac{z}{f(z)}\frac{zf'(z)}{f(z)}\\ &=\frac{1-w(z)}{1+(1-2\alpha)w(z)}+\frac{(1-2\alpha)zw'(z)\big(1-w(z)\big)}{\Big(1+(1-2\alpha)w(z)\Big)^2}+\frac{zw'(z)}{1+(1-2\alpha)w(z)}\\ &=\frac{\big(1-w(z)\big)\Big(1+(1-2\alpha)w(z)\Big)+(2-2\alpha)zw'(z)}{\Big(1+(1-2\alpha)w(z)\Big)^2}\\ &=\frac{1-2\alpha w(z)-(1-2\alpha)w^2(z)+2(1-\alpha)zw'(z)}{\Big(1+(1-2\alpha)w(z)\Big)^2},\end{aligned}$$

and hence

$$\mathscr{U}_f(z)=\left(\frac{z}{f(z)}\right)^2 f'(z)-1$$

$$= \frac{1-2\alpha w(z)-(1-2\alpha)w^2(z)+2(1-\alpha)zw'(z)-\Big(1+(1-2\alpha)w(z)\Big)^2}{\Big(1+(1-2\alpha)w(z)\Big)^2}$$
$$= \frac{(-2+2\alpha)w(z)-(1-2\alpha)(2-2\alpha)w^2(z)+2(1-\alpha)zw'(z)}{\Big(1+(1-2\alpha)w(z)\Big)^2}$$
$$= \frac{2(1-\alpha)\Big(\big(-w(z)+zw'(z)\big)-(1-2\alpha)w^2(z)\Big)}{\Big(1+(1-2\alpha)w(z)\Big)^2}.$$

It follows that

$$|\mathscr{U}_f(z)| \leq \frac{2(1-\alpha)\big(|zw'(z)-w(z)|+|1-2\alpha||w(z)|^2\big)}{\big(1-|1-2\alpha||w(z)|\big)^2}.$$

Writing $|w(z)|=t$, $|z|=r$, $0\leq t\leq r$, $|1-2\alpha|=a$, and substituting (2.15) leads to

$$\begin{aligned}|\mathscr{U}_f(z)| &\leq \frac{2(1-\alpha)\big(\frac{r^2-t^2}{1-r^2}+at^2\big)}{(1-at)^2}\\ &= \frac{2(1-\alpha)\Big(r^2+\big(a(1-r^2)-1\big)t^2\Big)}{(1-at)^2(1-r^2)}\\ &:= \Phi(t,r).\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial\Phi(t,r)}{\partial t} &= \frac{2(1-\alpha)\Big(2\big(a(1-r^2)-1\big)(1-at)^2t+2a(1-at)\big(r^2+\big(a(1-r^2)-1\big)t^2\big)\Big)}{(1-r^2)(1-at)^4}\\ &= \frac{4(1-\alpha)\Big(\big(a(1-r^2)-1\big)t-\big(a(1-r^2)-1\big)at^2+ar^2+\big(a(1-r^2)-1\big)at^2\Big)}{(1-r^2)(1-at)^3}\\ &= \frac{4(1-\alpha)\big(ar^2-\big(1-a(1-r^2)\big)t\big)}{(1-r^2)(1-at)^3}\\ &:= \frac{4(1-\alpha)}{(1-r^2)(1-at)^3}\phi(t,r),\end{aligned}$$

where

$$\phi(t,r) = ar^2-\big(1-a(1-r^2)\big)t.$$

Thus the critical points of $\Phi(t,r)$ over $t\in[0,r]$ occurs at $t=0$, $t=r$ and possibly at

$t_0 = ar^2/(1-a+ar^2)$, where $\phi(t_0,r) = 0$. Indeed t_0 is a critical point in $[0,r]$ whenever

$$g(r) = ar^2 - ar + 1 - a \tag{2.16}$$

is nonnegative.

For $a \in [0,4/5]$, it is evident that $g(r) \geq 0$ in $(0,1)$. Hence the maximum value of $\Phi(t,r)$ is the largest value of $\{\Phi(0,r); \Phi(t_0,r); \Phi(r,r)\}$. Since

$$\begin{aligned}
\Phi(t_0,r) &= \frac{2(1-\alpha)\Big(r^2 + \big(a(1-r^2)-1\big)\frac{a^2r^4}{(1-a+ar^2)^2}\Big)}{(1-r^2)(1-a\frac{ar^2}{1-a+ar^2})^2} \\
&= \frac{2(1-\alpha)r^2(1-a+ar^2-a^2r^2)(1-a+ar^2)}{(1-r^2)\big(1-a+ar^2-a^2r^2\big)^2} \\
&= \frac{2(1-\alpha)(1-a+ar^2)r^2}{(1-a)(1-r^2)(1+ar^2)},
\end{aligned}$$

and

$$\begin{aligned}
\Phi(r,r) &= \frac{2(1-\alpha)\Big(r^2 + \big(a(1-r^2)-1\big)r^2\Big)}{(1-r^2)(1-ar)^2} \\
&= \frac{2a(1-\alpha)r^2}{(1-ar)^2},
\end{aligned}$$

it follows that

$$\begin{aligned}
\Phi(t_0,r) - \Phi(r,r) &= \frac{2(1-\alpha)(1-a+ar^2)r^2}{(1-a)(1-r^2)(1+ar^2)} - \frac{2(1-\alpha)ar^2}{(1-ar)^2} \\
&:= \frac{2(1-\alpha)r^2}{(1-a)(1-r^2)(1+ar^2)(1-ar)^2} h(r),
\end{aligned}$$

where

$$\begin{aligned}
h(r) &= (1-a+ar^2)(1-ar)^2 - a(1-a)(1-r^2)(1+ar^2) \\
&= (1-2a+a^2) + (a^2r^4 - 2a^2r^3 + a^2r^2) + 2(ar^2 - ar - a^2r^2 + a^2r) \\
&= (1-a)^2 + a^2r^2(1-r)^2 - 2ar(1-r)(1-a) \\
&= \big((1-a) - ar(1-r)\big)^2 \geq 0.
\end{aligned}$$

Then

$$\Phi(t_0,r)-\Phi(r,r)=\frac{2(1-\alpha)r^2\big((1-a)-ar(1-r)\big)^2}{(1-a)(1-r^2)(1+ar^2)(1-ar)^2}\geq 0. \tag{2.17}$$

Also,

$$\begin{aligned}\Phi(t_0,r)-\Phi(0,r)&=\frac{2(1-\alpha)(1-a+ar^2)r^2}{(1-a)(1-r^2)(1+ar^2)}-\frac{2(1-\alpha)r^2}{(1-r^2)}\\&=\frac{2a^2(1-\alpha)r^4}{(1-a)(1-r^2)(1+ar^2)}\geq 0.\end{aligned} \tag{2.18}$$

It is evident from (2.18) and (2.17) that $\max\Phi(t,r)=\Phi(t_0,r)$. Thus

$$|\mathscr{U}_f(z)|\leq\Phi(t_0,r)=\frac{2(1-\alpha)(1-a+ar^2)r^2}{(1-a)(1-r^2)(1+ar^2)}<1$$

provided $|z|<r(\alpha)$, where $r(\alpha)$ is the root of the equation

$$2(1-\alpha)(1-a+ar^2)r^2-(1-a)(1-r^2)(1+ar^2)=0,$$

which is equivalent to

$$a\,(3-2\alpha-a)r^4+(1-a)(3-2\alpha-a)r^2-(1-a)=0.$$

Hence

$$r(\alpha)=\sqrt{\frac{\sqrt{(1-a)^2(3-2\alpha-a)^2+4a(1-a)(3-2\alpha-a)}-(1-a)(3-2\alpha-a)}{2a(3-2\alpha-a)}}.$$

There are two cases to consider for $a=|1-2\alpha|\in[0,4/5]$. First, when $1/10\leq\alpha\leq 1/2$. In this case $a=1-2\alpha$. Further simplification leads to

$$\begin{aligned}r(\alpha)&=\sqrt{\frac{\sqrt{16\alpha^2+16(1-2\alpha)\alpha}-4\alpha}{4(1-2\alpha)}}\\&=\sqrt{\frac{\sqrt{\alpha(1-\alpha)}-\alpha}{1-2\alpha}}.\end{aligned}$$

The second case occurs when $1/2 \le \alpha \le 9/10$. Then $a = 2\alpha - 1$. It follows that

$$\begin{aligned} r(\alpha) &= \sqrt{\frac{\sqrt{64(1-\alpha)^4 + 32(2\alpha-1)(1-\alpha)^2} - 8(1-\alpha)^2}{8(2\alpha-1)(1-\alpha)}} \\ &= \sqrt{\frac{\sqrt{2-4\alpha(1-\alpha)} - 2(1-\alpha)}{2(2\alpha-1)}}. \end{aligned}$$

For $a \in [4/5, 1]$, the roots of g in (2.16) occurs at

$$r_1 = \frac{a - \sqrt{a(5a-4)}}{2a}, \qquad r_2 = \frac{a + \sqrt{a(5a-4)}}{2a}. \tag{2.19}$$

Evidently $g(r) \ge 0$ over the intervals $[0, r_1]$ and $[r_2, 1)$, and so the maximum of $\Phi(t, r)$ occurs at $\Phi(t_0, r)$. On the other hand, $g(r) < 0$ over (r_1, r_2). Since t_0 is not a critical point, the maximum of $\Phi(t, r)$ occurs at either $\Phi(0, r)$ or $\Phi(r, r)$.

Consider

$$\begin{aligned} K(r) = \Phi(0,r) - \Phi(r,r) &= \frac{2(1-\alpha)r^2}{(1-r^2)} - \frac{2(1-\alpha)ar^2}{(1-ar)^2} \\ &= \frac{2r^2(1-\alpha)\big(a(1+a)r^2 - 2ar + 1 - a\big)}{(1-r^2)(1-ar)^2} \\ &:= \frac{2r^2(1-\alpha)}{(1-r^2)(1-ar)^2} k(r), \end{aligned}$$

where

$$k(r) = a(1+a)r^2 - 2ar + 1 - a, \quad a \in [4/5, 1].$$

The roots of k are

$$r_1' = \frac{a - \sqrt{a^3 + a(a-1)}}{a(1+a)}, \qquad r_2' = \frac{a + \sqrt{a^3 + a(a-1)}}{a(1+a)}.$$

Observe that $K(r) \le 0$ over (r_1', r_2'). Now

$$r_1 - r_1' = \frac{a - \sqrt{a(5a-4)}}{2a} - \frac{a - \sqrt{a^3 + a(a-1)}}{a(1+a)}$$

$$= \frac{a(a-1)-(1+a)\sqrt{a(5a-4)}+2\sqrt{a^3+a^2-a)}}{2a(1+a)}.$$

It is clear that $a(1+a)>0$ when $a\in[4/5,1]$ and a Mathematica plot in Figure 2.1 shows that $y_1(a)=a(a-1)-(1+a)\sqrt{a(5a-4)}+2\sqrt{a^3+a^2-a)}\geq 0$. This leads to $r_1\geq r_1'$.

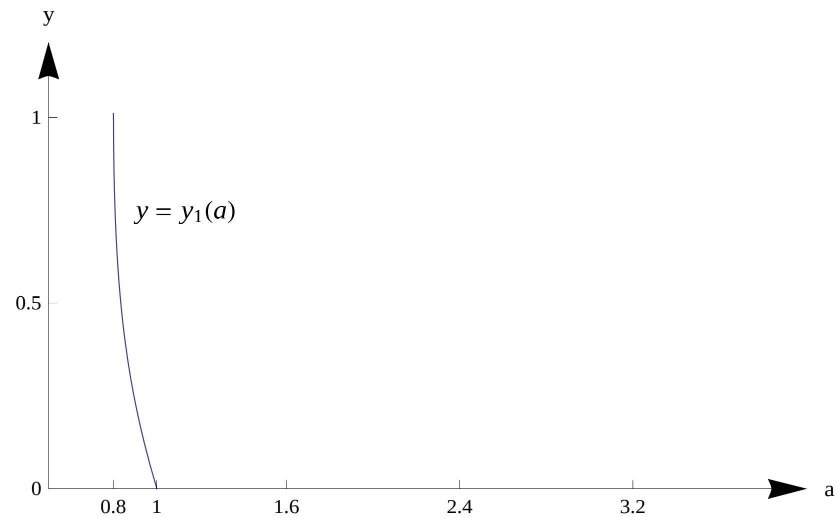

Figure 2.1: Graph of $y_1(a)=a(a-1)-(1+a)\sqrt{a(5a-4)}+2\sqrt{a^3+a^2-a)}$.

On the other hand,

$$\begin{aligned} r_2'-r_2 &= \frac{a+\sqrt{a^3+a(a-1)}}{a(1+a)}-\frac{a+\sqrt{a(5a-4)}}{2a} \\ &= \frac{a(1-a)-(1+a)\sqrt{a(5a-4)}+2\sqrt{a^3+a^2-a)}}{2a(1+a)}. \end{aligned}$$

Figure 2.2 shows that $y_2(a)=a(1-a)-(1+a)\sqrt{a(5a-4)}+2\sqrt{a^3+a^2-a)}\geq 0$ when $a\in[4/5,1]$, and thus $r_2'\geq r_2$. It follows that $(r_1,r_2)\subseteq(r_1',r_2')$, where r_1,r_2 are given by (2.19). Thus $K(r)\leq 0$ over (r_1,r_2), and the maximum value of $\Phi(t,r)$ is $\Phi(r,r)$.

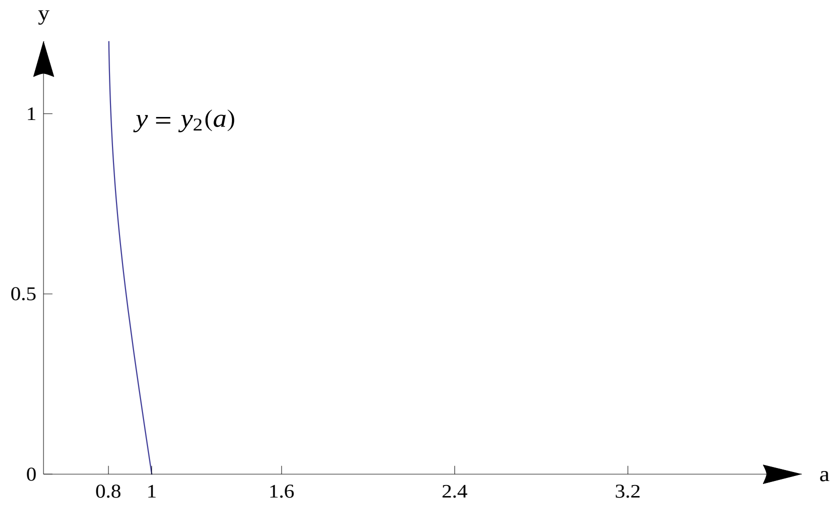

Figure 2.2: Graph of $y_2(a) = a(1-a) - (1+a)\sqrt{a(5a-4)} + 2\sqrt{a^3+a^2-a}$.

There are also two cases to consider for $a = |1-2\alpha| \in [4/5, 1]$, that is, $\alpha \in [0, 1/10]$ and $\alpha \in [9/10, 1)$. Consider first when $\alpha \in [0, 1/10]$, in this case $a = 1-2\alpha$, and it follows that

$$\begin{aligned} r_1(\alpha) &= \frac{(1-2\alpha) - \sqrt{(1-2\alpha)\big(5(1-2\alpha)-4\big)}}{2(1-2\alpha)} \\ &= \frac{(1-2\alpha) - \sqrt{(1-2\alpha)(1-10\alpha)}}{2(1-2\alpha)}, \end{aligned}$$

and

$$\begin{aligned} r_2(\alpha) &= \frac{(1-2\alpha) + \sqrt{(1-2\alpha)(5-10\alpha-4)}}{2(1-2\alpha)} \\ &= \frac{(1-2\alpha) + \sqrt{(1-2\alpha)(1-10\alpha)}}{2(1-2\alpha)}. \end{aligned}$$

If $r \in [0, r_1]$, then g given by (2.16) satisfies $g(r) \geq 0$. Thus

$$|\mathscr{U}_f(z)| \leq \Phi(t_0, r) < 1$$

for all $|z| < R_1(\alpha)$, where

$$R_1(\alpha) = \sqrt{\frac{\sqrt{\alpha(1-\alpha)} - \alpha}{1-2\alpha}}$$

is the root of the equation $\Phi(t_0, r) = 1$. A Mathematica plot in Figure 2.3 shows that $R_1(\alpha) \geq r_1(\alpha)$. Then

$$|\mathscr{U}_f(z)| < 1$$

whenever $|z| < r_1(\alpha)$.

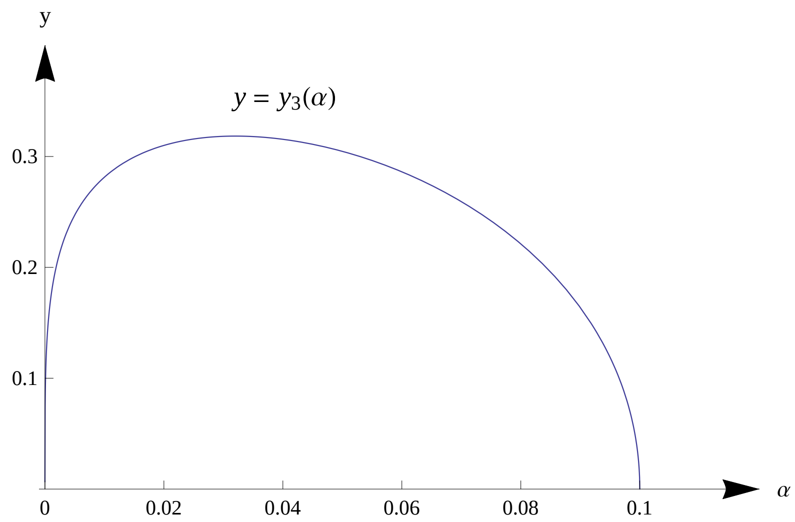

Figure 2.3: Graph of $y_3(\alpha) = \sqrt{\frac{\sqrt{\alpha(1-\alpha)}-\alpha}{1-2\alpha} - \frac{(1-2\alpha)-\sqrt{(1-2\alpha)(1-10\alpha)}}{2(1-2\alpha)}}$.

When $r \in (r_1, r_2)$, then $g(r) < 0$ and

$$|\mathscr{U}_f(z)| \leq \Phi(r, r) < 1$$

for all $|z| < R_2(\alpha)$, where $R_2(\alpha)$ is the root of the equation $\Phi(r, r) = 1$, that is,

$$\begin{aligned} R_2(\alpha) &= \frac{\sqrt{2(1-\alpha)(1-2\alpha)} - (1-2\alpha)}{(1-2\alpha)\big(2(1-\alpha) - (1-2\alpha)\big)} \\ &= \sqrt{\frac{2(1-\alpha)}{1-2\alpha}} - 1. \end{aligned}$$

Figure 2.4 and Figure 2.5 show that $r_1(\alpha) < R_2(\alpha) < r_2(\alpha)$, hence $|\mathscr{U}_f(z)| < 1$ for all $|z| < R_2(\alpha)$ when $\alpha \in [0, 1/10]$.

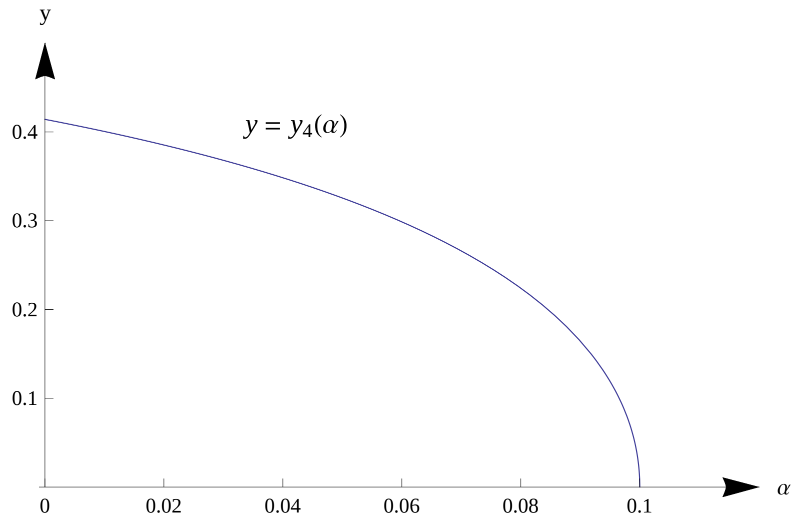

Figure 2.4: Graph of $y_4(\alpha) = \sqrt{\frac{2(1-\alpha)}{1-2\alpha}} - 1 - \frac{(1-2\alpha)-\sqrt{(1-2\alpha)(1-10\alpha)}}{2(1-2\alpha)}$.

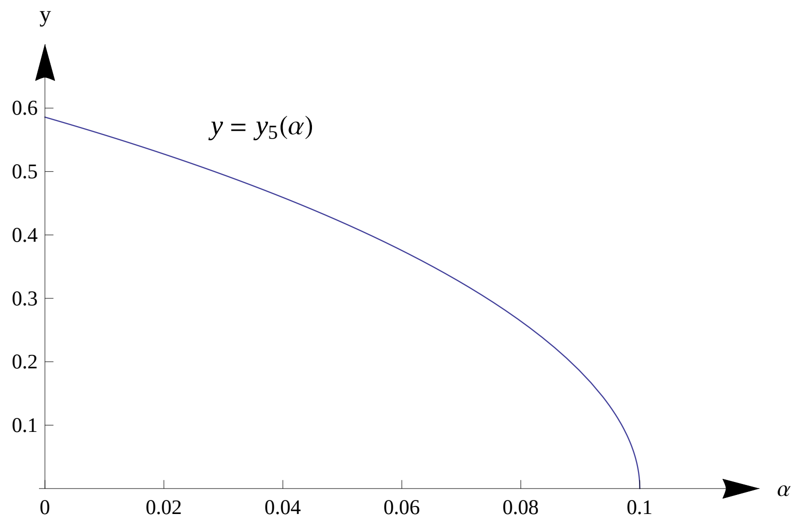

Figure 2.5: Graph of $y_5(\alpha) = \frac{(1-2\alpha)+\sqrt{(1-2\alpha)(1-10\alpha)}}{2(1-2\alpha)} - \sqrt{\frac{2(1-\alpha)}{1-2\alpha}} + 1$.

Next, consider the second case when $\alpha \in [9/10, 1)$, then $a = 2\alpha - 1$. It follows that

$$\begin{aligned} r_1(\alpha) &= \frac{(2\alpha-1) - \sqrt{(2\alpha-1)\big(5(2\alpha-1)-4\big)}}{2(2\alpha-1)} \\ &= \frac{(2\alpha-1) - \sqrt{(2\alpha-1)(10\alpha-9)}}{2(2\alpha-1)}, \end{aligned} \tag{2.20}$$

and

$$r_2(\alpha) = \frac{(2\alpha-1) + \sqrt{(2\alpha-1)(10\alpha-5-4)}}{2(2\alpha-1)}$$

$$= \frac{(2\alpha-1)+\sqrt{(2\alpha-1)(10\alpha-9)}}{2(2\alpha-1)}. \tag{2.21}$$

Likewise as in the first case, if $r \in [0, r_1]$, where r_1 is given by (2.20), then $g(r) \geq 0$ and

$$|\mathscr{U}_f(z)| \leq \Phi(t_0, r) < 1$$

for all $|z| < R_1'(\alpha)$, where

$$R_1'(\alpha) = \sqrt{\frac{\sqrt{2-4\alpha(1-\alpha)}-2(1-\alpha)}{2(2\alpha-1)}} \tag{2.22}$$

is the root of the equation $\Phi(t_0, r) = 1$. A Mathematica plot in Figure 2.6 shows that $R_1'(\alpha) > r_1(\alpha)$, then

$$|\mathscr{U}_f(z)| < 1$$

whenever $|z| < r_1(\alpha)$.

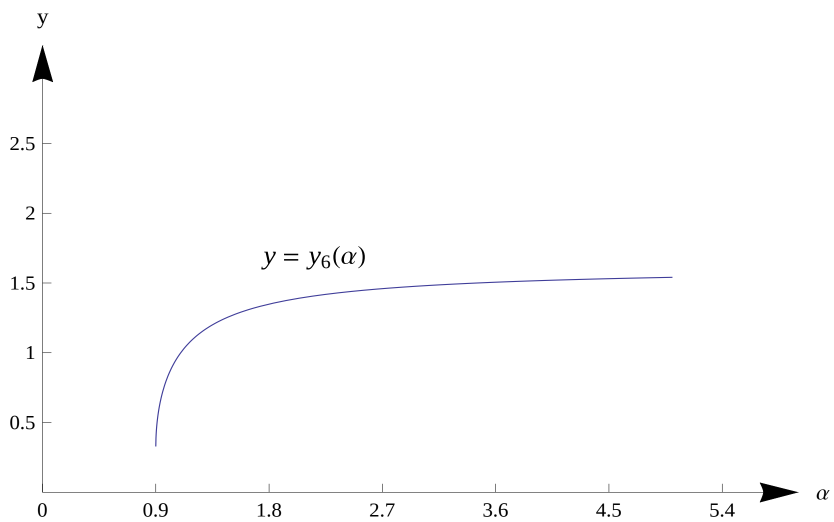

Figure 2.6: Graph of $y_6(\alpha) = \sqrt{\frac{\sqrt{2-4\alpha(1-\alpha)}-2(1-\alpha)}{2(2\alpha-1)}} - \frac{(2\alpha-1)-\sqrt{(2\alpha-1)(10\alpha-9)}}{2(2\alpha-1)}$.

If $r \in (r_1, r_2)$, where r_1 and r_2 are given by (2.20) and (2.21). Then $g(r) < 0$ and

$$|\mathscr{U}_f(z)| \leq \Phi(r, r) < 1$$

for all $|z| < R_2'(\alpha)$, where $R_2'(\alpha)$ is the root of the equation $\Phi(r, r) = 1$, that is,

$$\begin{aligned} R_2'(\alpha) &= \frac{\sqrt{2(1-\alpha)(2\alpha-1)}-(2\alpha-1)}{(2\alpha-1)\big(2(1-\alpha)-(2\alpha-1)\big)} \\ &= \frac{1}{4\alpha-3}\left(1-\sqrt{\frac{2(1-\alpha)}{2\alpha-1}}\right). \end{aligned}$$

A closer scrutiny of $R_2'(\alpha)$ from Figure 2.7 and Figure 2.8, evidently, $r_1(\alpha) < R_2'(\alpha) < r_2(\alpha)$ whenever $\alpha \in (\tau_\alpha, 1)$, where $\tau_\alpha = (8-11/\sqrt[3]{71+6\sqrt{177}}+\sqrt[3]{71+6\sqrt{177}})/12 \approx 0.93804$ is the root of the equation $R_2'(\alpha)-r_2(\alpha)=0$, or equivalently τ_α is the root of the equation

$$4\alpha-2-\left((2\alpha-1)+\sqrt{(2\alpha-1)(10\alpha-9)}\right)\left((2\alpha-1)+\sqrt{2(1-\alpha)(2\alpha-1)}\right)=0.$$

Thus in this case, $|\mathscr{U}_f(z)|<1$ for $|z|<R_2'(\alpha)$.

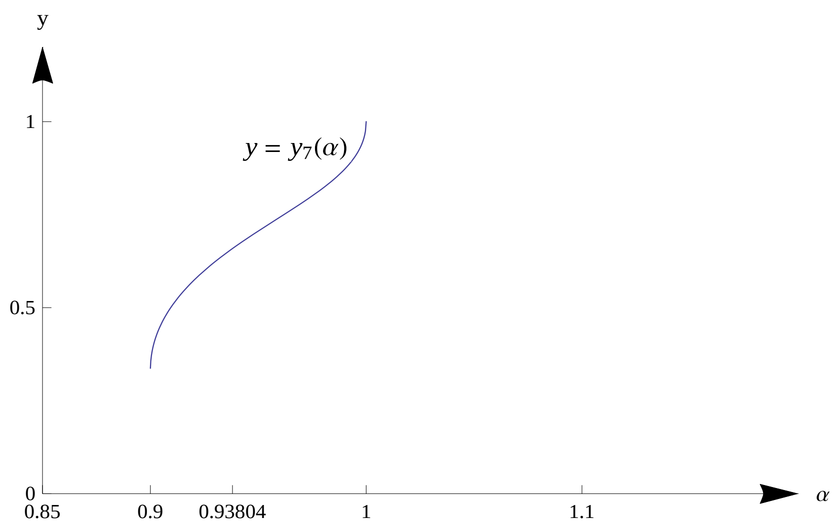

Figure 2.7: Graph of $y_7(\alpha) = \frac{1}{4\alpha-3}\left(1-\sqrt{\frac{2(1-\alpha)}{2\alpha-1}}\right) - \frac{(2\alpha-1)-\sqrt{(2\alpha-1)(10\alpha-9)}}{2(2\alpha-1)}$.

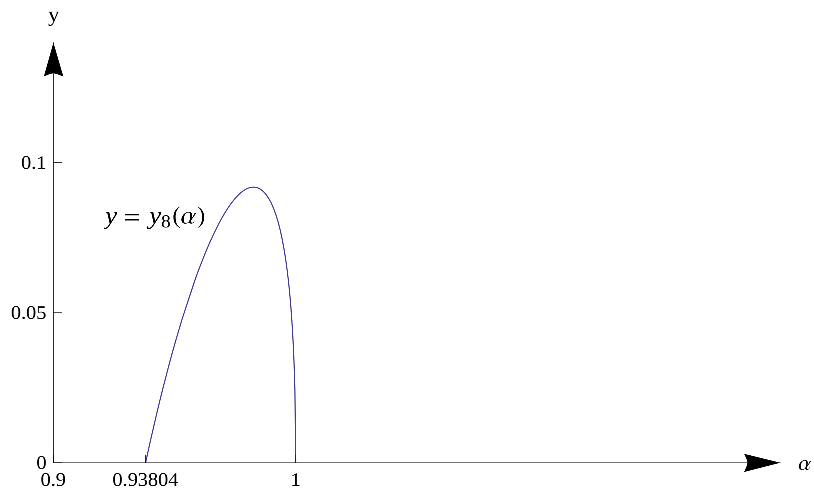

Figure 2.8: Graph of $y_8(\alpha) = \frac{(2\alpha-1)+\sqrt{(2\alpha-1)(10\alpha-9)}}{2(2\alpha-1)} - \frac{1}{4\alpha-3}\left(1-\sqrt{\frac{2(1-\alpha)}{2\alpha-1}}\right)$.

On the other hand, Figure 2.9 shows that $R_2'(\alpha) \geq r_2(\alpha)$ whenever $\alpha \in [9/10, \tau_\alpha]$. Thus if $r \in [r_2, 1)$, then $g(r) \geq 0$ and

$$|\mathscr{U}_f(z)| \leq \Phi(t_0, r) < 1$$

for all $|z| < R_1'(\alpha)$, where $R_1'(\alpha)$ is given by (2.22). A Mathematica plot in Figure 2.10 and Figure 2.11 show that $r_2(\alpha) \leq R_1'(\alpha) < 1$, it follows that $|\mathscr{U}_f(z)| < 1$ for $|z| < R_1'(\alpha)$ when $\alpha \in [9/10, \tau_\alpha]$.

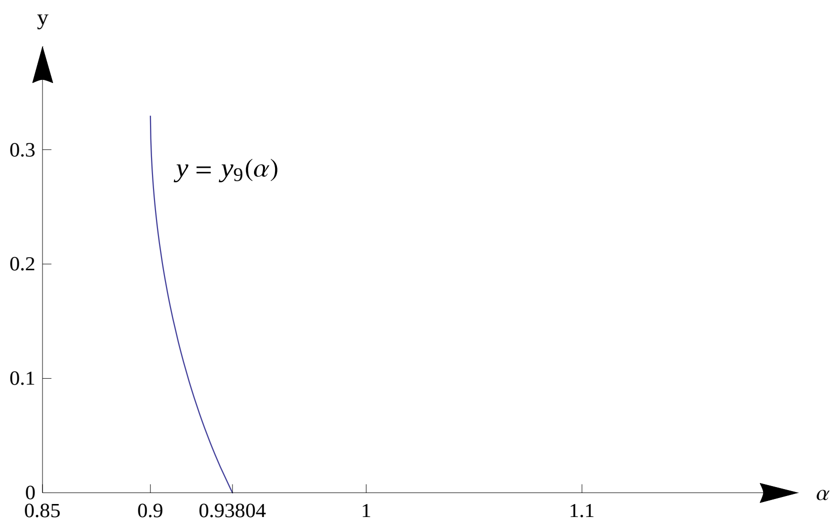

Figure 2.9: Graph of $y_9(\alpha) = \frac{1}{4\alpha-3}\left(1-\sqrt{\frac{2(1-\alpha)}{2\alpha-1}}\right) - \frac{(2\alpha-1)+\sqrt{(2\alpha-1)(10\alpha-9)}}{2(2\alpha-1)}$.

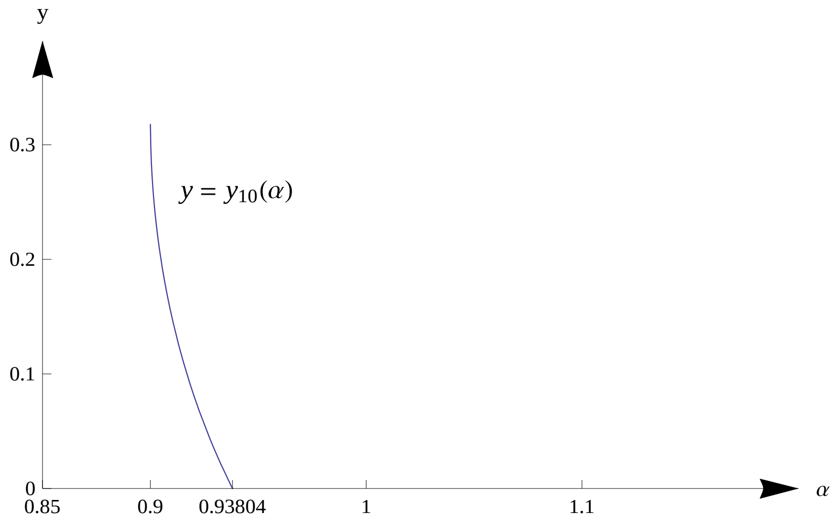

Figure 2.10: Graph of $y_{10}(\alpha) = \sqrt{\frac{\sqrt{2-4\alpha(1-\alpha)}-2(1-\alpha)}{2(2\alpha-1)}} - \frac{(2\alpha-1)+\sqrt{(2\alpha-1)(10\alpha-9)}}{2(2\alpha-1)}$.

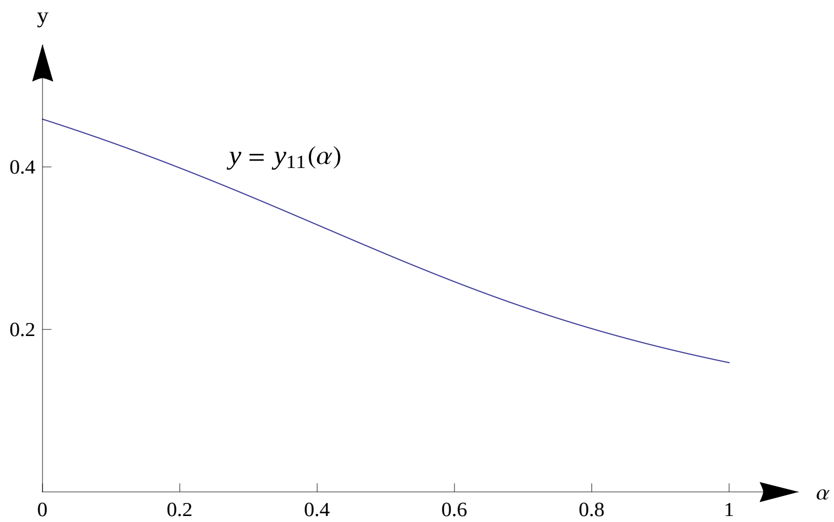

Figure 2.11: Graph of $y_{11}(\alpha) = 1 - \sqrt{\frac{\sqrt{2-4\alpha(1-\alpha)}-2(1-\alpha)}{2(2\alpha-1)}}$.

For $\alpha \in [0, 1/10]$, the extremal function is $f_0(z) = z\big(1-(1-2\alpha)z\big)/(1+z)$. In this case,

$$\begin{aligned}\frac{z}{f_0(z)} &= \frac{(1+z)}{1-(1-2\alpha)z} \\ &= (1+z)\sum_{n=0}^{\infty}(1-2\alpha)^n z^n \\ &= 1+\sum_{n=1}^{\infty}(1-2\alpha)^n z^n + \sum_{n=1}^{\infty}(1-2\alpha)^{n-1}z^n\end{aligned}$$

$$= 1 + 2(1-\alpha)\sum_{n=1}^{\infty}(1-2\alpha)^{n-1}z^n.$$

Thus

$$\frac{Rz}{f_0(Rz)} = 1 + 2(1-\alpha)\sum_{n=1}^{\infty}(1-2\alpha)^{n-1}R^n z^n = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

for $0 < R \leq 1$. Evidently,

$$\begin{aligned}
\sum_{n=2}^{\infty}(n-1)b_n &= 2(1-\alpha)(1-2\alpha)R^2\sum_{n=2}^{\infty}(n-1)((1-2\alpha)R)^{n-2} \\
&= 2(1-\alpha)(1-2\alpha)R^2\sum_{n=1}^{\infty}n((1-2\alpha)R)^{n-1} \\
&= \frac{2(1-\alpha)(1-2\alpha)R^2}{(1-(1-2\alpha)R)^2} \leq 1
\end{aligned}$$

if and only if $R \leq R(\alpha)$, where $R(\alpha) = \sqrt{2(1-\alpha)/(1-2\alpha)} - 1$ is the root of the equation

$$(1-2\alpha)R^2 + 2(1-2\alpha)R - 1 = 0.$$

It follows from Lemma 2.3 that $R^{-1}f_0(Rz) \in \mathcal{U}$ if $R \leq \sqrt{2(1-\alpha)/(1-2\alpha)} - 1$. $\square$

2.3 Product of Univalent Functions

For $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are appropriate subsets of $\mathcal{S}$, it is interesting to consider the function

$$F(z) = \frac{f(z)g(z)}{z}, \quad z \in \mathbb{D}, \tag{2.23}$$

and determine the largest radius of univalence for F. Also, it is interesting to find the radius r so that the function F belongs to particular subclasses of $\mathcal{S}$, such as $\mathcal{ST}$ and $\mathcal{U}$.

In [113], Obradović and Ponnusamy considered functions defined by (2.23), where $f \in \mathcal{F}_1 = \mathcal{U}$ and $g \in \mathcal{F}_2 = \mathcal{U}$, and proved that the $\mathcal{U}$-radius for F is $|z| = 1/3$, and that this radius is sharp. They also showed that the $\mathcal{U}$-radius for F is $|z| = r_0$, where $r_0 \approx 0.30294$, whenever $f, g \in \mathcal{S}$. In [112], they improved the value of r_0 to $r_0 \approx 0.326302$, where r_0 is the positive root of a certain equation. Additionally, they [113] showed that whenever $f, g \in \mathcal{ST}$, then the product F is starlike in the disk $|z| < 1/3$, and that this radius is sharp. Indeed if $f, g \in \mathcal{S}$, they conjectured that F is also univalent in the disk $|z| < 1/3$, and that the radius $1/3$ is best possible. In this section we shall validate the conjecture. Also, the radius of starlikeness for such functions F is shown to be $1/3$.

The following lemma is required.

Lemma 2.5. *If $f \in \mathcal{S}$, then*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{1-r}{1+r},$$

for $|z| = r < \tanh(1/2) \approx 0.46212.$

Proof. Let $f \in \mathcal{S}$. It follows from Lemma 2.1 that for $|z| \leq r < 1$, the region of values of $\zeta = \log(zf'(z)/f(z))$ is the disk

$$\mathcal{D}_r = \left\{\zeta : |\zeta| \leq \log\frac{1+r}{1-r}\right\}.$$

The function $w(z) = e^z$ is univalent in $\mathbb{D}$. Thus if r is chosen so that $\log((1+r)/(1-r)) < 1$, that is, $r < \tanh(1/2)$, then w is univalent in $\mathcal{D}_r$. Evidently the function $q(z) = w(z) - 1$ is convex in $\mathbb{D}$, that is, w is a convex function with positive coefficients in its series expansion. Thus

$$\begin{aligned}\inf_{\zeta\in\mathcal{D}_r} \mathrm{Re}\ w(\zeta) &= \inf_{\zeta\in\partial\mathcal{D}_r} \mathrm{Re}\ w(\zeta) = \inf_{\zeta\in\partial\mathcal{D}_r} w(\mathrm{Re}\ \zeta)\\ &= \inf_{\zeta\in\partial\mathcal{D}_r} |w(\zeta)| = \inf_{0\leq\theta\leq 2\pi}\left|\exp\left(\log\left(\frac{1+r}{1-r}\right)e^{i\theta}\right)\right|\end{aligned}$$

$$
\begin{aligned}
&= \inf_{0\le\theta\le 2\pi} \exp\left(\log\left(\frac{1+r}{1-r}\right)\cos\theta\right) \\
&= \exp\left(-\left(\log\frac{1+r}{1-r}\right)\right) = \frac{1-r}{1+r}
\end{aligned}
$$

for $|z| = r < \tanh(1/2)$. $\square$

Remark 2.2. Upon completion of the proof of Lemma 2.5, we found a similar result by Krzyż and Reade in [70]. However, our proof differs from Krzyż.

Theorem 2.8. *If $f, g \in \mathcal{S}$, then the function F defined by* (2.23) *is starlike in the disk $|z| < 1/3$. The radius $1/3$ is sharp.*

Proof. It follows from Lemma 2.5 that

$$
\mathrm{Re}\left(\frac{zF'(z)}{F(z)}\right) = \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) + \mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) - 1 = 2\left(\frac{1-r}{1+r}\right) - 1 = \frac{1-3r}{1+r}.
$$

Thus F is starlike in the disk $|z| < 1/3$.

To demonstrate the sharpness, let $f_0(z) = z/(1-z)^2 = g_0(z)$. Then $F_0(z) = z/(1-z)^4$. It follows that for $|z| = r$,

$$
\mathrm{Re}\left(\frac{zF_0'(z)}{F_0(z)}\right) = \frac{1+2r\cos\theta - 3r^2}{1-2r\cos\theta + r^2} > \frac{1-2r-3r^2}{(1+r)^2} = \frac{(1-3r)(1+r)}{(1+r)^2} = \frac{1-3r}{1+r}.
$$

Hence $\mathrm{Re}(zF_0'(z)/F_0(z)) > 0$ for $r < 1/3$. Furthermore, $F_0'(-1/3) = 0$, and thus F_0 is not univalent in any disk $r \ge 1/3$. $\square$

CHAPTER 3

ON THE SECOND HANKEL DETERMINANT FOR THE *K*TH-ROOT TRANSFORM OF ANALYTIC FUNCTIONS

3.1 Introduction

For positive integers q and n, the Hankel determinant $H_q(n)$ for an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is defined by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

The Hankel determinants play an important role in the study of singularities [37] as well as in the study of power series with integral coefficients [32]. Earlier investigations include those of [42, 43, 44, 58, 71, 91, 92, 93, 95, 96, 97, 98, 99, 100, 101, 102, 121] and [122], while recent works are those of [26, 29, 54, 55, 56, 63, 64, 72, 85, 87] and [88]. In [72], Lee *et al.* provided a brief survey on the Hankel determinants for analytic univalent functions and obtained bounds for $H_2(2)$ for functions belonging to several classes defined by subordination.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ with $f(z) \neq 0$ in $\mathbb{D}\backslash\{0\}$. Further, let $k \geq 2$ be a fixed integer. The *kth-root transform* of f is defined by

$$F(z) := \left(f\left(z^k\right)\right)^{\frac{1}{k}} = z\left(\frac{f\left(z^k\right)}{z^k}\right)^{\frac{1}{k}}.$$

Since $f\left(z^k\right)/z^k = 1+\sum_{n=2}^{\infty} a_n z^{k(n-1)} := 1+x$, and

$$(1+x)^{\frac{1}{k}} = 1+\frac{1}{k}x-\frac{(k-1)}{2k^2}x^2+\frac{(k-1)(2k-1)}{3!k^3}x^3+\cdots,$$

it follows that

$$\begin{aligned}
F(z) = z\left(\frac{f\left(z^k\right)}{z^k}\right)^{\frac{1}{k}} &= z\left(1+\frac{1}{k}\sum_{n=2}^{\infty} a_n z^{k(n-1)} - \frac{(k-1)}{2k^2}\left(\sum_{n=2}^{\infty} a_n z^{k(n-1)}\right)^2\right.\\
&\qquad \left. +\frac{(k-1)(2k-1)}{3!k^3}\left(\sum_{n=2}^{\infty} a_n z^{k(n-1)}\right)^3+\cdots\right)\\
&= z\left(1+\frac{1}{k}\left(a_2 z^k + a_3 z^{2k} + a_4 z^{3k}+\cdots\right) - \frac{(k-1)}{2k^2}\left(a_2^2 z^{2k} + 2a_2 a_3 z^{3k}+\ldots\right)\right.\\
&\qquad \left. +\frac{(k-1)(2k-1)}{3!k^3}\left(a_2^3 z^{3k}+\cdots\right)+\cdots\right)\\
&= z+\frac{a_2}{k}z^{k+1}+\frac{1}{k}\left(a_3-\frac{(k-1)}{2k}a_2^2\right)z^{2k+1}\\
&\qquad +\frac{1}{k}\left(a_4-\frac{(k-1)}{k}a_2 a_3+\frac{(k-1)(2k-1)}{3!k^2}a_2^3\right)z^{3k+1}+\cdots\\
&= z+\sum_{n=2}^{\infty} b_{(n-1)k+1} z^{(n-1)k+1},
\end{aligned}$$

where the initial coefficients are

$$\begin{aligned}
b_{k+1} &= \frac{a_2}{k}, \quad b_{2k+1} = \frac{1}{k}\left(a_3-\frac{(k-1)}{2k}a_2^2\right)\\
b_{3k+1} &= \frac{1}{k}\left(a_4-\frac{(k-1)}{k}a_2 a_3+\frac{(k-1)(2k-1)}{3!k^2}a_2^3\right).
\end{aligned} \tag{3.1}$$

Thus the Hankel determinants $H_q(n)$ for $F(z) = z+\sum_{n=2}^{\infty} b_{(n-1)k+1} z^{(n-1)k+1} = z+\sum_{n=2}^{\infty} B_n z^{(n-1)k+1}$ is

$$H_q(n) := \begin{vmatrix} B_n & B_{n+1} & \cdots & B_{n+q-1} \\ B_{n+1} & B_{n+2} & \cdots & B_{n+q} \\ \vdots & \vdots & & \vdots \\ B_{n+q-1} & B_{n+q} & \cdots & B_{n+2q-2} \end{vmatrix}$$

$$= \begin{vmatrix} b_{(n-1)k+1} & b_{nk+1} & \cdots & b_{(n+q-2)k+1} \\ b_{nk+1} & b_{(n+1)k+1} & \cdots & b_{(n+q-1)k+1} \\ \vdots & \vdots & & \vdots \\ b_{(n+q-2)k+1} & b_{(n+q-1)k+1} & \cdots & b_{(n+2q-3)k+1} \end{vmatrix}.$$

With $n = 2$, the Hankel determinants $H_q(2)$ for F is

$$H_q(2) := \begin{vmatrix} b_{k+1} & b_{2k+1} & \cdots & b_{qk+1} \\ b_{2k+1} & b_{3k+1} & \cdots & b_{(q+1)k+1} \\ \vdots & \vdots & & \vdots \\ b_{qk+1} & b_{(q+1)k+1} & \cdots & b_{(2q-1)k+1} \end{vmatrix}.$$

It follows that the second Hankel determinant $H_2(2)$ for F is

$$H_2(2) := \begin{vmatrix} b_{k+1} & b_{2k+1} \\ b_{2k+1} & b_{3k+1} \end{vmatrix} = b_{k+1}b_{3k+1} - b_{2k+1}^2.$$

In this chapter, bounds for the second Hankel determinant of the kth-root transform are obtained for several classes of functions defined via subordination. These classes can be seen as belonging to the genre of Ma-Minda starlike and convex functions, which will be made apparent in the next section. The results in this chapter are derived through several meticulous lengthy computations, and thus in several instances, these computations were validated by using of the computer algebra system Mathematica.

The following lemmas are needed to establish the results in subsequent sections.

Lemma 3.1. (Lemma 1.1) *If* $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \mathcal{P}$, *then*

$$|c_n| \leq 2.$$

This bound is sharp.

Lemma 3.2. [50, p.152] *If* $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \mathcal{P}$, *then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{3.2}$$

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)y, \tag{3.3}$$

for some $x, y \in \overline{\mathbb{D}}$.

The following result on the optimal value of a quadratic expression is also needed in the sequel. These estimates are obtained by standard calculus computations.

$$\max_{0 \leq t \leq 4} (Lt^2 + Mt + N) = \begin{cases} \frac{4LN - M^2}{4L}, & M > 0, L \leq -\frac{M}{8}, \\ N, & M \leq 0, L \leq -\frac{M}{4}, \\ 16L + 4M + N, & M \geq 0, L \geq -\frac{M}{8} \text{ or } M \leq 0, L \geq -\frac{M}{4}. \end{cases} \tag{3.4}$$

3.2 The Second Hankel Determinant of The *k*th-root Transform of Ma-Minda Starlike and Convex Functions

This section considers the class of Ma-Minda starlike and convex functions. For each class, a bound is obtained for its second Hankel determinant of the kth-root transform.

Definition 3.1. [77] *Let* $\varphi \in \mathcal{P}$ *be given by*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0, z \in \mathbb{D}). \tag{3.5}$$

Further, assume that φ is univalent in $\mathbb{D}$, maps $\mathbb{D}$ onto a region starlike with respect to 1*, and $\varphi(\mathbb{D})$ is symmetric with respect to the real axis. The class $\mathcal{ST}(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying $zf'(z)/f(z) \prec \varphi(z)$.*

In the literature, this class $\mathcal{ST}(\varphi)$ is widely called the *Ma-Minda starlike functions with respect to* φ. For the particular case when φ is given by

$$\varphi_\alpha(z) := \frac{1+(1-2\alpha)z}{1-z} = 1+2(1-\alpha)z+2(1-\alpha)z^2+2(1-\alpha)z^3+\cdots, \quad 0 \le \alpha < 1,$$

the class $\mathcal{ST}(\varphi) := \mathcal{ST}(\varphi_\alpha)$ is the well-known class of *starlike functions of order* α.

For the function

$$\varphi_{PAR}(z) := 1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 = 1+\frac{8}{\pi^2}z+\frac{16}{3\pi^2}z^2+\frac{184}{45\pi^2}z^3+\cdots,$$

$\mathcal{ST}(\varphi_{PAR})$ is the class $\mathcal{ST}_P$ of *parabolic starlike* functions introduced by Rønning [131]:

$$\mathcal{ST}_P := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'z}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)}-1\right| \right\}.$$

Ali and Ravichandran [21] gave a survey on parabolic starlike functions and its related class of uniformly convex functions.

When

$$\varphi_\beta(z) := \left(\frac{1+z}{1-z}\right)^\beta = 1+2\beta z+2\beta^2 z^2+\frac{2}{3}\beta(1+2\beta^2)z^3+\cdots, \quad 0 < \beta \le 1,$$

the class $\mathcal{ST}(\varphi_\beta)$ is the familiar class $\mathcal{ST}_\beta$ of *strongly starlike functions of order* β:

$$\mathcal{ST}_\beta := \left\{ f \in \mathcal{A} : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2} \right\}.$$

The class $\mathcal{ST}(\sqrt{1+z})$ is the class of *lemniscate of Bernoulli starlike* functions studied in [136]:

$$\mathcal{ST}_L := \left\{ f \in \mathcal{A} : \left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \right\}.$$

The first result in this chapter finds the bound for the second Hankel determinant of the kth-root transform for Ma-Minda starlike functions.

Theorem 3.1. *Let φ be given by* (3.5)*, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{ST}(\varphi)$, and $F(z) = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}$ be its kth-root transform. Further, let $\delta = 1/k^2$.*

1. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| \leq B_1, \quad \text{and} \quad |4B_1B_3 - \delta B_1^4 - 3B_2^2| - 3B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|H_2(2)| = |b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq \frac{B_1^2}{4k^2}.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| \geq B_1, \quad \text{and} \quad |4B_1B_3 - \delta B_1^4 - 3B_2^2| - B_1|B_2| - 2B_1^2 \geq 0,$$

or the conditions

$$|B_2| \leq B_1, \quad \text{and} \quad |4B_1B_3 - \delta B_1^4 - 3B_2^2| - 3B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq \frac{1}{12k^2}|4B_1B_3 - \delta B_1^4 - 3B_2^2|.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| > B_1, \quad \text{and} \quad 2B_1^2 + B_1|B_2| - |4B_1B_3 - \delta B_1^4 - 3B_2^2| \geq 0,$$

then the second Hankel determinant satisfies

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq \frac{B_1^2}{12k^2} \left(\frac{3|4B_1B_3 - \delta B_1^4 - 3B_2^2| - 4B_1^2 - |B_2|^2 - 4B_1|B_2|}{|4B_1B_3 - \delta B_1^4 - 3B_2^2| - 2B_1|B_2| - B_1^2} \right).$$

Proof. Since $f \in \mathcal{ST}(\varphi)$, there exists an analytic self-map w of $\mathbb{D}$ with $w(0) = 0$ satisfying

$$\frac{zf'(z)}{f(z)} = \varphi(w(z)). \tag{3.6}$$

Define the function $P_1 \in \mathcal{P}$ by

$$P_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots,$$

or equivalently,

$$w(z) = \frac{P_1(z)-1}{P_1(z)+1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right). \tag{3.7}$$

By using (3.7) along with (3.5) lead to the expansion

$$\begin{aligned}
\varphi(w(z)) &= 1 + B_1 w(z) + B_2 w^2(z) + \cdots \\
&= 1 + B_1\left(\frac{P_1(z)-1}{P_1(z)+1}\right) + B_2\left(\frac{P_1(z)-1}{P_1(z)+1}\right)^2 + B_3\left(\frac{P_1(z)-1}{P_1(z)+1}\right)^3 + \cdots \\
&= 1 + \frac{B_1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right) \\
&\quad + \frac{B_2}{2}\left(\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right)\right)^2 \\
&\quad + \frac{B_3}{2}\left(\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right)\right)^3 + \cdots \\
&= 1 + \frac{B_1}{2}c_1 z + \left(\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2}{4}c_1^2\right)z^2 \\
&\quad + \left(B_1\left(\frac{c_3}{2} - \frac{c_1 c_2}{2} + \frac{c_1^3}{8}\right) + B_2 c_1\left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) + \frac{B_3 c_1^3}{8}\right)z^3 + \cdots.
\end{aligned} \tag{3.8}$$

Now

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + \left(2a_3 - a_2^2\right)z^2 + \left(3a_4 - 3a_2 a_3 + a_2^3\right)z^3 + \cdots. \tag{3.9}$$

Comparing with (3.6), (3.8) and (3.9), it follows that

$$\begin{aligned}
a_2 &= \frac{B_1 c_1}{2}, \\
a_3 &= \frac{1}{2}\left(\left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) + \frac{B_1^2 c_1^2}{4}\right),
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\Big(\big(B_1^2 - B_1 + B_2\big)c_1^2 + 2B_1c_2\Big),\\
a_4 &= \frac{1}{3}\Big(\frac{-B_1^3c_1^3}{8} + \frac{3B_1c_1^3}{16}\big(B_1^2 - B_1 + B_2\big) + \frac{3B_1^2c_1c_2}{8} + B_1\big(\frac{4c_3 - 4c_1c_2 + c_1^3}{8}\big)\\
&\quad + B_2c_1\big(\frac{2c_2 - c_1^2}{4}\big) + \frac{B_3c_1^3}{8}\Big),\\
&= \frac{1}{48}\Big(\big(-4B_2 + 2B_1 + B_1^3 - 3B_1^2 + 3B_1B_2 + 2B_3\big)c_1^3\\
&\quad + 2\big(3B_1^2 - 4B_1 + 4B_2\big)c_1c_2 + 8B_1c_3\Big). \qquad (3.10)
\end{aligned}
$$

Consequently (3.10) and (3.1) yield

$$
\begin{aligned}
b_{k+1} &= \frac{B_1c_1}{2k},\\
b_{2k+1} &= \frac{1}{8k}\Big(\big(B_1^2 - B_1 + B_2\big)c_1^2 + 2B_1c_2\Big) + \frac{(1-k)}{8k^2}B_1^2c_1^2,\\
b_{3k+1} &= \frac{1}{48k}\Big(\big(B_1^3 - 3B_1^2 + 2B_1 - 4B_2 + 3B_1B_2 + 2B_3\big)c_1^3 + 2\big(3B_1^2 - 4B_1 + 4B_2\big)c_1c_2\\
&\quad + 8B_1c_3\Big) + \frac{(1-k)B_1c_1}{16k^2}\Big(\big(B_1^2 - B_1 + B_2\big)c_1^2 + 2B_1c_2\Big) + \frac{(1-k)(1-2k)}{48k^3}B_1^3c_1^3.
\end{aligned}
$$

Lengthy computations, validated by Mathematica, show that

$$
\begin{aligned}
b_{k+1}b_{3k+1} - b_{2k+1}^2 &= \left(\frac{1}{96k^2} - \frac{1}{64k^2} + \frac{(1-k)^2}{64k^4} + \frac{(k-1)(2k-1)}{96k^4}\right)B_1^4c_1^4\\
&\quad + \left(\frac{1}{32k^2} - \frac{1}{32k^2} + \frac{(1-k)}{32k^3} + \frac{(k-1)}{32k^3}\right)B_1^2B_2c_1^4\\
&\quad + \left(\frac{1}{48k^2} - \frac{1}{64k^2}\right)B_1^2c_1^4 + \left(\frac{1}{32k^2} - \frac{1}{24k^2}\right)B_1B_2c_1^4 +\\
&\quad + \left(\frac{1}{32k^2} + \frac{(1-k)}{32k^3} - \frac{(1-k)}{32k^3} - \frac{1}{32k^2}\right)B_1^3c_1^4\\
&\quad + \frac{1}{48k^2}B_1B_3c_1^4 - \frac{1}{64k^2}B_2^2c_1^4 + \left(-\frac{1}{12k^2} + \frac{1}{16k^2}\right)B_1^2c_1^2c_2\\
&\quad + \left(-\frac{1}{16k^2} + \frac{(1-k)}{16k^3} + \frac{1}{16k^2} + \frac{(k-1)}{16k^3}\right)B_1^3c_1^2c_2\\
&\quad + \left(\frac{1}{12k^2} - \frac{1}{16k^2}\right)B_1B_2c_1^2c_2 - \frac{1}{16k^2}B_1^2c_2^2 + \frac{1}{12k^2}B_1^2c_1c_3.
\end{aligned}
$$

Thus

$$b_{k+1}b_{3k+1}-b_{2k+1}^2=\frac{B_1}{192k^2}\Big(4\big(B_2-B_1\big)c_1^2c_2+\big(B_1-\frac{1}{k^2}B_1^3-2B_2+4B_3-3\frac{B_2^2}{B1}\big)c_1^4$$
$$+16B_1c_1c_3-12B_1c_2^2\Big).$$

Next, for ease of computations, let

$$d_1=16B_1,\quad d_2=4\left(B_2-B_1\right),\quad d_3=-12B_1,$$
$$d_4=B_1-\delta B_1^3-2B_2+4B_3-3\frac{B_2^2}{B_1},\quad \text{and}\quad T=\frac{B_1}{192k^2}. \tag{3.11}$$

It follows that,

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|=T\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|. \tag{3.12}$$

Substituting the values of c_2 and c_3 respectively from (3.2) and (3.3) in (3.12), then

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|=T\Big|\frac{d_1c_1}{4}\big(c_1^3+2c_1x(4-c_1^2)-c_1x^2(4-c_1^2)+2(4-c_1^2)(1-|x|^2)y\big)$$
$$+\frac{d_2c_1^2}{2}\big(c_1^2+x(4-c_1^2)\big)+\frac{d_3}{4}\big(c_1^4+2c_1^2x(4-c_1^2)+x^2(4-c_1^2)^2\big)$$
$$+d_4c_1^4\Big|$$
$$=\frac{T}{4}\Big|d_1\big(c_1^4+2c_1^2x(4-c_1^2)-c_1^2x^2(4-c_1^2)+2c(4-c_1^2)(1-|x|^2)y\big)$$
$$+d_2\big(2c_1^4+2xc_1^2(4-c_1^2)\big)+d_3\big(c_1^4+2xc_1^2(4-c_1^2)+x^2(4-c_1^2)^2\big)$$
$$+4d_4c_1^4\Big|.$$

Since the function $p\left(e^{i\theta}z\right)$ $(\theta\in\mathbb{R})$ is in the class $\mathcal{P}$ for any $p\in\mathcal{P}$, there is no loss of generality in assuming $c_1=c>0$, $c\in[0,2]$, it follows that

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|=\frac{T}{4}\Big|c^4\big(d_1+2d_2+d_3+4d_4\big)+2xc^2(4-c^2)(d_1+d_2+d_3)$$
$$+(4-c^2)x^2\big(-d_1c^2+d_3(4-c^2)\big)+2d_1c(4-c^2)(1-|x|^2)y\Big|$$

for some $x, y \in \overline{\mathbb{D}}$. With $s = |x|$, (3.11) yields

$$\begin{aligned}
\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leq \frac{T}{4}&\left(c^4\left|16B_3 - 4\delta B_1^3 - 12\frac{B_2^2}{B_1}\right| + 8sc^2(4-c^2)\,|B_2|\right.\\
&\left.+ s^2(4-c^2)(4B_1c^2 + 48B_1) + 32B_1c(4-c^2)(1-s^2)\right)\\
= T&\left(c^4\left|4B_3 - \delta B_1^3 - 3\frac{B_2^2}{B_1}\right| + 8B_1c(4-c^2) + 2\,|B_2|\,sc^2(4-c^2)\right.\\
&\left.+ B_1s^2(4-c^2)(c-2)(c-6)\right)\\
:= F&(c,s),
\end{aligned}$$

where $(c,s) \in [0,2]\times[0,1]$. Now

$$\frac{\partial F}{\partial s} = T\left(2\,|B_2|\,c^2(4-c^2) + 2B_1s(4-c^2)(c-2)(c-6)\right),$$

For $0 < s < 1$ and for any fixed c with $0 < c < 2$, it is evident that $\partial F/\partial s > 0$, and thus $F(c,s)$ is an increasing function of s. Hence

$$\max_{0\leq s\leq 1} F(c,s) = F(c,1) := G(c).$$

Upon simplification, we find that,

$$G(c) = \frac{B_1}{192k^2}\left(c^4\left(\left|4B_3 - \delta B_1^3 - \frac{3B_2^2}{B_1}\right| - 2\,|B_2| - B_1\right) + 8c^2\left(|B_2| - B_1\right) + 48B_1\right).$$

Writing $c^2 = t$ and

$$L = \left|4B_3 - \delta B_1^3 - \frac{3B_2^2}{B_1}\right| - 2\,|B_2| - B_1, \quad M = 8\left(|B_2| - B_1\right), \quad \text{and} \quad N = 48B_1,$$

it follows from (3.4) that,

$$\frac{192k^2}{B_1}\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| \leq \begin{cases} \frac{4LN-M^2}{4L}, & M>0, L\leq -\frac{M}{8}, \\ N, & M\leq 0, L\leq -\frac{M}{4}, \\ 16L+4M+N, & M\geq 0, L\geq -\frac{M}{8} \text{ or } M\leq 0, L\geq -\frac{M}{4}. \end{cases}$$

These conditions now lead to the desired bounds for the second Hankel determinant.

Thus for $|B_2|\leq B_1$ and $|4B_1B_3-\delta B_1^4-3B_2^2|-3B_1^2\leq 0$, the second Hankel determinant satisfies

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|\leq \frac{B_1^2}{4k^2}.$$

For $|B_2|\geq B_1$ and $|4B_1B_3-\delta B_1^4-3B_2^2|-B_1|B_2|-2B_1^2\geq 0$, or for $|B_2|\leq B_1$ and $|4B_1B_3-\delta B_1^4-3B_2^2|-3B_1^2\geq 0$, the second Hankel determinant satisfies

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|\leq \frac{1}{12k^2}|4B_1B_3-\delta B_1^4-3B_2^2|.$$

For $|B_2|>B_1$ and $2B_1^2+B_1|B_2|-|4B_1B_3-\delta B_1^4-3B_2^2|\geq 0$, the second Hankel determinant satisfies

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|\leq \frac{B_1^2}{12k^2}\left(\frac{3|4B_1B_3-\delta B_1^4-3B_2^2|-4B_1^2-|B_2|^2-4B_1|B_2|}{|4B_1B_3-\delta B_1^4-3B_2^2|-2B_1|B_2|-B_1^2}\right). \quad \square$$

The special case $k=1$ in Theorem 3.1 reduces to the result obtained by Lee *et al.* [72].

Corrollary 3.1. [72, Theorem 1] *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{ST}(\varphi)$.

1. *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$|B_2|\leq B_1, \quad \textit{and} \quad |4B_1B_3-B_1^4-3B_2^2|-3B_1^2\leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{4}.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| \geq B_1, \quad \textit{and} \quad |4B_1B_3 - B_1^4 - 3B_2^2| - B_1|B_2| - 2B_1^2 \geq 0,$$

or the conditions

$$|B_2| \leq B_1, \quad \textit{and} \quad |4B_1B_3 - B_1^4 - 3B_2^2| - 3B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_2 - a_3^2| \leq \frac{1}{12}|4B_1B_3 - B_1^4 - 3B_2^2|.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| > B_1, \quad \textit{and} \quad 2B_1^2 + B_1|B_2| - |4B_1B_3 - B_1^4 - 3B_2^2| \geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{12}\left(\frac{3|4B_1B_3 - B_1^4 - 3B_2^2| - 4B_1^2 - |B_2|^2 - 4B_1|B_2|}{|4B_1B_3 - B_1^4 - 3B_2^2| - 2B_1|B_2| - B_1^2}\right).$$

With $k = 1$ and the choice $\varphi(z) = (1+z)/(1-z)$, that is, $B_1 = B_2 = B_3 = 2$, Theorem 3.1 reduces to the following corollary.

Corrollary 3.2. [64, Theorem 3.1] *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{ST}$. Then $|a_2a_4 - a_3^2| \leq$ 1.*

Judicious choices of φ in Theorem 3.1 lead to the following results for the special cases.

Corrollary 3.3.

1. *If $f \in \mathcal{ST}(\alpha)$, then $|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq (1-\alpha)^2/k^2$.*
2. *If $f \in \mathcal{ST}_L$, then $|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq 1/(16k^2)$.*

3. *If $f \in \mathcal{ST}_P$, then $|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq 16/(\pi^4 k^2)$.*

4. *If $f \in \mathcal{ST}_\beta$, then $|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq \beta^2/k^2$.*

Proof. 1. If $f \in \mathcal{ST}(\alpha)$, then $\varphi(z) = (1+(1-2\alpha)z)/(1-z)$. Thus $B_1 = B_2 = B_3 = 2(1-\alpha)$, by applying Theorem 3.1, $|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq (1-\alpha)^2/k^2$.

2. If $f \in \mathcal{ST}_L$, then $\varphi(z) = \sqrt{1+z}$. Thus $B_1 = 1/2$, $B_2 = -1/8$, $B_3 = 1/16$, by applying Theorem 3.1, $|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq 1/(16k^2)$.

3. If $f \in \mathcal{ST}_P$, then $\varphi(z) = 1 + 2/\pi^2 \Big(\log\big((1+\sqrt{z})/(1-\sqrt{z})\big)\Big)^2$. Thus $B_1 = 8/\pi^2$, $B_2 = 16/3\pi^2$, $B_3 = 184/45\pi^2$, by applying Theorem 3.1, $|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq 16/(\pi^4 k^2)$.

4. If $f \in \mathcal{ST}_\beta$, then $\varphi(z) = \big((1+z)/(1-z)\big)^\beta$. Thus $B_1 = 2\beta$, $B_2 = 2\beta^2$, $B_3 = 2\beta(1+2\beta^2)/3$, by applying Theorem 3.1, $|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq \beta^2/k^2$. □

Definition 3.2. [77] *Let $\varphi \in \mathcal{P}$ satisfy the condition in Definition 3.1. The class $\mathcal{CV}(\varphi)$ of* Ma-Minda convex functions with respect to φ *consists of functions f satisfying the subordination*

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z).$$

The bound for the second Hankel determinant of the kth-root transform for Ma-Minda convex functions is determined in the next theorem.

Theorem 3.2. *Let φ be given by* (3.5)*, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{CV}(\varphi)$, and $F(z) = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}$ be its kth-root transform.*

1. *If B_1, B_2 and B_3 satisfy the conditions*

$$\frac{(3k^2+4k-6)}{k}B_1^2 + 4|B_2| - 2B_1 \leq 0, \quad and$$

$$\left|\frac{6(k+1)-(k+3)^2}{4k^2}B_1^4+(3k^2-5k+2)B_1^3+\frac{(9k-8)}{k}B_1^2B_2+6B_1B_3-4B_2^2\right|$$
$$-4B_1^2\leq 0,$$

then the second Hankel determinant satisfies

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|\leq\frac{B_1^2}{36k^2}.$$

2. *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$\frac{(3k^2+4k-6)}{k}B_1^2+4|B_2|-2B_1\geq 0,\quad \textit{and}$$
$$2\left|\frac{6(k+1)-(k+3)^2}{4k^2}B_1^4+(3k^2-5k+2)B_1^3+\frac{(9k-8)}{k}B_1^2B_2+6B_1B_3-4B_2^2\right|$$
$$-\frac{(3k^2+4k-6)}{k}B_1^3-4B_1|B_2|-6B_1^2\geq 0,$$

or the conditions

$$\frac{(3k^2+4k-6)}{k}B_1^2+4|B_2|-2B_1\leq 0,\quad \textit{and}$$
$$\left|\frac{6(k+1)-(k+36)^2}{4k^2}B_1^4+(3k^2-5k+2)B_1^3+\frac{(9k-8)}{k}B_1^2B_2+6B_1B_3-4B_2^2\right|$$
$$-4B_1^2\geq 0,$$

then the second Hankel determinant satisfies

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|$$
$$\leq\frac{B_1}{144k^2}\left|\frac{(6(k+1)-(k+3)^2)}{4k^2}B_1^3+(3k^2-5k+2)B_1^2+\frac{(9k-8)}{k}B_1B_2+6B_3-4\frac{B_2^2}{B_1}\right|.$$

3. *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$(3k^2+4k-6)B_1^2+4|B_2|-2B_1>0\quad \textit{and}$$
$$2\left|\frac{(6(k+1)-(k+3)^2)}{4k^2}B_1^3+(3k^2-5k+2)B_1^2+\frac{(9k-8)}{k}B_1B_2+6B_3-4\frac{B_2^2}{B_1}\right|$$
$$-\frac{(3k^2+4k-6)}{k}B_1^3-4B_1|B_2|-6B_1^2\leq 0,$$

then the second Hankel determinant satisfies

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|$$
$$\leq \frac{B_1^2}{576k^2}\left(\frac{\begin{array}{r}16\left|\frac{6(k+1)-(k+3)^2}{4k^2}B_1^4+(3k^2-5k+2)B_1^3+\frac{(9k-8)}{k}B_1^2B_2+6B_1B_3-4B_2^2\right|-16|B_2|^2\\ -48B_1|B_2|-36B_1^2-\frac{12(3k^2+4k-6)}{k}B_1^3-\frac{(3k^2+4k-6)^2}{k^2}B_1^4-\frac{8(3k^2+4k-6)}{k}B_1^2|B_2|\end{array}}{\begin{array}{r}\left|\frac{6(k+1)-(k+3)^2}{4k^2}B_1^4+(3k^2-5k+2)B_1^3+\frac{(9k-8)}{k}B_1^2B_2+6B_1B_3-4B_2^2\right|\\ -\frac{(3k^2+4k-6)}{k}B_1^3-4B_1|B_2|-2B_1^2\end{array}}\right).$$

Proof. Since $f\in\mathcal{CV}(\varphi)$, there exists an analytic self-map w of $\mathbb{D}$ with $w(0)=0$ satisfying

$$1+\frac{zf''(z)}{f'(z)}=\varphi(w(z)). \tag{3.13}$$

Define the function $P_1\in\mathcal{P}$ by

$$P_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\cdots.$$

Now

$$1+\frac{zf''(z)}{f'(z)}=1+2a_2z+(-4a_2^2+6a_3)z^2+(8a_2^3-18a_2a_3+12a_4)z^3+\cdots. \tag{3.14}$$

Comparing with (3.8), (3.13) and (3.14), it follows that

$$\begin{aligned}a_2&=\frac{B_1c_1}{4},\\ a_3&=\frac{1}{24}\big((B_1^2-B_1+B_2)c_1^2+2B_1c_2\big),\\ a_4&=\frac{1}{192}\big((-4B_2+2B_1+B_1^3-3B_1^2+3B_1B_2+2B_3)c_1^3\\ &\qquad+2(3B_1^2-4B_1+4B_2)c_1c_2+8B_1c_3\big).\end{aligned} \tag{3.15}$$

Consequently (3.15) and (3.1) yield

$$\begin{aligned}b_{k+1}&=\frac{B_1c_1}{4k},\\ b_{2k+1}&=\frac{1}{24k}\left((B_2-B_1)c_1^2+2B_1c_2\right)+\frac{(k+3)}{96k^2}B_1^2c_1^2,\end{aligned}$$

$$b_{3k+1}=\frac{1}{192k}\left(\left(\frac{(4k^2-3k+1)}{2k^2}B_1^3-3B_1^2+2B_1-4B_2+3B_1B_2+2B_3\right)c_1^3\right)$$
$$+\frac{1}{96k}\left(\left(\frac{(k+2)}{k}B_1^2-4B_1+4B_2\right)c_1c_2\right)+\frac{1}{24k}B_1c_3$$
$$+\frac{(1-k)B_1c_1^3}{96k^2}\left(B_1^2-B_1+B_2\right).$$

Lengthy computations, validated by Mathematica, show that

$$b_{k+1}b_{3k+1}-b_{2k+1}^2=\frac{B_1}{768k^2}\Bigg(c_1^4\Big(\frac{(6(k+1)-(k+3)^2)}{12k^2}B_1^3+\frac{(8k-9k^2)}{3k^2}(B_1^2-B_1B_2)$$
$$-\frac{4}{3}(B_2+\frac{B_2^2}{B_1})+\frac{2}{3}(B_1+3B_3)\Big)$$
$$+\frac{2c_1^2c_2}{3}\left(\frac{(3k^2+4k-6)}{k}B_1^2-4B_1+4B_2\right)+8B_1c_1c_3-\frac{16}{3}B_1c_2^2\Bigg).$$

Next, for ease of computations, let

$$d_1=8B_1,\quad d_2=\frac{2}{3}\left(\frac{(3k^2+4k-6)B_1^2}{k}+4(B_2-B_1)\right),\quad d_3=-\frac{16B_1}{3},$$
$$d_4=4\Big(\frac{(6(k+1)-(k+3)^2)}{48k^2}B_1^3+\frac{(8k-9k^2)}{12k^2}(B_1^2-B_1B_2)$$
$$-\frac{1}{3}(B_2+\frac{B_2^2}{3B_1}+\frac{1}{6}(B_1+3B_3)\Big),\quad \text{and}\quad T=\frac{B_1}{768k^2}. \tag{3.16}$$

Then

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|=T\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|. \tag{3.17}$$

Since the function $p\left(e^{i\theta}z\right)$ $(\theta\in\mathbb{R})$ is in the class $\mathcal{P}$ for any $p\in\mathcal{P}$, there is no loss of generality in assuming $c_1=c>0$, $c\in[0,2]$. Substituting the values of c_2 and c_3 respectively from (3.2) and (3.3) in (3.17), it follows that

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|=\frac{T}{4}\big|c^4\left(d_1+2d_2+d_3+4d_4\right)+2xc^2(4-c^2)(d_1+d_2+d_3)$$
$$+(4-c^2)x^2(-d_1c^2+d_3(4-c^2))+2d_1c(4-c^2)(1-|x|^2)y\big|$$

for some $x, y \in \overline{\mathbb{D}}$. With $s = |x|$, (3.16) yields

$$\begin{aligned}
\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leq \frac{T}{4}\Bigg(& 16c^4\left|\frac{(6(k+1)-(k+3)^2)}{48k^2}B_1^3 + \frac{(3k^3-5k^2+2k)}{12k}B_1^2\right. \\
& \left. - \frac{B_2^2}{3B_1} + \frac{(9k^2-8k)}{12k^2}B_1B_2 + \frac{B_3}{2}\right| + 32sc^2(4-c^2) \\
& \left(\frac{|B_2|}{6} + \frac{(3k^2+4k-6)B_1^2}{24k}\right) + 16s^2\left(4-c^2\right)\left(\frac{B_1}{6}c^2 + \frac{4B_1}{3}\right) \\
& + 16B_1c\left(4-c^2\right)\left(1-s^2\right)\Bigg) \\
= T\Bigg(& \frac{c^4}{3}\left|\frac{(6(k+1)-(k+3)^2)}{4k^2}B_1^3 + (3k^2-5k+2)B_1^2\right. \\
& \left. + (9k-8)B_1B_2 + 6B_3 - 4\frac{B_2^2}{B_1}\right| + 4B_1c\left(4-c^2\right) \\
& + \frac{1}{3}sc^2\left(4-c^2\right)\left((3k^2+4k-6)B_1^2 + 4|B_2|\right) \\
& + \frac{2B_1}{3}s^2\left(4-c^2\right)(c-2)(c-4)\Bigg) \\
:= & F(c,s),
\end{aligned}$$

$(c,s) \in [0,2] \times [0,1]$. Now

$$\frac{\partial F}{\partial s} = T\left(\frac{c^2}{3}\left(4-c^2\right)\left(\frac{(3k^2+4k-6)}{k}B_1^2 + 4|B_2|\right) + \frac{4B_1}{3}s\left(4-c^2\right)(c-2)(c-4)\right),$$

it is clear that $\partial F/\partial s > 0$ for $0 < s < 1$ and for any fixed c with $0 < c < 2$. Thus $F(c,s)$ is an increasing function of s Hence

$$\max_{0 \leq s \leq 1} F(c,s) = F(c,1) := G(c).$$

Upon simplification, we find that

$$\begin{aligned}
G(c) = T\Bigg(\frac{c^4}{3}\Bigg(\Bigg| & \frac{(6(k+1)-(k+3)^2)}{4k^2}B_1^3 + (3k^2-5k+2)B_1^2 + \frac{(9k-8)}{k}B_1B_2 \\
& + 6B_3 - 4\frac{B_2^2}{B_1}\Bigg| - \frac{(3k^2+4k-6)}{k}B_1^2 - 4|B_2| - 2B_1\Bigg)
\end{aligned}$$

$$+\frac{4}{3}c^2\Big(\frac{(3k^2+4k-6)}{k}B_1^2+4|B_2|-2B_1\Big)+\frac{64B_1}{3}\Bigg).$$

Writing $c^2=t$ and

$$\begin{aligned}
L &= \frac{1}{3}\Bigg(\Bigg|\frac{(6(k+1)-(k+3)^2)}{4k^2}B_1^3+(3k^2-5k+2)B_1^2+\frac{(9k-8)}{k}B_1B_2\\
&\qquad +6B_3-4\frac{B_2^2}{B_1}\Bigg|-\frac{(3k^2+4k-6)}{k}B_1^2-4|B_2|-2B_1\Bigg),\\
M &= \frac{4}{3}\Big(\frac{(3k^2+4k-6)}{k}B_1^2+4|B_2|-2B_1\Big), \quad \text{and}\\
N &= \frac{64B_1}{3},
\end{aligned}$$

it follows from (3.4) that

$$\frac{768k^2}{B_1}\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| \leq \begin{cases} \frac{4LN-M^2}{4L}, & M>0, L\leq -\frac{M}{8},\\ N, & M\leq 0, L\leq -\frac{M}{4},\\ 16L+4M+N, & M\geq 0, L\geq -\frac{M}{8} \text{ or } M\leq 0, L\geq -\frac{M}{4}. \end{cases}$$

These conditions now lead to the desired bounds for the second Hankel determinant. □

The special case $k=1$ in Theorem 3.2 reduces to the following corollary.

Corrollary 3.4. [72, Theorem 2] *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{CV}(\varphi)$.

1. *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$B_1^2+4|B_2|-2B_1\leq 0, \quad \textit{and} \quad |6B_1B_3+B_1^2B_2-B_1^4-4B_2^2|-4B_1^2\leq 0,$$

then the second Hankel determinant of f *satisfies*

$$|a_2a_4-a_3^2|\leq\frac{B_1^2}{36}.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$B_1^2+4|B_2|-2B_1 \geq 0, \quad \textit{and} \quad 2|6B_1B_3+B_1^2B_2-B_1^4-4B_2^2|-B_1^3-4B_1|B_2|-6B_1^2 \geq 0,$$

or the conditions

$$B_1^2+4|B_2|-2B_1 \leq 0, \quad \textit{and} \quad |6B_1B_3+B_1^2B_2-B_1^4-4B_2^2|-4B_1^2 \geq 0,$$

then the second Hankel determinant of f satisfies

$$|a_2a_4-a_3^2| \leq \frac{1}{144}|6B_1B_3+B_1^2B_2-B_1^4-4B_2^2|.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$B_1^2+4|B_2|-2B_1 > 0, \quad \textit{and} \quad 2|6B_1B_3+B_1^2B_2-B_1^4-4B_2^2|-B_1^3-4B_1|B_2|-6B_1^2 \leq 0,$$

then the second Hankel determinant of f satisfies

$$\begin{aligned}&|a_2a_4-a_3^2|\\ \leq&\frac{B_1^2}{576}\left(\frac{16|6B_1B_3+B_1^2B_2-B_1^4-4B_2^2|-12B_1^3-48B_1|B_2|-36B_1^2-B_1^4-8B_1^2|B_2|-16|B_2|^2}{|6B_1B_3+B_1^2B_2-B_1^4-4B_2^2|-B_1^3-4B_1|B_2|-2B_1^2}\right).\end{aligned}$$

With $k=1$ and the choice $\varphi(z)=(1+z)/(1-z)$, that is, $B_1=B_2=B_3=2$, Theorem 3.2 reduces to the following corollary.

Corrollary 3.5. [64, Theorem 3.2] *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n \in \mathcal{CV}((1+z)/(1-z))=\mathcal{CV}$. Then $|a_2a_4-a_3^2| \leq 1/8$.*

3.3 Further Results on The Second Hankel Determinant

In this section, several classes of normalized analytic functions are considered. For each class, a bound is obtained for its second Hankel determinant.

Definition 3.3. *Let $\varphi \in \mathcal{P}$ satisfy the condition in Definition 3.1, and b be a non-zero complex number. The class $R_b(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the* *subordination*

$$1+\frac{1}{b}\left(f'(z)-1\right) \prec \varphi(z).$$

This class was considered in [22] for the more general case of p-valent functions. The case $b=1$ and $\varphi(z)=(1+z)/(1-z)$ gives the subclass of close-to-convex functions studied by MacGregor [78], consisting of functions whose derivative has positive real part. Al Amiri *et al.* [18] introduced the general class of analytic functions satisfying $\operatorname{Re}\{1+1/b((zf'(z)/g(z))\ -1)\}>0$, for some starlike function g. It is evident that $R_b(\varphi)$ coincides with this class for $g(z)=z$ and $\varphi(z)=(1+z)/(1-z)$.

The following theorem gives the bound for the second Hankel determinant of the kth-root transform for functions in class $R_b(\varphi)$.

Theorem 3.3. *Let φ be given by* (3.5)*, $f(z)=z+\sum_{n=2}^{\infty}a_nz^n \in R_b(\varphi)$, and $F(z)=z+\sum_{n=1}^{\infty}b_{kn+1}z^{kn+1}$ be its kth-root transform. Further, let*

$$\lambda=\left|9B_1B_3-8B_2^2+\delta B_1^4\right|, \quad \textit{and} \quad \delta=\frac{3\left(k^2-1\right)}{2^3k^2}b^2.$$

1. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| \leq \frac{7}{2}B_1, \quad \textit{and} \quad 8B_1^2-\lambda \geq 0,$$

   then the second Hankel determinant satisfies

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| \leq \frac{B_1^2b^2}{9k^2}.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| \geq \frac{7}{2}B_1, \quad \textit{and} \quad \frac{9}{2}B_1^2+B_1|B_2|-\lambda \leq 0,$$

   or the conditions

$$|B_2| \leq \frac{7}{2}B_1, \quad \textit{and} \quad 8B_1^2-\lambda \leq 0,$$

then the second Hankel determinant satisfies

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^{2}\right| \leq \frac{b^{2}\lambda}{2^{3}3^{2}k^{2}}.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| > \frac{7}{2}B_1, \quad \textit{and} \quad \frac{9}{2}B_1^2 + B_1\,|B_2| - \lambda \geq 0,$$

then the second Hankel determinant satisfies

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^{2}\right| \leq \frac{b^{2}B_1^2}{2^{5}3^{2}k^{2}}\left(\frac{32\lambda-36B_1\,|B_2|-81B_1^2-4|B_2|^2}{\lambda-2B_1\,|B_2|-B_1^2}\right).$$

Proof. The proof is similar to Theorem 3.2. There exists an analytic self-map w of $\mathbb{D}$ satisfying

$$1+\frac{1}{b}\left(f'(z)-1\right)=\varphi\left(w(z)\right). \tag{3.18}$$

Now

$$1+\frac{1}{b}\left(f'(z)-1\right)=1+\frac{2}{b}a_2z+\frac{3}{b}a_3z^2+\frac{4}{b}a_4z^3+\cdots. \tag{3.19}$$

Comparing with (3.8), (3.18) and (3.19), we find that

$$\begin{aligned}
a_2 &= b\frac{B_1c_1}{4},\\
a_3 &= b\frac{\left(2B_1c_2-B_1c_1^2+B_2c_1^2\right)}{12},\\
a_4 &= b\frac{\left(4B_1c_3-4B_1c_1c_2+B_1c_1^3+4B_2c_1c_2-2B_2c_1^3+B_3c_1^3\right)}{32}.
\end{aligned} \tag{3.20}$$

Consequently (3.20) and (3.1) yield

$$\begin{aligned}
b_{k+1} &= b\frac{B_1c_1}{4k},\\
b_{2k+1} &= \frac{b}{12k}\left(2B_1c_2-B_1c_1^2+B_2c_1^2\right)+\frac{(1-k)b^2}{32k^2}B_1^2c_1^2,\\
b_{3k+1} &= \frac{b}{32k}\left(4B_1c_3-4B_1c_1c_2+B_1c_1^3+4B_2c_1c_2-2B_2c_1^3+B_3c_1^3\right)
\end{aligned}$$

$$+\frac{(1-k)b^2}{48k^2}\left(2B_1^2c_1c_2-B_1^2c_1^3+B_1B_2c_1^3\right)+\frac{(k-1)(2k-1)}{384}b^3B_1^3c_1^3.$$

Thus

$$\begin{aligned}b_{k+1}b_{3k+1}-b_{2k+1}^2=\frac{b^2}{2^73^2k^2}\Bigg(&\left(B_1^2-2B_1B_2+3^2B_1B_3-2^3B_2^2+\frac{3\left(k^2-1\right)}{2^3k^2}b^2B_1^4\right)c_1^4\\&+2^2\left(B_1B_2-B_1^2\right)c_1^2c_2+2^23^2B_1^2c_1c_3-2^5B_1^2c_2^2\Bigg).\end{aligned}$$

Writing

$$\begin{aligned}&d_1=36B_1^2,\quad d_2=4B_1\left(B_2-B_1\right),\quad d_3=-32B_1^2,\\&d_4=B_1^2-2B_1B_2+9B_1B_3-8B_2^2+\delta B_1^4,\quad\text{and}\quad T=\frac{|b|^2}{2^73^2k^2},\end{aligned}\tag{3.21}$$

then

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|=T\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|.$$

Equations (3.2) and (3.3) show that

$$\begin{aligned}\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|=\frac{T}{4}\big|&c^4(d_1+2d_2+d_3+4d_4)+2xc^2(4-c^2)(d_1+d_2+d_3)\\&+(4-c^2)x^2\left(-d_1c^2+d_3(4-c^2)\right)+2d_1c(4-c^2)(1-|x|^2)y\big|\end{aligned}$$

for some $x,y\in\overline{\mathbb{D}}$. With $s=|x|$, (3.21) yields

$$\begin{aligned}\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|&\le T\Big(c^4\lambda+2sc^2(4-c^2)B_1|B_2|+18B_1^2c(4-c^2)(1-s^2)\\&\qquad+(4-c^2)s^2B_1^2\big(9c^2+8(4-c^2)-18c\big)\Big)\\&=T\Big(c^4\lambda+2sc^2(4-c^2)B_1|B_2|+18B_1^2c(4-c^2)\\&\qquad+(4-c^2)s^2B_1^2(c-2)(c-16)\Big)\\&:=F(c,s),\end{aligned}$$

$(c,s)\in[0,2]\times[0,1]$. Now

$$\frac{\partial F}{\partial s}=T\left(2c^2(4-c^2)B_1|B_2|+2sB_1^2(4-c^2)(c-2)(c-16)\right),$$

it is evident that $\partial F/\partial s>0$ for $0<s<1$ and for any fixed c with $0<c<2$. Thus

$F(c,s)$ is an increasing function of s, and hence

$$\max_{0\le s\le 1} F(c,s) = F(c,1) := G(c).$$

Routine simplifications yield

$$G(c) = T\Big(c^4(\lambda - 2B_1|B_2| - B_1^2) + 4c^2B_1\big(2|B_2| - 7B_1\big) + 128B_1^2\Big).$$

With $c^2 = t$ and

$$L = \lambda - 2B_1|B_2| - B_1^2, \quad M = 4B_1(2|B_2| - 7B_1), \quad \text{and} \quad N = 128B_1^2,$$

it follows from (3.4) that

$$\frac{2^7 3^2 k^2}{b^2}\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \le \begin{cases} \frac{4LN-M^2}{4L}, & M>0, L\le -\frac{M}{8}, \\ N, & M\le 0, L\le -\frac{M}{4}, \\ 16L+4M+N, & M\ge 0, L\ge -\frac{M}{8} \text{ or } M\le 0, L\ge -\frac{M}{4}. \end{cases}$$

Inserting the values of the parameters L, M and N yield the desired conditions and bounds. □

When $k=1$ and $b=\tau$, Theorem 3.3 is equivalent to Theorem 3 in [72] for $\gamma = 0$.

Corrollary 3.6. [72, Theorem 3] *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R_\tau(\varphi)$. Further, let $\lambda = \left|9B_1B_3 - 8B_2^2\right|$. Then*

1. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| \le \frac{7}{2}B_1, \quad \textit{and} \quad 8B_1^2 - \lambda \ge 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \le \frac{B_1^2|b|^2}{9}.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| \ge \frac{7}{2}B_1, \quad \textit{and} \quad \frac{9}{2}B_1^2 + B_1|B_2| - \lambda \le 0,$$

or the conditions

$$|B_2| \le \frac{7}{2}B_1, \quad \text{and} \quad 8B_1^2 - \lambda \le 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \le \frac{|b|^2\lambda}{2^3 3^2}.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| > \frac{7}{2}B_1, \quad \text{and} \quad \frac{9}{2}B_1^2 + B_1|B_2| - \lambda \ge 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \le \frac{|b|^2B_1^2}{2^5 3^2}\left(\frac{32\lambda - 36B_1|B_2| - 81B_1^2 - 4|B_2|^2}{\lambda - 2B_1|B_2| - B_1^2}\right).$$

For $k = 1$, $b = 1$ and the choice $\varphi(z) = (1+z)/(1-z)$, that is, $B_1 = B_2 = B_3 = 2$, Theorem 3.3 is equivalent to Theorem 2.1 in [63] for $\gamma = 0$.

Corrollary 3.7. [63, Theorem 3.1] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R_1((1+z)/(1-z)) = R$. *Then* $\left|a_2a_4 - a_3^2\right| \le \frac{4}{9}$.

For $k = 1$, $b = \tau$ and the choice $\varphi(z) = (1+Az)/(1+Bz)$, $-1 \le B < A \le 1$, Theorem 3.3 is equivalent to Theorem 2.1 in [29] for $\gamma = 0$.

Corrollary 3.8. [29, Theorem 2.1] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R_\tau((1+Az)/(1+Bz))$. *Then* $\left|a_2a_4 - a_3^2\right| \le |\tau|^2(A-B)^2/9$.

Definition 3.4. *Let $\varphi \in \mathcal{P}$ satisfy the condition in Definition 3.1, and* $\alpha \ge 0$. *The class* $\mathcal{ST}(\alpha, \varphi)$ *consists of functions $f \in \mathcal{A}$ satisfying the subordination*

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2f''(z)}{f(z)} \prec \varphi(z).$$

Padmanabhan [117] introduced the class $\mathcal{ST}(\alpha,\varphi)$ in 2001 and investigated sufficient conditions for starlikeness. The special case when $\alpha = 1$ and $\varphi = (1+z)/(1-z)$ was considered in [127]. It is evident that $\mathcal{ST}(0,\varphi)$ reduces to the class $\mathcal{ST}(\varphi)$ treated in Definition 3.1.

Theorem 3.4. *Let φ be given by* (3.5), *$f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in \mathcal{ST}(\alpha,\varphi)$, and $F(z) = z+\sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}$ be its kth-root transform. Further, let*

$$\lambda = \left|\delta B_1^4 - 2\alpha B_1^2 B_2 + 4(1+2\alpha)(1+3\alpha)^3 B_1 B_3 - 3(1+2\alpha)^2(1+4\alpha)B_2^2\right|,$$

$$u = (1+2\alpha)(12\alpha^2+6\alpha+1), \quad \text{and}$$
$$\delta = \frac{-1-10\alpha+\left(5k^2-33\right)\alpha^2+12\left(k^2-3\right)\alpha^3}{((1+2\alpha)k)^2}.$$

1. *If B_1, B_2 and B_3 satisfy the conditions*

$$u\,|B_2| - (1+2\alpha)\left(1+6\alpha+6\alpha^2\right)B_1 + \alpha B_1^2 \le 0, \quad \text{and}$$

$$\lambda - 3(1+2\alpha)^2(1+4\alpha)B_1^2 \le 0,$$

then the second Hankel determinant satisfies

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \le \frac{B_1^2}{(2(1+3\alpha)k)^2}.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$u\,|B_2| - (1+2\alpha)\left(1+6\alpha+6\alpha^2\right)B_1 + \alpha B_1^2 \ge 0, \quad \text{and}$$

$$\lambda - B_1\left(u\,|B_2| + \alpha B_1^2 - 2(1+2\alpha)\left(1+6\alpha+9\alpha^2\right)B_1\right) \ge 0,$$

or the conditions

$$u\,|B_2| - (1+2\alpha)\left(1+6\alpha+6\alpha^2\right)B_1 + \alpha B_1^2 \le 0, \quad \text{and}$$

$$\lambda - 3(1+2\alpha)^2(1+4\alpha)B_1^2 \ge 0,$$

then the second Hankel determinant satisfies

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| \leq \frac{\lambda}{3(1+4\alpha)\Big(2(1+2\alpha)(1+3\alpha)k\Big)^2}.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$u\left|B_2\right|-(1+2\alpha)\left(1+6\alpha+6\alpha^2\right)B_1+\alpha B_1^2>0, \quad \textit{and}$$

$$\lambda-B_1\Big(u\left|B_2\right|+\alpha B_1^2-2(1+2\alpha)\left(1+6\alpha+9\alpha^2\right)B_1\Big)\leq 0,$$

then the second Hankel determinant satisfies

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| \leq \frac{P}{Q},$$

where

$$P=B_1^2\Big(-3(1+2\alpha)(1+6\alpha+8\alpha^2)\big(-\lambda+(u+2\alpha B_1)B_1^2+2uB_1|B_2|\big)$$
$$-\big(\alpha B_1^2-(1+2\alpha)(1+6\alpha+6\alpha^2)B_1+u|B_2|\big)^2\Big),$$

and

$$Q=12k^2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)\big(\lambda-B_1^2(u+2\alpha B_1)-2uB_1|B_2|\big).$$

Proof. Let

$$\frac{zf'(z)+\alpha^2z^2f''(z)}{f(z)}=\varphi\left(w(z)\right) \tag{3.22}$$

for an analytic self-map w of $\mathbb{D}$. Since

$$\frac{zf'(z)+\alpha^2z^2f"(z)}{f(z)}=1+(1+2\alpha)a_2z+\left(2(1+3\alpha)a_3-(1+2\alpha)a_2^2\right)z^2$$
$$+\left(3(1+4\alpha)a_4-(3+8\alpha)a_2a_3+(1+2\alpha)a_2^3\right)z^3+\cdots, \tag{3.23}$$

it follows from (3.8), (3.22) and (3.23) that

$$a_2=\frac{1}{2(1+2\alpha)}B_1c_1,$$
$$a_3=\frac{1}{2^3(1+2\alpha)(1+3\alpha)}\Big((1+2\alpha)\big(2B_1c_2-(B_1-B_2)c_1^2\big)+B_1^2c_1^2\Big),$$

$$a_4=\frac{1}{2^4\cdot 3(1+4\alpha)}\Bigg(8(B_1c_3-B_1c_1c_2+B_2c_1c_2)-4B_2c_1^3+2(B_1c_1^3+B_3c_1^3)$$
$$+\frac{(3+8\alpha)}{(1+2\alpha)^2(1+3\alpha)}\left((1+2\alpha)(2B_1^2c_1c_2-B_1^2c_1^3+B_1B_2c_1^3)+B_1^3c_1^3\right)-\frac{2B_1^3c_1^3}{(1+2\alpha)^2}\Bigg).$$

Consequently (3.1) yields

$$\begin{aligned}
b_{k+1}&=\frac{B_1c_1}{2k(1+2\alpha)},\\
b_{2k+1}&=\frac{\left(2B_1c_2+(B_2-B_1)\,c_1^2\right)}{2^3k(1+3\alpha)}+\frac{(3-k)\alpha+1}{8k^2(1+2\alpha)^2(1+3\alpha)}B_1^2c_1^2,\\
b_{3k+1}&=\frac{\left(4\,(B_1c_3-B_1c_1c_2+B_2c_1c_2)-2B_2c_1^3+B_1c_1^3+B_3c_1^3\right)}{2^33k(1+4\alpha)}\\
&\quad+\frac{\left(4\alpha(3-k)+3\right)\left(2B_1^2c_1c_2-B_1^2c_1^3+B_1B_2c_1^3\right)}{2^4\cdot 3k^2(1+2\alpha)(1+3\alpha)(1+4\alpha)}\\
&\quad+\frac{\left(12\alpha^2+7\alpha+1\right)-\alpha k\,(4\alpha(3-k)+3)}{2^4\cdot 3k^3(1+2\alpha)^3(1+3\alpha)(1+4\alpha)}B_1^3c_1^3.
\end{aligned}$$

Routine computations show that

$$\begin{aligned}
b_{k+1}b_{3k+1}-b_{2k+1}^2&=\frac{(12\alpha^3k^2+5\alpha^2k^2-36\alpha^3-33\alpha^2-10\alpha-1)}{2^6\cdot 3k^4(1+2\alpha)^4(1+3\alpha)^2(1+4\alpha)}B_1^4c_1^4\\
&\quad+\frac{\alpha k\left(B_1^2(B_1-B_2)c_1^4-2B_1^3c_1^2c_2\right)}{2^5\cdot 3k^3(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}\\
&\quad+\frac{(12\alpha^2+6\alpha+1)\left(B_1(B_1-2B_2)c_1^4+4B_1(B_2-B_1)c_1^2c_2\right)}{2^6\cdot 3k^2(1+2\alpha)(1+3\alpha)^2(1+4\alpha)}\\
&\quad+\frac{(4B_1^2c_1c_3+B_1B_3c_1^4)}{2^4\cdot 3k^2(1+2\alpha)(1+4\alpha)}-\frac{(4B_1^2c_2^2+B_2^2c_1^4)}{2^6k^2(1+3\alpha)^2},
\end{aligned}$$

which yields

$$\begin{aligned}
\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|=T\Big|&c_1^4\Big(B_1\left(u(B_1-2B_2)+4(1+2\alpha)(1+3\alpha)^2B_3\right.\\
&\left.+2\alpha B_1(B_1-B_2)+\delta B_1^3\right)-3(1+2\alpha)^2(1+4\alpha)B_2^2\Big)\\
&+c_1^2c_2\Big(4B_1\left(u(B_2-B_1)-\alpha B_1^2\right)\Big)\\
&+16(1+2\alpha)(1+3\alpha)^2B_1^2c_1c_3-12(1+2\alpha)^2(1+4\alpha)B_1^2c_2^2\Big|,
\end{aligned}$$

where

$$T=\frac{1}{2^6 3k^2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}.$$

By writing

$$\begin{aligned}
d_1 &= 2^4(1+2\alpha)(1+3\alpha)^2B_1^2,\\
d_2 &= 4B_1\big(u(B_2-B_1)-\alpha B_1^2\big),\\
d_3 &= -12(1+2\alpha)^2(1+4\alpha)B_1^2,\\
d_4 &= B_1\big(u(B_1-2B_2)+4(1+2\alpha)(1+3\alpha)^2B_3\\
&\qquad +2\alpha B_1(B_1-B_2)+\delta B_1^3\big)-3(1+2\alpha)^2(1+4\alpha)B_2^2, \qquad (3.24)
\end{aligned}$$

then

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| = T\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|.$$

Proceeding similarly as in the previous proofs, it can be shown that

$$\begin{aligned}
\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| = \frac{T}{4}\Big|&c^4(d_1+2d_2+d_3+4d_4)+2xc^2(4-c^2)(d_1+d_2+d_3)\\
&+(4-c^2)x^2\big(-d_1c^2+d_3(4-c^2)\big)+2d_1c(4-c^2)(1-|x|^2)y\Big|
\end{aligned}$$

for some $x,y\in\overline{\mathbb{D}}$. With $s=|x|$, (3.24) yields

$$\begin{aligned}
\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| &\le T\Big(c^4\lambda+2B_1sc^2(4-c^2)(u|B_2|+\alpha B_1^2)\\
&\qquad +s^2(1+2\alpha)(4-c^2)\big(4(1+3\alpha)^2B_1^2c^2\\
&\qquad +3(1+2\alpha)(1+4\alpha)B_1^2(4-c^2)\big)\\
&\qquad +8(1+2\alpha)(1+3\alpha)^2B_1^2c(4-c^2)(1-s^2)\Big)\\
&= T\Big(c^4\lambda+2sB_1c^2\,(4-c^2)(u|B_2|+\alpha B_1^2)\\
&\qquad +8(1+2\alpha)(1+3\alpha)^2B_1^2c(4-c^2)\\
&\qquad +s^2u(4-c^2)B_1^2(c-2)(c-p)\Big)\\
&:= F(c,s),
\end{aligned}$$

where

$$p = \frac{6(1+6\alpha+8\alpha^2)}{1+6\alpha+12\alpha^2} > 2.$$

It is evident that $\partial F/\partial s > 0$ for $0 < s < 1$ and for any fixed c with $0 < c < 2$. Thus $F(c,s)$ is an increasing function of s. Hence

$$\max_{0\le s\le 1} F(c,s) = F(c,1) := G(c),$$

with

$$\begin{aligned} G(c) = T\Big(& c^4\Big(\lambda - B_1\big(u(B_1+2|B_2|)+2\alpha B_1^2\big)\Big) \\ & + 8B_1c^2\big(u|B_2| - (1+2\alpha)(1+6\alpha+6\alpha^2)B_1 + \alpha B_1^2\big) \\ & + 48(1+2\alpha)(1+6\alpha+8\alpha^2)B_1^2\Big). \end{aligned}$$

Writing $c^2 = t$ and

$$\begin{aligned} L &= \lambda - B_1\left(u(B_1+2|B_2|)+2\alpha B_1^2\right), \\ M &= 8B_1\left(u|B_2| - (1+2\alpha)(1+6\alpha+6\alpha^2)B_1 + \alpha B_1^2\right), \quad \text{and} \\ N &= 48(1+2\alpha)(1+6\alpha+8\alpha^2)B_1^2. \end{aligned} \tag{3.25}$$

Now, (3.4) yields

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \le \frac{1}{2^6 3k^2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \times \begin{cases} \frac{4LN-M^2}{4L}, & M > 0, L \le -\frac{M}{8}, \\ N, & M \le 0, L \le -\frac{M}{4}, \\ 16L+4M+N, & M \ge 0, L \ge -\frac{M}{8} \text{ or } M \le 0, L \ge -\frac{M}{4}. \end{cases}$$

where L, M and N are given by (3.25). □

Definition 3.5. *Let $\varphi \in \mathcal{P}$ satisfy the condition in Definition 3.1, and $\alpha \in [0,1]$. The class $L(\alpha,\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the subordination*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z).$$

This class is analogous to the α-logarithmically convex functions L_α introduced by Lewandowski *et al.* [73]. In [35], Darus *et al.* found bounds for $|a_2|$, $|a_3|$ and $|a_3-\mu a_2|$, μ real, for $f \in L(\alpha,(1+z)/(1-z)) = L_\alpha$. Evidently, $L(1,\varphi)$ reduces to the class $\mathcal{ST}(\varphi)$ considered in Definition 3.1 and $L(0,\varphi)$ reduces to the class $\mathcal{CV}(\varphi)$ considered in Definition 3.2.

Theorem 3.5. *Let φ be given by* (3.5)*, $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in L(\alpha,\varphi)$, and $F(z) = z+\sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}$ be its kth-root transform. Further, let*

$$\lambda = \left|\delta B_1^4 + vB_1^2B_2 + 4(2-\alpha)(3-2\alpha)^2B_1B_3 - 3(2-\alpha)^2(4-3\alpha)B_2^2\right|,$$

$$u = (2-\alpha)(7\alpha^2-18\alpha+12),$$

$$v = (7\alpha^3-13\alpha^2-6\alpha+12), \quad \textit{and}$$

$$\delta = \Big(4(2\alpha-3)\big(2k^2\alpha(15-17\alpha+6\alpha^2-4\alpha^3)-9k(8-10\alpha-\alpha^2+3\alpha^3)$$
$$-6(12-17\alpha+6\alpha^2)\big)+9(3\alpha-4)\big(-6+4\alpha+k(-2+\alpha+\alpha^2)\big)^2\Big)\Big/3\big(2k(2-\alpha)\big)^2.$$

1. If B_1, B_2 and B_3 satisfy the conditions

$$vB_1^2+2u|B_2|-2(2-\alpha)(6-6\alpha+\alpha^2)B_1 \le 0, \quad \textit{and}$$

$$\lambda + 3(2-\alpha)^2(3\alpha-4)B_1^2 \le 0,$$

then the second Hankel determinant satisfies

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| \leq \left(\frac{B_1}{2(3-2\alpha)k}\right)^2.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$vB_1^2+2u|B_2|-2(2-\alpha)(6-6\alpha+\alpha^2)B_1 \geq 0, \quad \textit{and}$$

$$2\lambda - B_1\left(vB_1^2+2u|B_2|+4(2-\alpha)(3-2\alpha)^2B_1\right) \geq 0,$$

or the conditions

$$vB_1^2+2u|B_2|-2(2-\alpha)(6-6\alpha+\alpha^2)B_1 \leq 0, \quad \textit{and}$$

$$\lambda+3(2-\alpha)^2(3\alpha-4)B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| \leq \frac{\lambda}{3(4-3\alpha)\left(2(2-\alpha)(3-2\alpha)k\right)^2}.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$vB_1^2+2u|B_2|-2(2-\alpha)(6-6\alpha+\alpha^2)B_1 > 0, \quad \textit{and}$$

$$2\lambda - B_1\left(vB_1^2+2u|B_2|+4(2-\alpha)(3-2\alpha)^2B_1\right) \leq 0,$$

then the second Hankel determinant satisfies

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| \leq \frac{P}{Q},$$

where

$$P = B_1^2\Big(-\left(vB_1^2+2(\alpha-2)(6-6\alpha+\alpha^2)B_1+2u|B_2|\right)^2+12(\alpha-2)^2(3\alpha-4)$$
$$\times\left(uB_1^2+vB_1^3-\lambda+2uB_1|B_2|\right)\Big),$$

and

$$Q = 3(4-3\alpha)\left(2^2(2-\alpha)(3-2\alpha)k\right)^2\left(\lambda-uB_1^2-vB_1^3-2uB_1|B_2|\right).$$

Proof. Since $f \in L(\alpha, \varphi)$, there exists an analytic self-map w of $\mathbb{D}$ with $w(0) = 0$ satisfying

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{(1-\alpha)} = \varphi\left(w(z)\right).$$

Now

$$\begin{aligned}\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{(1-\alpha)} =& 1+(1-2\alpha)a_2z+\left(2(3-2\alpha)a_3+\frac{(2-\alpha)^2-3(4-3\alpha)}{2}a_2^2\right)z^2\\ &+\Big(3(4-3\alpha)a_3-\frac{(\alpha^3+21\alpha^2+20\alpha-48)}{6}a_2^3\\ &+(4\alpha^2+11\alpha-18)a_2a_3\Big)z^3+\cdots,\end{aligned}$$

the above equation and (3.8) yield

$$\begin{aligned} a_2 &= \frac{B_1c_1}{2(2-\alpha)},\\ a_3 &= \frac{1}{2(3-2\alpha)}\left(\frac{-(\alpha^2+5\alpha-8)}{2\cdot 4(2-\alpha)^2}B_1^2c_1^2+\frac{B_1c_2}{2}-\frac{B_1c_1^2}{4}+\frac{B_2c_1^2}{4}\right),\\ a_4 &= \frac{1}{2^3\cdot 3(4-3\alpha)}\left(4B_2c_1c_2-2B_2c_1^3+B_3c_1^3+B_3c_1^3+4B_1c_3-4B_1c_1c_2+B_1c_1^3\right)\\ &\quad+\frac{(\alpha^3+21\alpha^2+20\alpha-48)}{2^4\cdot 3^2(2-\alpha)^3(4-3\alpha)}B_1^3c_1^3+\frac{\left(4\alpha^2+11\alpha-18\right)}{2^5\cdot 3(2-\alpha)^3(3-2\alpha)(4-3\alpha)}\\ &\quad\big((\alpha^2+5\alpha-8)B_1^3c_1^3-4(2-\alpha)^2B_1^2c_1c_2+2(2-\alpha)^2B_1^2c_1^3\\ &\quad-2(2-\alpha)^2B_1B_2c_1^3\big).\end{aligned} \tag{3.26}$$

From (3.26) and (3.1), and after some lengthy computations (validated by Mathematica), we find that

$$\begin{aligned}\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| = T\Big(&c^4\big(\delta B_1^4c_1^4+u(B_1^2c_1^4-4B_1^2c_1^2c_2-2B_1B_2c_1^4+4B_1B_3c_1^2c_2)\\ &+v(B_1^2B_2c_1^4-B_1^3c_1^4+2B_1^3c_1^2c_2)+4(2-\alpha)(3-2\alpha)^2B_1B_3c_1^4\\ &-12(2-\alpha)^2(4-3\alpha)B_1^2c_2^2+16(2-\alpha)(3-2\alpha)^2B_1^2c_1c_3\\ &-3(2-\alpha)^2(4-3\alpha)B_2^2c_1^4\big)\end{aligned}$$

$$
\begin{aligned}
&= T\Big|c_1^4\big(\delta B_1^4 + uB_1(B_1-2B_2) + vB_1^2(B_2-B_1) \\
&\quad + 4(2-\alpha)(3-2\alpha)^2B_1B_3 - 3(2-\alpha)^2(4-3\alpha)B_2^2\big) \\
&\quad + 2B_1c_1^2c_2\left(2u(B_2-B_1)+vB_1^2\right) + 16c_1c_3\left((2-\alpha)(3-2\alpha)^2B_1^2\right) \\
&\quad - 12c_2^2\left((2-\alpha)^2(4-3\alpha)B_1^2\right)\Big|,
\end{aligned}
$$

where

$$
T = \frac{1}{3(4-3\alpha)\left(2^3(2-\alpha)(3-2\alpha)k\right)^2}.
$$

Let

$$
\begin{aligned}
d_1 &= 16(2-\alpha)(3-2\alpha)^2B_1^2, \\
d_2 &= 2\left(2u(B_2-B_1)+vB_1^2\right)B_1, \\
d_3 &= -12(2-\alpha)^2(4-3\alpha)B_1^2, \\
d_4 &= \delta B_1^4 + uB_1(B_1-2B_2) + vB_1^2(B_2-B_1) + 4(2-\alpha)(3-2\alpha)^2B_1B_3 \\
&\quad - 3(2-\alpha)^2(4-3\alpha)B_2^2.
\end{aligned}
\tag{3.27}
$$

It follows that

$$
\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| = T(d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4).
$$

Consequently,

$$
\begin{aligned}
\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| &= \frac{T}{4}\Big|c^4(d_1+2d_2+d_3+4d_4) + 2xc^2(4-c^2)(d_1+d_2+d_3) \\
&\quad + (4-c^2)x^2\big(-d_1c^2 + d_3(4-c^2)\big) + 2d_1c(4-c^2)(1-|x|^2)y\Big|
\end{aligned}
$$

for some $x, y \in \overline{\mathbb{D}}$. With $s = |x|$, (3.27) yields

$$
\begin{aligned}
\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| &\leq T\Big(c^4\lambda + sc^2(4-c^2)B_1(2u|B_2| + vB_1^2) \\
&\quad + s^2(4-c^2)(2-\alpha)B_1^2\big((7\alpha^2-18\alpha+12)c^2 \\
&\quad + 12(2-\alpha)(4-3\alpha)\big) + 8c(4-c^2)(2-\alpha)(3-2\alpha)^2B_1^2(1-s^2)\Big)
\end{aligned}
$$

$$= T\Big(c^4\lambda + sc^2(4-c^2)B_1(2u|B_2| + vB_1^2) + 8c(4-c^2)(2-\alpha)$$

$$(3-2\alpha)^2B_1^2 + s^2u(4-c^2)B_1^2(c-2)(c-p)\Big)$$

$$:= F(c,s),$$

where $u > 0$, $v > 0$, and

$$p = \frac{6(8-10\alpha+3\alpha^2)}{12-18\alpha+7\alpha^2} > 2.$$

Since $\partial F/\partial s > 0$ for $0 < s < 1$ and for any fixed c with $0 < c < 2$, it follows that $F(c,s)$ is increasing relative to s, and so

$$\max_{0\le s\le 1} F(c,s) = F(c,1) := G(c),$$

with

$$\begin{aligned} G(c) = T\Bigg(& c^4\Big(\lambda - B_1\big(vB_1^2 + u(2|B_2| + B_1)\big)\Big) \\ &+ c^2\Big(4B_1\big(2u|B_2| + vB_1^2 - 2(2-\alpha)(\alpha^2-6\alpha+6)B_1\big)\Big) \\ &+ 48(2-\alpha)^2(4-3\alpha)B_1^2\Bigg). \end{aligned}$$

Letting $c^2 = t$ and

$$\begin{aligned} L &= \lambda - B_1\left(vB_1^2 + u(2|B_2| + B_1)\right), \\ M &= 4B_1\Big(2\big(u|B_2| - (2-\alpha)(\alpha^2-6\alpha+6)B_1\big) + vB_1^2\Big), \quad \text{and} \\ N &= 48(2-\alpha)^2(4-3\alpha)B_1^2. \end{aligned} \tag{3.28}$$

Now, (3.4) leads to

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right| \le \frac{1}{3(4-3\alpha)\left(2^3(2-\alpha)(3-2\alpha)k\right)^2} \times \begin{cases} \frac{4LN-M^2}{4L}, & M>0, L\le -\frac{M}{8}, \\ N, & M\le 0, L\le -\frac{M}{4}, \\ 16L+4M+N, & M\ge 0, L\ge -\frac{M}{8} \text{ or } M\le 0, L\ge -\frac{M}{4}. \end{cases}$$

where L, M and N are given by (3.28). □

Remark 3.1.

1. For $\alpha = 1$, Theorem 3.5 reduces to Theorem 3.1, and for $\alpha = 0$, Theorem 3.5 reduces to Theorem 3.2.
2. If $k = 1$ and $\alpha = 1$, then $\delta = -1$, $u = 1$, $v = 0$ and $\lambda = |4B_1B_3 - B_1^4 - 3B_2^2|$. Thus Theorem 3.5 reduces to [72, Theorem 1].
3. If $k = 1$ and $\alpha = 0$, then $\delta = -12$, $u = 24$, $v = 12$ and $\lambda = |72B_1B_3 + 12B_1^2B_2 - 12B_1^4 - 48B_2^2|$. Thus Theorem 3.5 reduces to [72, Theorem 2].

With $k = 1$, $\alpha = 1$, and the choice of $\varphi(z) = (1+z)/(1-z)$, that is, $B_1 = B_2 = B_3 = 2$, Theorem 3.5 reduces to the following corollary.

Corrollary 3.9. [64, Theorem 3.1] *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in L(1,(1+z)/(1-z)) = \mathcal{ST}$. Then*

$$|a_2a_4 - a_3^2| \le 1.$$

With $k = 1$, $\alpha = 0$, and the choice of $\varphi(z) = (1+z)/(1-z)$, that is, $B_1 = B_2 = B_3 = 2$, Theorem 3.5 reduces to the following corollary.

Corrollary 3.10. [64, Theorem 3.2] *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in L(0,(1+z)/(1-z)) = \mathcal{CV}$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}.$$

The last theorem gives the bound for the second Hankel determinant of the kth-root transform for the class $M(\alpha, \varphi)$.

Definition 3.6. *Let $\varphi \in \mathcal{P}$ satisfy the condition in Definition 3.1, and $\alpha \geq 0$. The class $M(\alpha, \varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the subordination*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z).$$

The class $M(\alpha, \varphi)$ is analogous to the α-convex functions of Mocanu *et al.*[86], who investigated geometric properties of the class in the case $\varphi = (1+z)/(1-z)$.

Theorem 3.6. *Let φ be given by* (3.5)*, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M(\alpha, \varphi)$, and $F(z) = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}$ be its kth-root transform. Further, let*

$$\lambda = \left|\delta B_1^4 + 6\alpha B_1^2 B_2 + 4(1+2\alpha)^2 B_1 B_3 - 3(1+\alpha)(1+3\alpha)B_2^2\right|,$$
$$\delta = -\frac{\left(1 + \left(7+4k^2\right)\alpha + \left(16+7k^2\right)\alpha^2 + \left(12+k^2\right)\alpha^3\right)}{(1+\alpha)^3 k^2},$$

and $u = (7\alpha^2 + 4\alpha + 1)$.

1. *If B_1, B_2 and B_3 satisfy the conditions*

$$u|B_2| + 3\alpha B_1^2 - (\alpha^2 + 4\alpha + 1)B_1 \leq 0, \quad \text{and} \quad \lambda - 3(1+\alpha)(1+3\alpha)B_1^2 \leq 0,$$

   then the second Hankel determinant satisfies

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leq \left(\frac{B_1}{2(1+2\alpha)k}\right)^2.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$u|B_2| + 3\alpha B_1^2 - (\alpha^2 + 4\alpha + 1)B_1 \geq 0, \quad \text{and}$$

$$\lambda - B_1\left(2(1+2\alpha)^2 B_1 + 3\alpha B_1^2 + u|B_2|\right) \geq 0,$$

or the conditions

$$u|B_2| + 3\alpha B_1^2 - (\alpha^2+4\alpha+1)B_1 \leq 0, \quad \textit{and} \quad \lambda - 3(1+\alpha)(1+3\alpha)B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leq \frac{\lambda}{3(1+\alpha)(1+3\alpha)\left(2(1+2\alpha)k\right)^2}.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$u|B_2| + 3\alpha B_1^2 - (\alpha^2+4\alpha+1)B_1 > 0, \quad \textit{and}$$

$$\lambda - B_1\left(2(1+2\alpha)^2 B_1 + 3\alpha B_1^2 + u|B_2|\right) \leq 0,$$

then the second Hankel determinant satisfies

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leq \frac{P}{Q},$$

where

$$\begin{aligned} P = B_1^2\Big(&-\left(6\alpha B_1^2 - (1+4\alpha+\alpha^2)B_1 + u|B_2|\right)^2 \\ &- 3(1+\alpha)(1+3\alpha)\left(-\lambda + (u+12\alpha B_1)B_1^2 + 2uB_1|B_2|\right)\Big), \quad \textit{and} \\ Q = 3&(1+\alpha)(1+3\alpha)\left(4(1+2\alpha)k\right)^2\left(\lambda - (u+12\alpha B_1)B_1^2 - 2uB_1|B_2|\right). \end{aligned}$$

Proof. Let

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) = \varphi\left(w(z)\right) \tag{3.29}$$

for an analytic self-map w of $\mathbb{D}$. Since

$$\begin{aligned} (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) &= 1 + (1+\alpha)a_2 z + \left((2+4\alpha)a_3 - (1+3\alpha)a_2^2\right)z^2 \\ &\quad + \left((3+9\alpha)a_4 - (3+15\alpha)a_2a_3 + (1+7\alpha)a_2^3\right)z^3 \\ &\quad + \cdots, \end{aligned} \tag{3.30}$$

it follows from (3.8), (3.29) and (3.30) that

$$\begin{aligned}
a_2 &= \frac{1}{2(1+\alpha)} B_1 c_1, \\
a_3 &= \frac{1}{(1+2\alpha)} \Big(\frac{(1+3\alpha) B_1^2 c_1^2}{4(1+\alpha)^2} + \frac{B_1 c_2}{2} - \frac{B_1 c_1^2}{4} + \frac{B_2 c_1^4}{4} \Big), \\
a_4 &= \frac{1}{3(1+3\alpha)} \left(\frac{(3+15\alpha)}{4(1+\alpha)(1+2\alpha)} \left(\frac{(1+3\alpha) B_1^3 c_1^3}{4(1+\alpha)^2} + \frac{B_1^2 c_1 c_2}{2} - \frac{B_1^2 c_1^3}{4} + \frac{B_1 B_2 c_1^3}{4} \right) \right. \\
&\quad \left. - \frac{(1+7\alpha) B_1^3 c_1^3}{8(1+\alpha)^3} + \frac{B_1 c_3}{2} - \frac{B_1 c_1 c_2}{2} + \frac{B_1 c_1^3}{8} + \frac{B_2 c_1 c_2}{2} - \frac{B_2 c_1^3}{4} + \frac{B_3 c_1^3}{8} \right).
\end{aligned} \tag{3.31}$$

From (3.31) and (3.1), direct lengthy computations (validated by Mathematica) reveal that

$$\begin{aligned}
\left| b_{k+1} b_{3k+1} - b_{2k+1}^2 \right| = T \Big| & \delta B_1^4 c_1^4 - 6\alpha B_1^3 + 12\alpha B_1^3 c_1^2 c_2 + 6\alpha B_1^2 B_2 c^4 - 4u B_1^2 c_1^2 c_2 \\
& + u B_1^2 c_1^4 + 4u B_1 B_2 c_1^2 c_2 - 2u B_1 B_2 c_1^4 + 16(1+2\alpha)^2 B_1^2 c_1 c_3 \\
& + 4(1+2\alpha)^2 B_1 B_3 c_1^4 - 12(1+\alpha)(1+3\alpha) B_1^2 c_2^2 \\
& - 3(1+\alpha)(1+3\alpha) B_2^2 c_1^4 \Big| \\
= T \Big| & c_1^4 \big(\delta B_1^4 + 6\alpha B_1^2 (B_2 - B_1) + u B_1 (B_1 - 2B_2) \\
& + 4(1+2\alpha)^2 B_1 B_3 - 3(1+\alpha)(1+3\alpha) B_2^2 \big) \\
& + 4 c_1^2 c_2 B_1 \big(3\alpha B_1^2 + u(B_2 - B_1) \big) + 16(1+2\alpha)^2 B_1^2 c_1 c_3 \\
& - 12(1+\alpha)(1+3\alpha) B_1^2 c_2^2 \Big|,
\end{aligned}$$

where

$$T = \frac{1}{3(1+\alpha)(1+3\alpha)\left(2^3(1+2\alpha)k\right)^2}.$$

By writing

$$\begin{aligned}
d_1 &= 16(1+2\alpha)^2 B_1^2, \\
d_2 &= 4B_1\left(3\alpha B_1^2 + u(B_2 - B_1)\right), \\
d_3 &= -12(1+\alpha)(1+3\alpha)B_1^2, \\
d_4 &= \delta B_1^4 + 6\alpha B_1^2(B_2 - B_1) + uB_1(B_1 - 2B_2) \\
&\quad + 4(1+2\alpha)^2 B_1 B_3 - 3(1+\alpha)(1+3\alpha)B_2^2,
\end{aligned} \tag{3.32}$$

then

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| = T\left|d_1 c_1 c_3 + d_2 c_1^2 c_2 + d_3 c_2^2 + d_4 c_1^4\right|.$$

Consequently,

$$\begin{aligned}
\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| = \frac{T}{4}\Big|&c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4-c^2)(d_1 + d_2 + d_3) \\
&+ (4-c^2)x^2\big(-d_1c^2 + d_3(4-c^2)\big) + 2d_1c(4-c^2)(1-|x|^2)y\Big|
\end{aligned}$$

for some $x, y \in \overline{\mathbb{D}}$. With $s = |x|$, (3.32) yields

$$\begin{aligned}
\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| &\le T\Big(c^4\lambda + 2sc^2(4-c^2)B_1\left(3\alpha B_1^2 + u|B_2|\right) \\
&\quad + s^2(4-c^2)B_1^2\big(uc^2 + 12(1+\alpha)(1+3\alpha)\big) \\
&\quad + 8c(4-c^2)(1+2\alpha)^2 B_1^2(1-s^2)\Big) \\
&= T\Big(c^4\lambda + 2sc^2(4-c^2)B_1\left(3\alpha B_1^2 + u|B_2|\right) \\
&\quad + 8c(4-c^2)(1+2\alpha)^2 B_1^2 + s^2u(4-c^2)B_1^2(c-2)(c-p)\Big) \\
&:= F(c,s),
\end{aligned}$$

where

$$p = \frac{6(1+\alpha)(1+3\alpha)}{u} > 2.$$

It is clear that $\partial F/\partial s > 0$ for $0 < s < 1$ and for any fixed c with $0 < c < 2$. Thus $F(c,s)$ is an increasing function of s. Hence

$$\max_{0\le s\le 1} F(c,s) = F(c,1) := G(c),$$

with

$$\begin{aligned}G(c) = T\Bigg(& c^4\left(\lambda - B_1\left(6\alpha B_1^2 + u(B_1+2|B_2|)\right)\right)\\ & + c^2\Big(4B_1\left(6\alpha B_1^2 + 2u|B_2| - 2(1+4\alpha+\alpha^2)B_1\right)\Big)\\ & + 48(1+\alpha)(1+3\alpha)B_1^2\Bigg).\end{aligned}$$

Next let $c^2 = t$ and

$$\begin{aligned} & L = \lambda - B_1\left(6\alpha B_1^2 + u(B_1+2|B_2|)\right),\\ & M = 8B_1\left(3\alpha B_1^2 + u|B_2| - (1+4\alpha+\alpha^2)B_1\right), \quad \text{and} \\ & N = 48(1+\alpha)(1+3\alpha)B_1^2.\end{aligned} \tag{3.33}$$

From (3.4), it follows that

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \le \frac{1}{3(1+\alpha)(1+3\alpha)\left(2^3(1+2\alpha)k\right)^2} \times \begin{cases} \frac{4LN-M^2}{4L}, & M>0, L\le -\frac{M}{8},\\ N, & M\le 0, L\le -\frac{M}{4},\\ 16L+4M+N, & M\ge 0, L\ge -\frac{M}{8} \text{ or } M\le 0, L\ge -\frac{M}{4}.\end{cases}$$

where L, M and N are given by (3.33). The rest of the proof is now evident. □

Remark 3.2.

1. With $\alpha = 0$, Theorem 3.6 reduces to Theorem 3.1, and with $\alpha = 1$, Theorem 3.6 reduces to Theorem 3.2.
2. If $k=1$ and $\alpha=0$, then $\delta=-1$, $u=1$, and $\lambda = |4B_1B_3 - B_1^4 - 3B_2^2|$. Thus Theorem 3.6 reduces to [72, Theorem 1].
3. If $k=1$ and $\alpha=1$, then $\delta=-6$, $u=12$, and $\lambda = |36B_1B_3 + 6B_1^2B_2 - 6B_1^4 - 24B_2^2|$. Thus Theorem 3.6 reduces to [72, Theorem 2].

With $k = 1$, $\alpha = 0$, and the choice of $\varphi(z) = (1+z)/(1-z)$, that is, $B_1 = B_2 = B_3 = 2$, Theorem 3.6 reduces to the following corollary.

Corrollary 3.11. [64, Theorem 3.1] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M(0, (1+z)/(1-z)) = \mathcal{ST}$. *Then*

$$|a_2 a_4 - a_3^2| \leq 1.$$

With $k = 1$, $\alpha = 1$, and the choice of $\varphi(z) = (1+z)/(1-z)$, that is, $B_1 = B_2 = B_3 = 2$, Theorem 3.6 reduces to the following corollary.

Corrollary 3.12. [64, Theorem 3.2] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M(1, (1+z)/(1-z)) = \mathcal{CV}$. *Then*

$$|a_2 a_4 - a_3^2| \leq \frac{1}{8}.$$

CHAPTER 4

PRODUCT OF UNIVALENT LOGHARMONIC MAPPINGS

4.1 Introduction

Recall that $B(\mathbb{D})$ is the set of functions $a \in \mathcal{H}(\mathbb{D})$ satisfying $|a(z)| < 1$ for $z \in \mathbb{D}$, and B_0 is its subclass consisting of $a \in B$ with $a(0) = 0$.

Suppose $0 \notin f(\mathbb{D})$, and $\log(f(z))$ is harmonic and sense-preserving in $\mathbb{D}$. Then f is a solution of the equation

$$\overline{\frac{\partial}{\partial \bar{z}} \log(f(z))} = a(z) \frac{\partial}{\partial z} \log(f(z))$$

for some $a \in B(\mathbb{D})$, that is,

$$\overline{\left(\frac{f_{\bar{z}}(z)}{f(z)}\right)} = a(z) \frac{f_z(z)}{f(z)}. \tag{4.1}$$

Thus a nonconstant function f in $\mathbb{D}$ which is a solution of the nonlinear elliptic partial differential equation (4.1) is called *loghamonic*. The function a is called *the second dilatation* of $\log f$.

In this work, emphasis is given on the class $\mathcal{S}_{Lh}$ of univalent and sense-preserving logharmonic mappings in $\mathbb{D}$ with respect to $a \in B_0$. These mappings are of the form

$$f(z) = zh(z)\overline{g(z)}, \tag{4.2}$$

normalized by $h(0) = g(0) = 1$, and $0 \notin (hg)(\mathbb{D})$. This class has been studied extensively in recent years, for instance, in the works of [1, 2, 3, 5, 6, 7, 9, 11, 13, 14, 15] and [82].

In addition, recall that $\mathcal{ST}_{Lh}$ is the subclass of univalent starlike logharmonic mappings. The classical family $\mathcal{ST}$ of univalent analytic starlike functions is evidently a subclass of $\mathcal{ST}_{Lh}$ [1, 4, 6], and [9]. The representation in (4.2) is essential to the present work as it allows the treatment of logharmonic mappings f through their associated analytic representations h and g. In 2006, Abdulhadi and AbuMuhanna [4] established a connection between starlike logharmonic mappings of order α and starlike analytic functions of order α. Studies on starlike logharmonic mapping is an active subject of investigation, several recent works can be found in [27, 28] and [120].

Taking the logarithmic differentiation on (4.2) gives

$$\frac{f_z(z)}{f(z)} = \frac{1}{z} + \frac{h'(z)}{h(z)}, \qquad \overline{\left(\frac{f_{\bar{z}}(z)}{f(z)}\right)} = \overline{\left(\frac{(\overline{g(z)})_{\bar{z}}}{\overline{g(z)}}\right)} = \frac{g'(z)}{g(z)}. \tag{4.3}$$

It follows from (4.1) and (4.3) that the functions h, g and the dilatation a satisfy

$$\frac{g'(z)}{g(z)} = a(z)\frac{(zh(z))'}{zh(z)}. \tag{4.4}$$

Thus

$$\frac{zg'(z)}{g(z)} = a(z)\left(1 + \frac{zh'(z)}{h(z)}\right). \tag{4.5}$$

Given an analytic function ψ with a specified geometric property and an $a \in B_0$, a common method to construct a logharmonic mapping $f(z) = zh(z)\overline{g(z)}$ in $\mathcal{S}_{Lh}$ is to solve for h and g via the equation $\psi(z) = zh(z)/g(z)$ and (4.5).

Since

$$\frac{\psi'(z)}{\psi(z)} = \frac{1}{z} + \frac{h'(z)}{h(z)} - \frac{g'(z)}{g(z)},$$

it follows from (4.5) that

$$\frac{\psi'(z)}{\psi(z)}=\frac{1}{a(z)}\frac{g'(z)}{g(z)}-\frac{g'(z)}{g(z)},$$

which in turn implies

$$\frac{g'(z)}{g(z)}=\frac{a(z)}{1-a(z)}\frac{\psi'(z)}{\psi(z)}.$$

Thus the solution is $f(z)=zh(z)\overline{g(z)}$ with

$$g(z)=\exp\int_0^z\frac{a(s)}{1-a(s)}\frac{\psi'(s)}{\psi(s)}ds,\quad \text{and}\quad h(z)=\frac{\psi(z)g(z)}{z}.$$

In this chapter, sufficient conditions for the function $F(z)=f(z)|f(z)|^{2\gamma}$ to be $\alpha-$spirallike logharmonic function are obtained. By taking the product combination of two functions $f_1(z)=zh_1(z)\overline{g_1(z)}$, and $f_2(z)=zh_2(z)\overline{g_2(z)}$ which are univalent starlike logharmonic with respect to the same $a\in B_0$, we construct a new univalent starlike logharmonic mapping $F(z)=f_1^{\lambda}(z)f_2^{1-\lambda}(z),\ 0\leq\lambda\leq 1$, with respect to the same a. In addition, if $f_1(z)=zh_1(z)\overline{g_1(z)}$ is a logharmonic mapping with respect to $a_1\in B_0$, and $f_2(z)=zh_2(z)\overline{g_2(z)}$ is a logharmonic mapping with respect to $a_2\in B_0$, sufficient conditions are obtained for the product combination $F(z)=f_1^{\lambda}(z)f_2^{1-\lambda}(z),\ 0\leq\lambda\leq 1$ to be univalent starlike logharmonic with respect to some $\mu\in B_0$. We conclude the work by providing several examples of univalent starlike logharmonic mappings constructed from this product.

4.2 Product of Logharmonic Mappings

Let Ω be a simply connected domain in $\mathbb{C}$ containing the origin. Then Ω is said to be $\alpha-$spirallike, $|\alpha|<\pi/2$, if $w\exp(-te^{i\alpha})\in\Omega$ for all $t\geq 0$ whenever $w\in\Omega$. Evidently, if $\alpha=0$, then Ω is starlike.

Lemma 4.1. (Theorem 1.20) [38, p. 52] *Let $f \in \mathcal{A}$, and $|\alpha| < \pi/2$. Then f is $\alpha-$spirallike in $\mathbb{D}$ if and only if*

$$\operatorname{Re}\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > 0, \qquad z \in \mathbb{D}.$$

Recall that $\mathcal{SP}^{\alpha}_{Lh}$ is the subclass of $\mathcal{S}_{Lh}$ consisting of all $\alpha-$spirallike logharmonic mappings. Further, recall that $\mathcal{SP}^{\alpha}$ is the subclass of $\mathcal{SP}^{\alpha}_{Lh}$ such that $f \in \mathcal{H}(\mathbb{D})$.

To prove the results in this section, the following lemmas are required.

Lemma 4.2. (Theorem 1.19) *Let $f(z) = zh(z)\overline{g(z)}$ be logharmonic in $\mathbb{D}$ with $0 \notin hg(\mathbb{D})$. Then $f \in \mathcal{ST}_{Lh}(\alpha)$ if and only if $\psi(z) = zh(z)/g(z) \in \mathcal{ST}(\alpha)$.*

Lemma 4.3. (Theorem 1.21) *Let $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ be logharmonic in $\mathbb{D}$ with $0 \notin hg(\mathbb{D})$, where $\beta = \overline{a(0)}(1+a(0))/(1-|a(0)|^2)$. Then $f \in \mathcal{SP}^{\alpha}_{Lh}$ if and only if $\psi(z) = zh(z)/(g(z))^{e^{2i\alpha}} \in \mathcal{SP}^{\alpha}$.*

The first result is to obtain sufficient conditions for the function $F(z) = f(z)|f(z)|^{2\gamma}$ to be $\alpha-$spirallike logharmonic mapping.

Theorem 4.1. *Let $f(z) = zh(z)\overline{g(z)} \in \mathcal{ST}_{Lh}$ with respect to $a \in B_0$, and γ be a constant with $\operatorname{Re}\gamma > -1/2$. Then $F(z) = f(z)|f(z)|^{2\gamma}$ is an $\alpha-$spirallike logharmonic mapping with respect to*

$$\hat{a}(z) = \frac{(1+\overline{\gamma})a(z)+\overline{\gamma}}{1+\gamma+\gamma a(z)},$$

where $\alpha = \tan^{-1}(2\operatorname{Im}\gamma/(1+2\operatorname{Re}\gamma))$.

Proof. Since f is logharmonic with respect to a, it follows that the function $F(z) = f(z)|f(z)|^{2\gamma} = f^{1+\gamma}\overline{f}^{\gamma}$ satisfies

$$\frac{F_z}{F} = (1+\gamma)\frac{f_z}{f} + \gamma\frac{(\overline{f})_z}{\overline{f}}$$

$$
\begin{aligned}
&= (1+\gamma)\frac{f_z}{f} + \gamma\overline{\left(\frac{(f)_{\bar z}}{f}\right)} \\
&= (1+\gamma)\frac{f_z}{f} + \gamma a(z)\frac{f_z}{f},
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\left(\frac{(F)_{\bar z}}{F}\right)} &= (1+\overline{\gamma})\overline{\left(\frac{(f)_{\bar z}}{f}\right)} + \overline{\gamma}\overline{\left(\frac{(\overline{f})_{\bar z}}{\overline{f}}\right)} \\
&= (1+\overline{\gamma})\overline{\left(\frac{(f)_{\bar z}}{f}\right)} + \overline{\gamma}\overline{\left(\frac{\overline{(f_z)}}{\overline{f}}\right)} \\
&= (1+\overline{\gamma})a(z)\frac{f_z}{f} + \overline{\gamma}\frac{f_z}{f}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\hat{a}(z) = \frac{\overline{\left(\frac{(F)_{\bar z}}{F}\right)}}{\frac{F_z}{F}} &= \frac{(1+\overline{\gamma})a(z)\frac{f_z}{f} + \overline{\gamma}\frac{f_z}{f}}{(1+\gamma)\frac{f_z}{f} + \gamma a(z)\frac{f_z}{f}} \\
&= \frac{1+\overline{\gamma}}{1+\gamma}\frac{a(z) + \dfrac{\overline{\gamma}}{1+\overline{\gamma}}}{1 + a(z)\dfrac{\gamma}{1+\gamma}}.
\end{aligned}
$$

Since $\operatorname{Re}\gamma > -1/2$, it follows that $|\gamma/(1+\gamma)|^2 < 1$. Then

$$
\left|\frac{\overline{\gamma}}{1+\overline{\gamma}}\right|^2 (1-|a(z)|^2) < 1-|a(z)|^2,
$$

and thus

$$
|a(z)|^2 + \left|\frac{\overline{\gamma}}{1+\overline{\gamma}}\right|^2 < 1 + |a(z)|^2\left|\frac{\gamma}{1+\gamma}\right|^2.
$$

It yields that

$$
\left|a(z) + \frac{\overline{\gamma}}{1+\overline{\gamma}}\right|^2 < \left|1 + a(z)\frac{\gamma}{1+\gamma}\right|^2.
$$

Evidently,

$$\left|\hat{a}(z)\right| = \left|\frac{a(z)+\dfrac{\overline{\gamma}}{1+\overline{\gamma}}}{1+a(z)\dfrac{\gamma}{1+\gamma}}\right| < 1,$$

and hence the function $F(z) = f(z)|f(z)|^{2\gamma}$ is logharmonic with respect to $\hat{a}(z)$. Now

$$F(z) = f(z)|f(z)|^{2\gamma} = f^{1+\gamma}(z)\overline{f}^{\gamma}(z) = z|z|^{2\gamma}h^{1+\gamma}(z)g^{\gamma}(z)\overline{h^{\overline{\gamma}}(z)g^{1+\overline{\gamma}}(z)} = z|z|^{2\gamma}H(z)\overline{G(z)},$$

where $H(z) = h^{1+\gamma}(z)g^{\gamma}(z)$, and $G(z) = h^{\overline{\gamma}}(z)g^{1+\overline{\gamma}}(z)$.

Consider the analytic function

$$\Psi(z) = zH(z)/(G(z))^{e^{2i\alpha}} = zh^{1+\gamma}(z)g^{\gamma}(z)/(h^{\overline{\gamma}}(z)g^{1+\overline{\gamma}}(z))^{e^{2i\alpha}}.$$

It is evident that $\Psi(0) = 0$ and $\Psi'(0) = 1$. Furthermore,

$$\begin{aligned}
e^{-i\alpha}\frac{z\Psi'(z)}{\Psi(z)} &= e^{-i\alpha}\left(1+(1+\gamma)\frac{zh'(z)}{h(z)}+\gamma\frac{zg'(z)}{g(z)}-e^{2i\alpha}\left(\overline{\gamma}\frac{zh'(z)}{h(z)}+(1+\overline{\gamma})\frac{zg'(z)}{g(z)}\right)\right)\\
&= e^{-i\alpha}\Big(1+\big((1+\gamma)-\overline{\gamma}e^{2i\alpha}\big)\frac{zh'(z)}{h(z)}-\big((1+\overline{\gamma})e^{2i\alpha}-\gamma\big)\frac{zg'(z)}{g(z)}\Big)\\
&= e^{-i\alpha}+\big((1+\gamma)e^{-i\alpha}-\overline{\gamma}e^{i\alpha}\big)\frac{zh'(z)}{h(z)}-\big((1+\overline{\gamma})e^{i\alpha}-\gamma e^{-i\alpha}\big)\frac{zg'(z)}{g(z)}.
\end{aligned}$$

Then

$$\operatorname{Re}\left(e^{-i\alpha}\frac{z\Psi'(z)}{\Psi(z)}\right) = \cos\alpha\left(1+\operatorname{Re}\left(\frac{\big((1+\gamma)e^{-i\alpha}-\overline{\gamma}e^{i\alpha}\big)}{\cos\alpha}\frac{zh'(z)}{h(z)}-\frac{\big((1+\overline{\gamma})e^{i\alpha}-\gamma e^{-i\alpha}\big)}{\cos\alpha}\frac{zg'(z)}{g(z)}\right)\right).$$

Since

$$\begin{aligned}
\frac{(1+\gamma)e^{-i\alpha}-\overline{\gamma}e^{i\alpha}}{\cos\alpha} &= \frac{(1+\gamma-\overline{\gamma})\cos\alpha - i(1+\gamma+\overline{\gamma})\sin\alpha}{\cos\alpha}\\
&= 1+2i\operatorname{Im}\gamma - i(1+2\ \operatorname{Re}\gamma)\tan\alpha,
\end{aligned}$$

and

$$\begin{aligned}
\frac{(1+\overline{\gamma})e^{i\alpha}-\gamma e^{-i\alpha}}{\cos\alpha} &= \frac{(1+\overline{\gamma}-\gamma)\cos\alpha + i(1+\overline{\gamma}+\gamma)\sin\alpha}{\cos\alpha}\\
&= 1-2i\operatorname{Im}\gamma + i(1+2\ \operatorname{Re}\gamma)\tan\alpha,
\end{aligned}$$

the condition on α ensures that

$$\frac{(1+\gamma)e^{-i\alpha}-\overline{\gamma}e^{i\alpha}}{\cos\alpha}=\frac{(1+\overline{\gamma})e^{i\alpha}-\gamma e^{-i\alpha}}{\cos\alpha}=1.$$

Therefore,

$$\begin{aligned}\operatorname{Re}\left(e^{-i\alpha}\frac{z\Psi'(z)}{\Psi(z)}\right)&=\cos\alpha\left(1+\operatorname{Re}\frac{zh'(z)}{h(z)}-\operatorname{Re}\frac{zg'(z)}{g(z)}\right)\\&=\cos\alpha\operatorname{Re}\left(1+\frac{zh'(z)}{h(z)}-\frac{zg'(z)}{g(z)}\right)=\cos\alpha\operatorname{Re}\left(\frac{z\psi'(z)}{\psi(z)}\right).\end{aligned}$$

Since $f\in\mathcal{ST}_{Lh}$, Lemma 4.2 shows that $\psi(z)=zh(z)/g(z)\in\mathcal{ST}$, and thus

$$\operatorname{Re}\left(e^{-i\alpha}\frac{z\Psi'(z)}{\Psi(z)}\right)>0.$$

Hence, Ψ is an $\alpha-$spirallike analytic function. Then Lemma 4.3 shows that F is an $\alpha-$spirallike logharmonic with respect to $\hat{a}(z)$. □

Corrollary 4.1. *Let* $f(z)=zh(z)\overline{g(z)}\in\mathcal{ST}_{Lh}$. *The function* $F(z)=f(z)|f(z)|^{2\gamma}$ *in Theorem 4.1 is a univalent starlike function if and only if* $\gamma>-1/2$.

Proof. Since

$$F(z)=f(z)|f(z)|^{2\gamma}=f^{1+\gamma}(z)\overline{f}^{\gamma}(z)=z|z|^{2\gamma}h^{1+\gamma}(z)g^{\gamma}(z)\overline{h^{\overline{\gamma}}(z)g^{1+\overline{\gamma}}(z)}=z|z|^{2\gamma}H(z)\overline{G(z)}$$

is a univalent logharmonic mapping, where $H(z)=h^{1+\gamma}(z)g^{\gamma}(z)$, and $G(z)=h^{\overline{\gamma}}(z)g^{1+\overline{\gamma}}(z)$, it follows from Theorem 1.18 that $\operatorname{Re}\gamma>-1/2$. Further, F in Theorem 4.1 is starlike implies $\alpha=0$. Thus $\operatorname{Im}\gamma=0$, that is, γ is real. Hence $\gamma>-1/2$.

Conversely, let $\gamma>-1/2$ in Theorem 4.1, that is, γ is real. Then $\tan\alpha=0$. Since $|\alpha|<\pi/2$, it follows that $\alpha=0$. Hence F is a univalent starlike function. □

The following result derives a sufficient condition for starlikeness for the product $F(z) = f_1^{\lambda}(z) f_2^{1-\lambda}(z)$.

Theorem 4.2. *Let* $f_k(z) = zh_k(z)\overline{g_k(z)} \in \mathcal{ST}_{Lh}$ $(k = 1, 2)$ *with respect to the same* $a \in B_0$*. Then* $F(z) = f_1^{\lambda}(z) f_2^{1-\lambda}(z)$, $0 \le \lambda \le 1$, *is a univalent starlike logharmonic mapping with respect to the same* a.

Proof. Let $\mu = \overline{((F)_{\bar{z}}/F)}/(F_z/F)$. Then

$$\mu = \frac{\lambda \overline{\left(\frac{(f_1)_{\bar{z}}}{f_1}\right)} + (1-\lambda)\overline{\left(\frac{(f_2)_{\bar{z}}}{f_2}\right)}}{\lambda \dfrac{(f_1)_z}{f_1} + (1-\lambda)\dfrac{(f_2)_z}{f_2}}.$$

Since

$$\frac{(f_1)_z}{f_1} = \frac{(zh_1)'}{zh_1}, \quad \frac{(f_2)_z}{f_2} = \frac{(zh_2)'}{zh_2}, \tag{4.6}$$

and

$$\overline{\left(\frac{(f_1)_{\bar{z}}}{f_1}\right)} = \frac{g_1'}{g_1}, \quad \overline{\left(\frac{(f_2)_{\bar{z}}}{f_2}\right)} = \frac{g_2'}{g_2}, \tag{4.7}$$

it follows from (4.4), (4.6) and (4.7) that

$$\begin{aligned}
\mu &= \frac{\lambda \dfrac{g_1'}{g_1} + (1-\lambda)\dfrac{g_2'}{g_2}}{\lambda \dfrac{(zh_1)'}{zh_1} + (1-\lambda)\dfrac{(zh_2)'}{zh_2}} \\
&= \frac{\lambda a \dfrac{(zh_1)'}{zh_1} + (1-\lambda) a \dfrac{(zh_2)'}{zh_2}}{\lambda \dfrac{(zh_1)'}{zh_1} + (1-\lambda)\dfrac{(zh_2)'}{zh_2}} = a.
\end{aligned}$$

Thus F is a logharmonic mapping with respect to a.

Now

$$\begin{aligned}
F(z) = f_1^{\lambda}(z) f_2^{1-\lambda}(z) &= \left(zh_1(z)\overline{g_1(z)}\right)^{\lambda} \left(zh_2(z)\overline{g_2(z)}\right)^{1-\lambda} \\
&= zh_1^{\lambda}(z) h_2^{1-\lambda}(z) \overline{g_1^{\lambda}(z) g_2^{1-\lambda}(z)} := zh(z)\overline{g(z)},
\end{aligned}$$

where $h(z)=h_1^{\lambda}(z)h_2^{1-\lambda}(z)$ and $g(z)=g_1^{\lambda}(z)g_2^{1-\lambda}(z)$.

The associated analytic function for F is

$$\psi(z)=\frac{zh(z)}{g(z)}=\frac{zh_1^{\lambda}(z)h_2^{1-\lambda}(z)}{g_1^{\lambda}(z)g_2^{1-\lambda}(z)}=\left(\frac{zh_1(z)}{g_1(z)}\right)^{\lambda}\left(\frac{zh_2(z)}{g_2(z)}\right)^{1-\lambda}=(\psi_1(z))^{\lambda}(\psi_2(z))^{1-\lambda}.$$

Then $\psi(0)=0$ and $\psi'(0)=1$. By Lemma 4.2, $\psi_k=zh_k/g_k\in\mathcal{ST}$, and thus

$$\operatorname{Re}\left(\frac{z\psi'(z)}{\psi(z)}\right)=\lambda\operatorname{Re}\left(\frac{z\psi_1'(z)}{\psi_1(z)}\right)+(1-\lambda)\operatorname{Re}\left(\frac{z\psi_2'(z)}{\psi_2(z)}\right)>0.$$

Hence ψ is a univalent starlike analytic function, and now again Lemma 4.2 yields the desired result. $\square$

The following corollary is an immediate consequence of Theorem 4.2.

Corrollary 4.2. *Let $f_k(z)=zh_k(z)\overline{g_k(z)}\in\mathcal{ST}_{Lh}$ $(k=1,2,\ldots,n)$ with respect to the same $a\in B_0$. Then $F=f_1^{\lambda_1}f_2^{\lambda_2}\cdots f_n^{\lambda_n}$ is a univalent starlike logharmonic mapping with respect to the same a, where $0\leq\lambda_k\leq 1$ and $\lambda_1+\lambda_2+\cdots+\lambda_n=1$.*

Sufficient conditions for the product $F(z)=f_1^{\lambda}(z)f_2^{1-\lambda}(z)$ to be univalent starlike logharmonic mapping is obtained in the following result.

Theorem 4.3. *Let $f_k(z)=zh_k(z)\overline{g_k(z)}\in\mathcal{ST}_{Lh}$ $(k=1,2)$ with respect to $a_k\in B_0$. Suppose also that*

$$\operatorname{Re}(1-a_1\overline{a_2})\frac{(zh_1)'}{zh_1}\overline{\left(\frac{(zh_2)'}{zh_2}\right)}\geq 0. \tag{4.8}$$

Then $F(z)=f_1^{\lambda}(z)f_2^{1-\lambda}(z)$, $0\leq\lambda\leq 1$, is a univalent starlike logharmonic mapping.

Proof. The argument is similar to the proof of Theorem 4.2. Let $(F_z/F)\mu(z)=\overline{(F_{\overline{z}}/F)}$.

In view of (4.4), it is follows that

$$|\mu(z)| = \left| \frac{\lambda \dfrac{g_1'}{g_1} + (1-\lambda)\dfrac{g_2'}{g_2}}{\lambda \dfrac{(zh_1)'}{zh_1} + (1-\lambda)\dfrac{(zh_2)'}{zh_2}} \right| = \left| \frac{\lambda a_1 \dfrac{(zh_1)'}{zh_1} + (1-\lambda) a_2 \dfrac{(zh_2)'}{zh_2}}{\lambda \dfrac{(zh_1)'}{zh_1} + (1-\lambda)\dfrac{(zh_2)'}{zh_2}} \right|. \tag{4.9}$$

By assumption,

$$\begin{aligned}
&\left| \lambda \frac{(zh_1)'}{zh_1} + (1-\lambda)\frac{(zh_2)'}{zh_2} \right|^2 - \left| \lambda a_1 \frac{(zh_1)'}{zh_1} + (1-\lambda) a_2 \frac{(zh_2)'}{zh_2} \right|^2 \\
&= \lambda^2 \left| \frac{(zh_1)'}{zh_1} \right|^2 + (1-\lambda)^2 \left| \frac{(zh_2)'}{zh_2} \right|^2 + 2\lambda(1-\lambda)\,\mathrm{Re}\left(\frac{(zh_1)'}{zh_1} \overline{\left(\frac{(zh_2)'}{zh_2} \right)} \right) \\
&\quad - \lambda^2 |a_1|^2 \left| \frac{(zh_1)'}{zh_1} \right|^2 - (1-\lambda)^2 |a_2|^2 \left| \frac{(zh_2)'}{zh_2} \right|^2 \\
&\quad - 2\lambda(1-\lambda)\,\mathrm{Re}\left(\frac{a_1 (zh_1)'}{zh_1} \overline{\left(\frac{a_2 (zh_2)'}{zh_2} \right)} \right) \\
&= \lambda^2 (1-|a_1|^2) \left| \frac{(zh_1)'}{zh_1} \right|^2 + (1-\lambda)^2 (1-|a_2|^2) \left| \frac{(zh_2)'}{zh_2} \right|^2 \\
&\quad + 2\lambda(1-\lambda)\ \mathrm{Re}\left((1-a_1\overline{a_2})\, \frac{(zh_1)'}{zh_1} \overline{\left(\frac{(zh_2)'}{zh_2} \right)} \right).
\end{aligned}$$

Since $|a_k| < 1$, it is evident from (4.8) that

$$\left| \lambda \frac{(zh_1)'}{zh_1} + (1-\lambda)\frac{(zh_2)'}{zh_2} \right|^2 - \left| \lambda a_1 \frac{(zh_1)'}{zh_1} + (1-\lambda) a_2 \frac{(zh_2)'}{zh_2} \right|^2 > 0.$$

Thus $|\mu(z)| < 1$, and hence F is logharmonic mapping with respect to μ.

The associated analytic function for F is $\psi(z) = (zh_1^{\lambda} h_2^{1-\lambda})/(g_1^{\lambda} g_2^{1-\lambda})$. Then $\psi(0) = 0$ and $\psi'(0) = 1$. It follows from Lemma 4.2 that $\psi_k = zh_k/g_k \in \mathcal{ST}$, and thus

$$\mathrm{Re}\left(\frac{z\psi'(z)}{\psi(z)} \right) = \lambda\,\mathrm{Re}\left(\frac{z\psi_1'(z)}{\psi_1(z)} \right) + (1-\lambda)\,\mathrm{Re}\left(\frac{z\psi_2'(z)}{\psi_2(z)} \right) > 0.$$

Hence ψ is a univalent starlike analytic function. It is now evident from Lemma 4.2 that F is a univalent starlike logharmonic mapping. $\square$

Another sufficient conditions for starlikeness of the product $F(z) = f_1^{\lambda}(z) f_2^{1-\lambda}(z)$ is derived in the following result.

Theorem 4.4. *Let $f_k = zh_k\overline{g_k} \in \mathcal{S}_{Lh}$ $(k = 1,2)$ with respect to $a_k \in B_0$ satisfying $zh_kg_k = z$. Then $F(z) = f_1^{\lambda}(z) f_2^{1-\lambda}(z)$, $0 \leq \lambda \leq 1$, is a univalent starlike logharmonic mapping.*

Proof. Since

$$\frac{(zh_k(z))'}{zh_k(z)} + \frac{g_k'(z)}{g_k(z)} = \frac{1}{z},$$

it follows from (4.4) that

$$\frac{(zh_k(z))'}{zh_k(z)} + a_k(z)\left(\frac{(zh_k(z))'}{zh_k(z)}\right) = \frac{1}{z}.$$

Equivalently,

$$\frac{(zh_k(z))'}{zh_k(z)} = \frac{1}{z(1+a_k(z))}. \tag{4.10}$$

Now $F(z) = f_1^{\lambda}(z) f_2^{1-\lambda}(z)$, then (4.9) and (4.10) readily yield

$$\begin{aligned}|\mu(z)| &= \left|\frac{\dfrac{\lambda a_1}{z(1+a_1)} + \dfrac{(1-\lambda)a_2}{z(1+a_2)}}{\dfrac{\lambda}{z(1+a_1)} + \dfrac{(1-\lambda)}{z(1+a_2)}}\right| = \left|\frac{\frac{\lambda a_1(1+a_2)+(1-\lambda)a_2(1+a_1)}{z(1+a_1)(1+a_2)}}{\frac{\lambda(1+a_2)+(1-\lambda)(1+a_1)}{z(1+a_1)(1+a_2)}}\right| \\ &= \left|\frac{\lambda a_1 + (1-\lambda)a_2 + a_1a_2}{1+(1-\lambda)a_1 + \lambda a_2}\right|.\end{aligned}$$

Evidently, $|\mu(z)| < 1$ is equivalent to

$$K(\lambda) = |1+(1-\lambda)a_1 + \lambda a_2|^2 - |\lambda a_1 + (1-\lambda)a_2 + a_1a_2|^2 > 0.$$

A further computation shows that

$$
\begin{aligned}
K(\lambda) &= (1+(1-\lambda)a_1+\lambda a_2)\,(1+(1-\lambda)\overline{a_1}+\lambda\overline{a_2}) \\
&\qquad -(\lambda a_1+(1-\lambda)a_2+a_1a_2)\,(\lambda\overline{a_1}+(1-\lambda)\overline{a_2}+\overline{a_1a_2}) \\
&= 1+(1-\lambda)\overline{a_1}+\lambda\overline{a_2}+(1-\lambda)a_1+(1-\lambda)^2|a_1|^2 \\
&\qquad +\lambda(1-\lambda)a_1\overline{a_2}+\lambda a_2+\lambda(1-\lambda)\overline{a_1}a_2+\lambda^2|a_2|^2 \\
&\qquad -\big(\lambda^2|a_1|^2+\lambda(1-\lambda)\overline{a_1}a_2+\lambda|a_1|^2a_2+\lambda(1-\lambda)a_1\overline{a_2}+(1-\lambda)^2|a_2|^2 \\
&\qquad +(1-\lambda)a_1|a_2|^2+\lambda|a_1|^2\overline{a_2}+(1-\lambda)|a_2|^2\overline{a_1}+|a_1|^2|a_2|^2\big) \\
&= 1+(1-\lambda)(\overline{a_1}+a_1)+\lambda(\overline{a_2}+a_2)+(1-2\lambda)(|a_1|^2-|a_2|^2) \\
&\qquad -(1-\lambda)|a_2|^2(\overline{a_1}+a_1)-\lambda|a_1|^2(\overline{a_2}+a_2)-|a_1|^2|a_2|^2 \\
&= 1+2(1-\lambda)\operatorname{Re}a_1+2\lambda\operatorname{Re}a_2-2\lambda|a_1|^2+2\lambda|a_2|^2-2(1-\lambda)|a_2|^2\operatorname{Re}a_1 \\
&\qquad -2\lambda|a_1|^2\operatorname{Re}a_2-|a_1|^2|a_2|^2+|a_1|^2-|a_2|^2 \\
&= 2\lambda\operatorname{Re}a_2-2\lambda|a_1|^2\operatorname{Re}a_2-2\lambda\operatorname{Re}a_1+2\lambda|a_2|^2\operatorname{Re}a_1-2\lambda|a_1|^2+2\lambda|a_2|^2 \\
&\qquad +\big(1-|a_2|^2\big)\,(1+2\operatorname{Re}a_1+|a_1|^2) \\
&= 2\lambda\Big(\big(1-|a_1|^2\big)\operatorname{Re}a_2-\big(1-|a_2|^2\big)\operatorname{Re}a_1-\big(|a_1|^2-|a_2|^2\big)\Big) \\
&\qquad +\big(1-|a_2|^2\big)\,|1+a_1|^2.
\end{aligned}
$$

It is evident that K is a continuous monotonic function of λ in the interval $[0,1]$. Also,

$$K(0)=\big(1-|a_2|^2\big)\,|1+a_1|^2>0,$$

and

$$
\begin{aligned}
K(1) &= 2\big((1-|a_1|^2)\operatorname{Re}a_2-(1-|a_2|^2)\operatorname{Re}a_1+|a_2|^2-|a_1|^2\big)+(1-|a_2|^2)|1+a_1|^2 \\
&= 2\operatorname{Re}a_2-2|a_1|^2\operatorname{Re}a_2-2\operatorname{Re}a_1+2|a_2|^2\operatorname{Re}a_1+2|a_2|^2-2|a_1|^2 \\
&\qquad +(1-|a_2|^2)(1+2\operatorname{Re}a_1+|a_1|^2) \\
&= 1+2\operatorname{Re}a_2+|a_2|^2-|a_1|^2-2|a_1|^2\operatorname{Re}a_2-|a_1|^2|a_2|^2
\end{aligned}
$$

$$= (1-|a_1|^2)(1+2\operatorname{Re} a_2+|a_2|^2)$$

$$= \left(1-|a_1|^2\right)|1+a_2|^2 > 0.$$

Thus $K(\lambda) > 0$ for all $\lambda \in [0,1]$, and hence F is logharmonic with respect to μ.

Let $\psi_k(z) = zh_k(z)/g_k(z)$. Then $\psi_k(0) = 0$ and $\psi_k'(0) = 1$. In view of (4.4) and (4.10), it is clear that

$$\begin{aligned}\operatorname{Re}\left(\frac{z\psi_k'(z)}{\psi_k(z)}\right) &= \operatorname{Re}\left(\frac{z(zh_k(z))'}{zh_k(z)} - \frac{zg_k'(z)}{g_k(z)}\right) = \operatorname{Re}\left(\frac{z(zh_k(z))'}{zh_k(z)} - a_k\frac{z(zh_k(z))'}{zh_k(z)}\right)\\ &= \operatorname{Re}\left((1-a_k(z))\frac{z(zh_k(z))'}{zh_k(z)}\right) = \operatorname{Re}\left(\frac{1-a_k(z)}{1+a_k(z)}\right) = \frac{1-|a_k(z)|^2}{|1+a_k(z)|^2} > 0.\end{aligned}$$

Hence ψ_k is a univalent starlike function.

The associated analytic function for F is given by $\psi(z) = (zh_1^{\lambda}h_2^{1-\lambda})/(g_1^{\lambda}g_2^{1-\lambda})$. Then $\psi(0) = 0$ and $\psi'(0) = 1$. Further,

$$\operatorname{Re}\left(\frac{z\psi'(z)}{\psi(z)}\right) = \lambda\operatorname{Re}\left(\frac{z\psi_1'(z)}{\psi_1(z)}\right) + (1-\lambda)\operatorname{Re}\left(\frac{z\psi_2'(z)}{\psi_2(z)}\right) > 0.$$

Thus ψ is a univalent starlike analytic function, and therefore Lemma 4.2 yields the required result. □

The proof of Theorem 4.4 gives the following result.

Corrollary 4.3. *Let* $f_k = zh_k\overline{g_k} \in \mathcal{S}_{Lh}$ $(k=1,2)$ *with respect to* $a_k \in B_0$, *and suppose that* $zh_kg_k = z$. *Then* $\psi_k(z) = z(h_k(z))^2 \in \mathcal{ST}$.

Proof. it follows from (4.10) that

$$\frac{(h_k(z))'}{h_k(z)} + \frac{1}{z} = \frac{1}{z(1+a_k(z))}.$$

Then

$$\frac{z(h_k(z))'}{h_k(z)} = \frac{1}{1+a_k(z)} - 1 = \frac{-a_k(z)}{1+a_k(z)}. \tag{4.11}$$

Since

$$\frac{z\psi_k'(z)}{\psi_k(z)} = 1 + 2\frac{z(h_k(z))'}{h_k(z)},$$

it follows from (4.11) that

$$\mathrm{Re}\left(\frac{z\psi_k'(z)}{\psi_k(z)}\right) = 1 + 2\,\mathrm{Re}\left(\frac{z(h_k(z))'}{h_k(z)}\right) = 1 - 2\,\mathrm{Re}\left(\frac{a_k(z)}{1+a_k(z)}\right) = \mathrm{Re}\left(\frac{1-a_k(z)}{1+a_k(z)}\right) > 0.$$

Thus $\psi_k \in \mathcal{ST}$. $\square$

4.3 Examples

This section gives several illustrative examples.

Example 4.1. Let

$$f(z) = z\left(\frac{1-\bar{z}}{1-z}\right).$$

The associated analytic function for f is $\psi(z) = z/(1-z)^2$. Evidently, $\psi(0) = 0$, $\psi'(0) = 1$, and $z\psi'(z)/\psi(z) = (1+z)/(1-z)$. Thus ψ is a univalent starlike analytic function. Then by Lemma 4.2, f is a univalent starlike logharmonic mapping with respect to

$$a(z) = \frac{\overline{\left(\frac{f_{\bar{z}}(z)}{f(z)}\right)}}{\frac{f_z(z)}{f(z)}} = \frac{\overline{\left(\frac{-1}{1-\bar{z}}\right)}}{\frac{1}{z(1-z)}} = -z.$$

Now Theorem 4.1 shows that the function $F(z) = f(z)|f(z)|^{2\gamma}$ is an $\alpha-$spirallike logharmonic mapping with respect to

$$\hat{a}(z) = \frac{1+\bar{\gamma}}{1+\gamma}\frac{-z + \dfrac{\bar{\gamma}}{1+\bar{\gamma}}}{1 - \dfrac{\gamma}{1+\gamma}z},$$

where $\alpha = \tan^{-1}(2\operatorname{Im}\gamma/(1+2\operatorname{Re}\gamma))$. In particular, if $\gamma = i$, then $\alpha = \tan^{-1}(2) = 0.352\pi$, and

$$\hat{a}(z) = \frac{1-i}{1+i}\left(\frac{-z-\frac{i}{1-i}}{1-\frac{iz}{1+i}}\right) = \frac{-i-(1-i)z}{1+i-iz}.$$

The image of circles in the unit disk under f is shown in Figure 4.1, and Figure 4.2 shows the image of the radial slits in $\mathbb{D}$ by F.

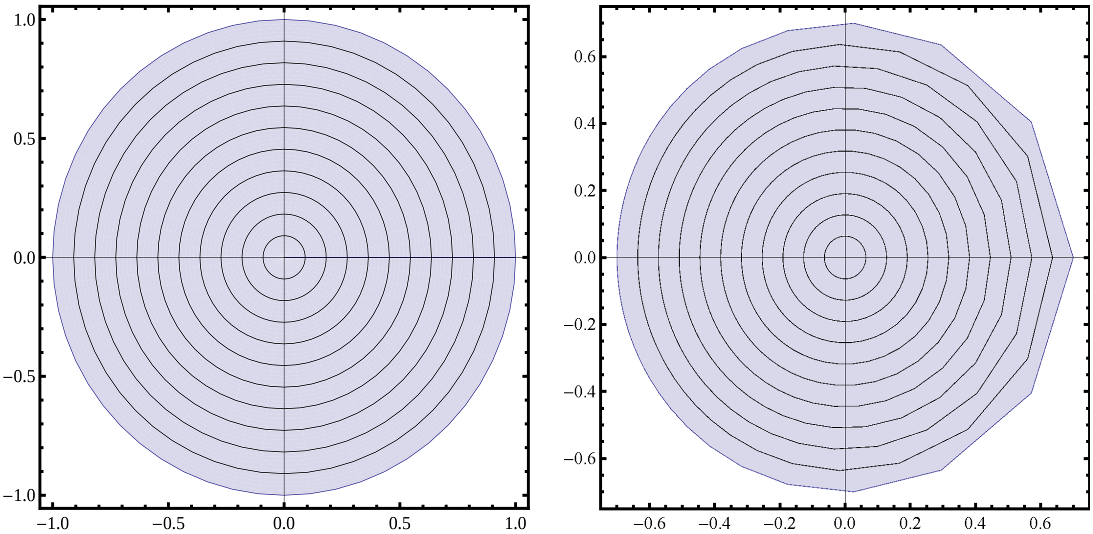

Figure 4.1: Graph of circles in $\mathbb{D}$ by $f(z) = \dfrac{z(1-\bar{z})}{1-z}$.

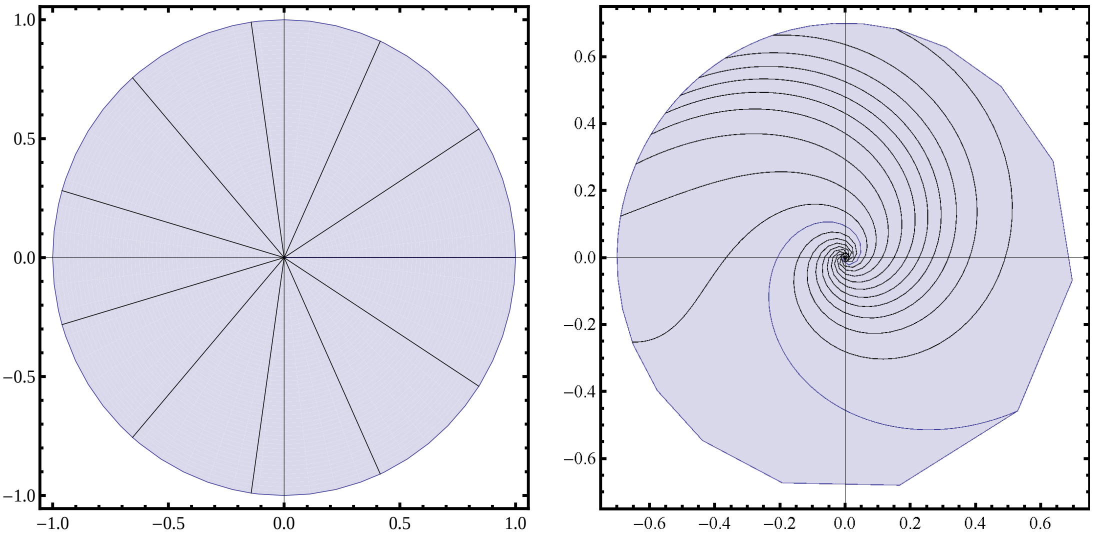

Figure 4.2: Graph of radial slits by $F(z) = f(z)|f(z)|^{2i}$, $f(z) = \dfrac{z(1-\bar{z})}{1-z}$.

Example 4.2. Consider the functions

$$f_1(z) = z\left(\frac{1-\bar{z}}{1-z}\right)\exp\left\{\operatorname{Re}\frac{4z}{1-z}\right\}, \quad \text{and} \quad f_2(z) = z\left(\frac{1+\bar{z}}{1+z}\right),$$

where

$$h_1(z) = \frac{\exp\left\{\frac{2z}{1-z}\right\}}{1-z}, \quad \text{and} \quad g_1(z) = \exp\left\{\frac{2z}{1-z}\right\}(1-z).$$

It is evident that $\psi_1(z) = z/(1-z)^2$. Then $\psi_1(0) = 0$, $\psi_1'(0) = 1$, and $z\psi_1'(z)/\psi_1(z) = (1+z)/(1-z)$. Thus ψ_1 is a univalent starlike analytic function, and it follows from Lemma 4.2 that f_1 is a univalent starlike logharmonic mapping with respect to

$$a_1(z) = \frac{\overline{\left(\frac{(f_1)_{\bar{z}}}{f_1}\right)}}{\frac{(f_1)_z}{f_1}} = \frac{\overline{\left(\frac{-1}{1-\bar{z}} + \frac{2}{(1-\bar{z})^2}\right)}}{\frac{1}{z} + \frac{1}{1-z} + \frac{2}{(1-z)^2}} = \frac{\overline{\left(\frac{1+\bar{z}}{(1-\bar{z})^2}\right)}}{\frac{1+z}{z(1-z)^2}} = z.$$

Further, $\psi_2(z) = z/(1+z)^2$. Then $\psi_2(0) = 0$, $\psi_2'(0) = 1$, and $z\psi_2'(z)/\psi_2(z) = (1-z)/(1+z)$. Thus ψ_2 is a univalent starlike analytic function, and it follows from Lemma 4.2 that f_2 is a univalent starlike logharmonic mapping with respect to

$$a_2(z) = \frac{\overline{\left(\frac{(f_2)_{\bar{z}}}{f_2}\right)}}{\frac{(f_2)_z}{f_2}} = \frac{\overline{\left(\frac{1}{1+\bar{z}}\right)}}{\frac{1}{z} - \frac{1}{1+z}} = \frac{\frac{1}{1+z}}{\frac{1}{z(1+z)}} = z.$$

Now Theorem 4.2 shows that $F(z) = f_1^{\lambda}(z)f_2^{1-\lambda}(z)$, $0 \le \lambda \le 1$, is a univalent starlike logharmonic mapping with respect to $a(z) = z$.

The image of F is shown in Figure 4.3 for $\lambda = 1/3$.

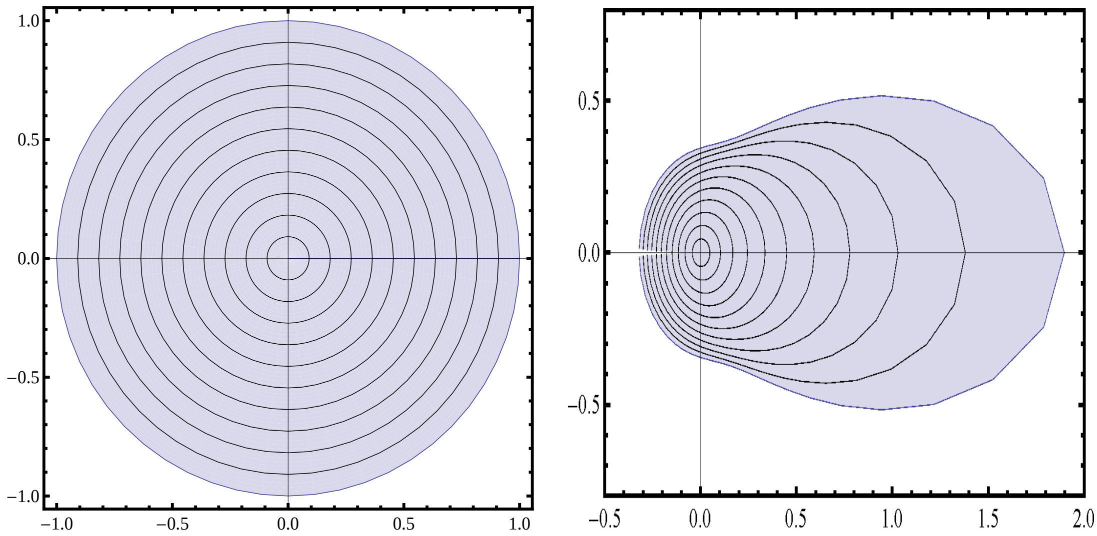

Figure 4.3: Graph of circles in $\mathbb{D}$ by $F(z) = z\left(\frac{1-\overline{z}}{1-z}\exp\left\{\operatorname{Re}\frac{4z}{1-z}\right\}\right)^{1/3}\left(\frac{1+\overline{z}}{1+z}\right)^{2/3}$.

Example 4.3. Let

$$f_1(z) = z\left(\frac{1-\overline{z}}{1-z}\right), \quad \text{and} \quad f_2(z) = z\left(\frac{1-\overline{z}}{1-z}\right)\exp\left\{\operatorname{Re}\frac{4z}{1-z}\right\}.$$

Simple calculations show that f_1 and f_2 are respectively starlike logharmonic with dilatations $a_1(z) = -z$ and $a_2(z) = z$. Also, $zh_1(z) = z/(1-z)$, and $zh_2(z) = z\exp\{2z/(1-z)\}$ $/(1-z)$. Then

$$\frac{(zh_1(z))'}{zh_1(z)} = \frac{1}{z(1-z)}, \quad \text{and} \quad \frac{(zh_2(z))'}{zh_2(z)} = \frac{1+z}{z(1-z)^2}.$$

Since

$$\begin{aligned}\operatorname{Re}\left((1-a_1\overline{a_2})\frac{(zh_1)'}{(zh_1)}\overline{\left(\frac{(zh_2)'}{(zh_2)}\right)}\right) &= \operatorname{Re}\left((1+|z|^2)\frac{1}{z(1-z)}\frac{1+\overline{z}}{\overline{z}(1-\overline{z})^2}\right)\\ &= \frac{(1+|z|^2)}{|z|^2|1-z|^2}\operatorname{Re}\frac{1+z}{1-z} > 0,\end{aligned}$$

the conditions of Theorem 4.3 are satisfied and thus $F(z) = f_1^{\lambda}(z)f_2^{1-\lambda}(z)$, $0 \le \lambda \le 1$

is a univalent starlike logharmonic mapping with respect to

$$\mu(z) = \frac{-z\lambda \frac{(zh_1(z))'}{(zh_1(z))} + (1-\lambda)z\frac{(zh_2(z))'}{(zh_2(z))}}{\lambda \frac{(zh_1(z))'}{(zh_1(z))} + (1-\lambda)\frac{(zh_2(z))'}{(zh_2(z))}} = \frac{\frac{-\lambda z}{z(1-z)} + \frac{(1-\lambda)z(1+z)}{z(1-z)^2}}{\frac{\lambda}{z(1-z)} + \frac{(1-\lambda)(1+z)}{z(1-z)^2}} = \frac{z((1-2\lambda)+z)}{1+(1-2\lambda)z}.$$

The image of circles in $\mathbb{D}$ under F for $\lambda = 1/3$ is shown in Figure 4.4.

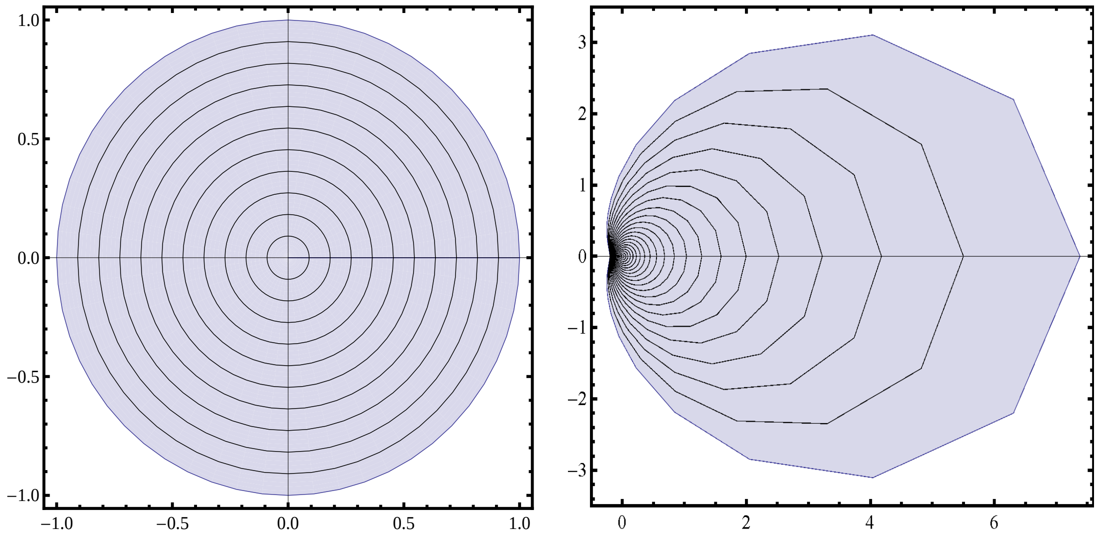

Figure 4.4: Graph of $F(z) = z\left(\frac{1-\bar{z}}{1-z}\right)^{1/3}\left(\frac{1-\bar{z}}{1-z}\exp\left\{\operatorname{Re}\frac{4z}{1-z}\right\}\right)^{2/3}$.

Example 4.4. Let $f_1(z) = zh_1(z)\overline{g_1(z)}$, where $zh_1(z)g_1(z) = z$, and $a_1(z) = z$. Then (4.4) shows that

$$h_1(z) = \frac{1}{1+z}, \quad \text{and} \quad g_1(z) = 1+z.$$

Thus

$$f_1(z) = \frac{z(1+\bar{z})}{1+z}.$$

Further, let $f_2(z) = zh_2(z)\overline{g_2(z)}$, where $zh_2(z)g_2(z) = z$, and $a_2(z) = z^2$. Then (4.4) shows that

$$h_2(z) = \frac{1}{\sqrt{1+z^2}}, \quad \text{and} \quad g_2(z) = \sqrt{1+z^2}.$$

In this case,

$$f_2(z) = \frac{z\sqrt{1+\bar{z}^2}}{\sqrt{1+z^2}}.$$

It is shown in Example 4.2 that f_1 is a univalent starlike logharmonic mapping with respect to z.

The associated analytic function for f_2 is $\psi_2(z) = z/(1+z^2)$. Then $\psi_2(0) = 0$, $\psi_2'(0) = 1$ and $z\psi_2'(z)/\psi_2(z) = (1-z^2)/(1+z^2)$. Thus ψ_2 is a univalent starlike analytic function, Lemma 4.2 shows that f_2 is a univalent starlike logharmonic mapping with respect to z^2.

Now f_1 and f_2 satisfy the conditions of Theorem 4.4, and thus $F(z) = f_1^{\lambda}(z) f_2^{1-\lambda}(z)$, $0 \le \lambda \le 1$ is a univalent starlike logharmonic mapping.

The image of $\mathbb{D}$ under F for $\lambda = 1/3$ is shown in Figure 4.5.

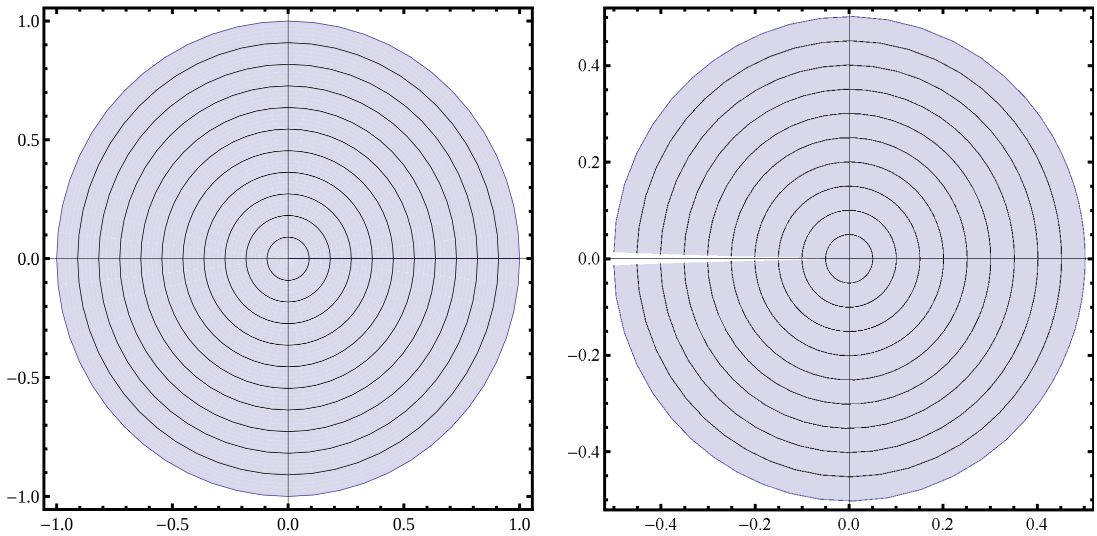

Figure 4.5: Graph of $F(z) = z\left(\frac{1+\bar{z}}{1+z}\right)^{1/3}\left(\frac{\sqrt{1+\bar{z}^2}}{\sqrt{1+z^2}}\right)^{2/3}$.

CHAPTER 5

ON ROTATIONALLY TYPICALLY REAL LOGHARMONIC MAPPINGS

5.1 Introduction

Recall that $B(\mathbb{D})$ is the set of self-maps $a \in \mathcal{H}(\mathbb{D})$, and B_0 is its subclass consisting of $a \in B$ with $a(0) = 0$. A logharmonic mapping in $\mathbb{D}$ with respect to a is a solution of the nonlinear elliptic partial differential equation

$$\overline{\left(\frac{f_{\bar{z}}(z)}{f(z)}\right)} = a(z)\frac{f_z(z)}{f(z)}, \tag{5.1}$$

where the second dilatation function a lies in B.

Recall that $\mathcal{S}_{Lh}$ is the class consisting of univalent logharmonic mappings f in $\mathbb{D}$ with respect to some $a \in B_0$ of the form

$$f(z) = zh(z)\overline{g(z)},$$

normalized by $h(0) = 1 = g(0)$, and h and g are nonvanishing analytic functions in $\mathbb{D}$.

In addition, recall that $\mathcal{ST}_{Lh}$ is the subclass of $\mathcal{S}_{Lh}$ consisting of all starlike logharmonic mappings.

An analytic function φ in $\mathbb{D}$ is said to be *typically real* if $\varphi(z)$ is real whenever z is real and nonreal elsewhere. Similarly, a logharmonic mapping f in $\mathbb{D}$ is said to be typically real if $f(z)$ is real whenever z is real and nonreal elsewhere. Typically real logharmonic mappings have been studied by Abdulhadi in [2].

Let HG denote the class consisting of functions $\varphi(z) = zh(z)g(z)$, where h and g are in $\mathcal{H}(\mathbb{D})$, and normalized by $h(0) = 1 = g(0)$. This chapter treats the class TLh of all logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ satisfy $\varphi(z) = zh(z)g(z) \in HG$ and is analytically typically real in $\mathbb{D}$. If $f_1(z) = zh_1(z)\overline{g_1(z)}$ and $f_2(z) = zh_2(z)\overline{g_2(z)}$ are logharmonic with respect to the same $a \in B$, then $f(z) = f_1^{\lambda}(z)f_2^{1-\lambda}(z)$, $0 \leq \lambda \leq 1$ is logharmonic with respect to a. It is evident that mappings $\varphi(z) = zh(z)g(z)$ in the class HG are rotations of the corresponding logharmonic mappings $f(z) = zh(z)\overline{g(z)}$.

In Section 5.2, mappings f in TLh are shown to admit an integral representation. Every mapping $f \in TLh$ is shown to be a product of two particular logharmonic mappings. The radius of starlikeness for this class, as well as an upper estimate for its arclength are determined.

In Section 5.3, we explore conditions on the dilatation a that would ensure univalent logharmonic mappings $f(z) = zh(z)\overline{g(z)} \in TLh$ necessarily satisfies $f(\mathbb{D})$ is a domain symmetric with respect to the real axis. Sufficient conditions for univalent logharmonic mappings to be in the class TLh are determined.

In Section 5.4, an integral representation and the radius of starlikeness for a subclass of TLh are obtained.

The following lemmas are needed to establish the results in subsequent sections.

Lemma 5.1. *If f has real coefficients, then $f(\mathbb{D})$ is a domain symmetric with respect to the real axis.*

Proof. Let $(f(\mathbb{D}))^* = \{w : \overline{w} \in f(\mathbb{D})\}$, and let $w = f(z) \in f(\mathbb{D})$. Since $\mathbb{D}$ is symmetric and f has real coefficients, it follows that $f(\overline{z}) = \overline{f(z)} \in f(\mathbb{D})$, that is, $\overline{w} \in f(\mathbb{D})$. Thus $w \in (f(\mathbb{D}))^*$. Hence $f(\mathbb{D}) \subset (f(\mathbb{D}))^*$.

Conversely, let $w \in (f(\mathbb{D}))^*$. Then $\overline{w} \in f(\mathbb{D})$, that is, $\overline{w} = f(z)$ for some $z \in \mathbb{D}$. Since $\mathbb{D}$ is symmetric and f has real coefficients, it follows that $f(\bar{z}) = \overline{f(z)} \in f(\mathbb{D})$, that is, $\overline{\overline{w}} = w \in f(\mathbb{D})$, and thus $(f(\mathbb{D}))^* \subset f(\mathbb{D})$. Therefore, $(f(\mathbb{D}))^* = f(\mathbb{D})$, that is, $f(\mathbb{D})$ is a domain symmetric with respect to the real axis. $\square$

Lemma 5.2. [8, Lemma 2.4] *Let Ω_1 be a bounded strictly starlike domain of $\mathbb{C}$ with respect to the origin, that is, each radial ray from 0 intersects the boundary $\partial\Omega$ of $\Omega = f(\mathbb{D})$ in exactly one point of $\mathbb{C}$. Suppose that*

$$f_j(z) = z|z|^{\beta} h_j(z)\overline{g_j(z)} \quad z \in \mathbb{D},\ h_j(0) > 0,\ g_j(0) = 1,\ j = 1,2$$

are two univalent logharmonic mappings with respect to the same a satisfying $f_1(\mathbb{D}) = \Omega_1$, and $f_2(\mathbb{D}) \subset f_1(\mathbb{D})$. Then $h_2(0) \leq h_1(0)$, and equality holds if and only if $f_1 = f_2$.

Lemma 5.3. (Theorem 1.19) *Let $f(z) = zh(z)\overline{g(z)}$ be logharmonic in $\mathbb{D}$ with $0 \notin hg(\mathbb{D})$. Then $f \in \mathcal{ST}_{Lh}(\alpha)$ if and only if $\psi(z) = zh(z)/g(z) \in \mathcal{ST}(\alpha)$ for $0 \leq \alpha < 1$.*

Lemma 5.4. [133] *Let $f \in \mathcal{P}(\alpha)$. Then*

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2r(1-\alpha)}{(1-r)(1+(1-2\alpha)r)}, \quad |z| \leq r.$$

Recall that $\mathcal{P}_{\mathbb{R}}$ is the class consisting of all normalized analytic functions with positive real part and with real coefficients in $\mathbb{D}$.

Lemma 5.5. (Theorem 1.13) *A function $\varphi \in T$ if and only if there exists a function $p \in \mathcal{P}_{\mathbb{R}}$ such that $\varphi(z) = zp(z)/(1-z^2)$.*

5.2 An Integral Representation and Radius of Starlikeness

The first result is to establish an integral representation for logharmonic mappings.

Lemma 5.6. *Let $f(z) = zh(z)\overline{g(z)}$ be a logharmonic mapping with respect to $a \in B$, and let $\varphi(z) = zh(z)g(z)$, where $h, g \in \mathcal{H}(\mathbb{D})$. Then*

$$f(z) = \varphi(z) \exp\left(-2i \operatorname{Im} \int_0^z \frac{a(s)}{1+a(s)} \frac{\varphi'(s)}{\varphi(s)} ds\right).$$

Proof. Since

$$f(z) = \varphi(z) \frac{\overline{g(z)}}{g(z)}, \tag{5.2}$$

taking the logarithmic differentiation on (5.2) gives

$$\frac{f_z(z)}{f(z)} = \frac{\varphi'(z)}{\varphi(z)} - \frac{g'(z)}{g(z)}, \tag{5.3}$$

and

$$\overline{\left(\frac{f_{\overline{z}}(z)}{f(z)}\right)} = \overline{\left(\frac{(\overline{g(z)})_{\overline{z}}}{\overline{g(z)}}\right)} = \frac{g'(z)}{g(z)}. \tag{5.4}$$

Substituting the equations (5.3) and (5.4) into (5.1) yields

$$\frac{g'(z)}{g(z)} = a(z)\left(\frac{\varphi'(z)}{\varphi(z)} - \frac{g'(z)}{g(z)}\right).$$

Thus

$$\frac{g'(z)}{g(z)} = \frac{a(z)}{1+a(z)} \frac{\varphi'(z)}{\varphi(z)}, \tag{5.5}$$

which leads to

$$g(z) = \exp \int_0^z \frac{a(s)}{1+a(s)} \frac{\varphi'(s)}{\varphi(s)} ds. \tag{5.6}$$

Substituting the equation (5.6) into (5.2) yields

$$
\begin{aligned}
f(z) &= \varphi(z)\frac{\overline{\exp\int_0^z \frac{a(s)}{1+a(s)}\frac{\varphi'(s)}{\varphi(s)}ds}}{\exp\int_0^z \frac{a(s)}{1+a(s)}\frac{\varphi'(s)}{\varphi(s)}ds} \\
&= \varphi(z)\exp\left(\overline{\int_0^z \frac{a(s)}{1+a(s)}\frac{\varphi'(s)}{\varphi(s)}ds} - \int_0^z \frac{a(s)}{1+a(s)}\frac{\varphi'(s)}{\varphi(s)}ds\right) \\
&= \varphi(z)\exp\left(-2i\,\mathrm{Im}\int_0^z \frac{a(s)}{1+a(s)}\frac{\varphi'(s)}{\varphi(s)}ds\right). \qquad \square
\end{aligned}
$$

Let TLh^0 denote the subclass of TLh consisting of logharmonic mappings f in $\mathbb{D}$ with respect to $a \in B$ of the form $f(z) = zh(z)\overline{g(z)}$ and satisfying $\varphi(z) = zh(z)g(z) = z/(1-z^2)$. It follows from Lemma 5.6 that

$$
f(z) = \frac{z}{1-z^2}\exp\left(-2i\,\mathrm{Im}\int_0^z \frac{a(s)}{1+a(s)}\frac{1+s^2}{s(1-s^2)}ds\right). \tag{5.7}
$$

Denote by $\mathcal{P}_{Lh}$ the class consisting of logharmonic mappings w with respect to $a \in B$ of the form $w(z) = h(z)\overline{g(z)}$, where h and g are analytic in $\mathbb{D}$, normalized by $h(0) = g(0) = 1$ and satisfy $p(z) = h(z)g(z) \in \mathcal{P}_{\mathbb{R}}$. Similar to the proof of Lemma 5.6, it can readily be established that

$$
w(z) = p(z)\exp\left(-2i\,\mathrm{Im}\int_0^z \frac{a(s)}{1+a(s)}\frac{p'(s)}{p(s)}ds\right). \tag{5.8}
$$

It is evident that the class $\mathcal{P}_{Lh}$ contains the class $\mathcal{P}_{\mathbb{R}}$.

The following result gives a representation formula for functions in the class TLh in terms of functions in TLh^0 and $\mathcal{P}_{Lh}$.

Theorem 5.1. *A function f belongs to TLh with respect to the same $a \in B$ if and only* *if* $f(z)=F(z)R(z)$ *for some* $F\in TLh^0$ *and* $R\in\mathcal{P}_{Lh}$ *with respect to the same* $a\in B$.

Proof. Let $f(z)=zh(z)\overline{g(z)}\in TLh$ be a logharmonic with respect to $a\in B$. It follows from Lemma 5.5 that every typically real analytic function φ has the form $(1-z^2)\varphi(z)=zp(z)$ for some $p\in\mathcal{P}_{\mathbb{R}}$. Thus Lemma 5.6 yields

$$\begin{aligned}
f(z) &= \frac{zp(z)}{1-z^2}\exp\left(-2i\mathrm{Im}\int_0^z \frac{a(s)}{1+a(s)}\left(\frac{1+s^2}{s(1-s^2)}+\frac{p'(s)}{p(s)}\right)ds\right)\\
&= \left(\frac{z}{1-z^2}\exp\left(-2i\mathrm{Im}\int_0^z \frac{a(s)}{1+a(s)}\frac{1+s^2}{s(1-s^2)}ds\right)\right)\\
&\quad\times\left(p(z)\exp\left(-2i\mathrm{Im}\int_0^z \frac{a(s)}{1+a(s)}\frac{p'(s)}{p(s)}ds\right)\right)\\
&:= F(z)R(z),
\end{aligned}$$

where

$$F(z)=\frac{z}{1-z^2}\exp\left(-2i\mathrm{Im}\int_0^z \frac{a(s)}{1+a(s)}\frac{1+s^2}{s(1-s^2)}ds\right)\in TLh^0,$$

and

$$R(z)=p(z)\exp\left(-2i\mathrm{Im}\int_0^z \frac{a(s)}{1+a(s)}\frac{p'(s)}{p(s)}ds\right)\in\mathcal{P}_{Lh}.$$

Conversely, let $f(z)=F(z)R(z)$ where $F\in TLh^0$ and $R\in\mathcal{P}_{Lh}$. Since F and R are logharmonic with respect to $a\in B$, it follows from Proposition 1.3 in Section 1.8 that f is logharmonic with respect to $a\in B$. Further, it is evident from (5.7) and (5.8) that

$$\begin{aligned}
f(z)=F(z)R(z) &= \frac{z}{1-z^2}\exp\left(-2i\mathrm{Im}\int_0^z \frac{a(s)}{1+a(s)}\frac{1+s^2}{s(1-s^2)}ds\right)\\
&\quad\times p(z)\exp\left(-2i\mathrm{Im}\int_0^z \frac{a(s)}{1+a(s)}\frac{p'(s)}{p(s)}ds\right)\\
&= \frac{zp(z)}{1-z^2}\exp\left(-2i\mathrm{Im}\int_0^z \frac{a(s)}{1+a(s)}\left(\frac{1+s^2}{s(1-s^2)}+\frac{p'(s)}{p(s)}\right)ds\right)\\
&= \frac{zp(z)}{1-z^2}\frac{\overline{\exp\int_0^z \frac{a(s)}{1+a(s)}\left(\frac{1+s^2}{s(1-s^2)}+\frac{p'(s)}{p(s)}\right)ds}}{\exp\int_0^z \frac{a(s)}{1+a(s)}\left(\frac{1+s^2}{s(1-s^2)}+\frac{p'(s)}{p(s)}\right)ds}
\end{aligned}$$

$$= zh(z)\overline{g(z)},$$

where

$$p(z) = h(z)(1-z^2)\exp\int_0^z \frac{a(s)}{1+a(s)}\left(\frac{1+s^2}{s(1-s^2)}+\frac{p'(s)}{p(s)}\right)ds,$$

and

$$g(z) = \exp\int_0^z \frac{a(s)}{1+a(s)}\left(\frac{1+s^2}{s(1-s^2)}+\frac{p'(s)}{p(s)}\right)ds.$$

Thus $\varphi(z) = zh(z)g(z) = zp(z)/(1-z^2)$, $p \in \mathcal{P}_{\mathbb{R}}$. It follows from Lemma 5.5 that $\varphi \in T$, and hence $f \in TLh$. $\square$

Corrollary 5.1. *A function f belongs to TLh with respect to the same $a \in B$ if and only if $F^2/f \in TLh$ for some $F \in TLh^0$ with respect to the same $a \in B$,and $f(z) = F(z)R(z)$ for some $R \in \mathcal{P}_{Lh}$ with respect to the same $a \in B$.*

Proof. Let $f(z) = F(z)R(z) \in TLh$, where $R = H\overline{G} \in \mathcal{P}_{Lh}$. Since

$$\overline{\left(\frac{\left(\frac{1}{R}\right)_{\bar{z}}}{\frac{1}{R}}\right)} = \overline{\left(\frac{\frac{-(R)_{\bar{z}}}{R^2}}{\frac{1}{R}}\right)} = \overline{\left(\frac{-(R)_{\bar{z}}}{R}\right)} = \frac{-a(R)_z}{R} = a\frac{\left(\frac{1}{R}\right)_z}{\frac{1}{R}},$$

it follows that $1/R$ is logharmonic with respect to the same a. Furthermore, $1/R = 1/(H\overline{G}) = (1/H)(\overline{1/G})$, and

$$\text{Re}\left(\frac{1}{HG}\right) = \text{Re}\left(\frac{\overline{HG}}{|HG|^2}\right) = \frac{\text{Re}\,(HG)}{|HG|^2} > 0.$$

Also, $1/R$ has real coefficients. Thus the function $1/R$ is in the class $\mathcal{P}_{Lh}$. Hence Theorem 5.1 shows that $F/R = F^2/f \in TLh$.

Conversely, let $F^2/f \in TLh$. Then by Theorem 5.1, $F^2/f = F_1R_1$, where $F_1 \in TLh^0$ and $R_1 \in \mathcal{P}_{Lh}$. Now $F^2/F_1 \in TLh^0$, and $1/R_1 \in \mathcal{P}_{Lh}$. Thus Theorem 5.1 shows that $f = F^2/(F_1R_1) \in TLh$. $\square$

Next, the radius of starlikeness of mappings $f \in TLh$ is determined.

Theorem 5.2. *Let $f(z) = zh(z)\overline{g(z)} \in TLh$. Then f maps the disk $|z| < 3 - 2\sqrt{2}$ onto a starlike domain.*

Proof. A function f maps the circle $|z| = r$ onto a starlike curve provided

$$\frac{\partial}{\partial \theta} \arg f(re^{i\theta}) = \operatorname{Re} \frac{zf_z - \bar{z}f_{\bar{z}}}{f} > 0.$$

Let $\varphi(z) = zh(z)g(z)$. In view of (5.3), (5.4) and (5.5), it is evident that

$$\begin{aligned}
\operatorname{Re} \frac{zf_z - \bar{z}f_{\bar{z}}}{f} &= \operatorname{Re} \left(\frac{z\varphi'(z)}{\varphi(z)} - \frac{zg'(z)}{g(z)} - \overline{\left(\frac{zg'(z)}{g(z)} \right)} \right) \\
&= \operatorname{Re} \left(\frac{z\varphi'(z)}{\varphi(z)} - \frac{2zg'(z)}{g(z)} \right) \\
&= \operatorname{Re} \left(\frac{z\varphi'(z)}{\varphi(z)} - \frac{2a(z)}{1+a(z)} \frac{z\varphi'(z)}{\varphi(z)} \right) \\
&= \operatorname{Re} \left(\frac{1-a(z)}{1+a(z)} \frac{z\varphi'(z)}{\varphi(z)} \right)
\end{aligned}$$

for some $a \in B$. Kirwan [68] has shown that the radius of starlikeness for typically real analytic functions φ is $\rho_0 = \sqrt{2} - 1$.

Now, let

$$q(z) = \frac{1-a(z)}{1+a(z)} \frac{z\varphi'(z)}{\varphi(z)},$$

and $\sigma(z) = \rho_0 z$, and whence $q(\sigma(z))$ is subordinate to $((1+z)/(1-z))^2$ in $\mathbb{D}$.

Writing $p(z) = (1+z)/(1-z)$, it follows from [48, p. 84] that

$$\left| p(z) - \frac{1+r^2}{1-r^2} \right| \leq \frac{2r}{1-r^2}.$$

Then $|\arg(p(z))| \leq \pi/4$ for $2r/(1+r^2) \leq 1/\sqrt{2}$, that is, $|\arg((1+z)/(1-z))| \leq \pi/4$ provided $|z| \leq \rho_0$, where ρ_0 is the root of the equation $r^2 - 2\sqrt{2}r + 1 = 0$. It follows that $|\arg((1+z)/(1-z))^2| < \pi/2$ for $|z| < \rho_0$. Therefore, Re $q(\sigma(z)) > 0$ for $|z| < \rho_0$, which is equivalent to Re $q(w) > 0$ for $|w| = |\rho_0 z| < \rho_0^2$. Hence $f(z) = zh(z)\overline{g(z)}$ is starlike in the disk $|z| < \rho_0^2 = 3-2\sqrt{2}$. $\square$

In the next result, an upper estimate is established for arclength of all mappings f in the class TLh.

Theorem 5.3. *Let $f(z) = zh(z)\overline{g(z)} \in TLh$ be a logharmonic mapping with respect to $a \in B$, and $|f(z)| \le M(r)$, $0 < r < 1$. Then an upper bound for its arclength $L(r)$ is given by*

$$L(r) \le 4\pi M(r)\frac{1+r+2r^2-2r^3}{(1-r)(1-r^2)}.$$

Proof. Let C_r denote the image of the circle $|z| = r < 1$ under the mapping $w = f(z)$. Then

$$\begin{aligned} L(r) = \int_{C_r} |df| \; &= \int_0^{2\pi} |zf_z - \bar{z}f_{\bar{z}}| d\theta \\ &\le M(r)\int_0^{2\pi} \left|\frac{zf_z - \bar{z}f_{\bar{z}}}{f}\right| d\theta. \end{aligned}$$

Since $\varphi(z) = zh(z)g(z) = zp(z)/(1-z^2)$ for some $p \in \mathcal{P}_{\mathbb{R}}$, it follows from (5.3), (5.4) and (5.5) that

$$\begin{aligned} \frac{zf_z - \bar{z}f_{\bar{z}}}{f} &= \frac{z\varphi'(z)}{\varphi(z)} - \frac{zg'(z)}{g(z)} - \overline{\left(\frac{zg'(z)}{g(z)}\right)} \\ &= \frac{z\varphi'(z)}{\varphi(z)} - 2\text{Re}\left(\frac{zg'(z)}{g(z)}\right) = \frac{z\varphi'(z)}{\varphi(z)} - 2\text{Re}\left(\frac{a(z)}{1+a(z)}\frac{z\varphi'(z)}{\varphi(z)}\right) \\ &= \text{Re}\left(\frac{1-a(z)}{1+a(z)}\frac{z\varphi'(z)}{\varphi(z)}\right) + i\,\text{Im}\left(\frac{z\varphi'(z)}{\varphi(z)}\right) \\ &= \text{Re}\left(\frac{1-a(z)}{1+a(z)}\left(\frac{zp'(z)}{p(z)} + \frac{1+z^2}{1-z^2}\right)\right) + i\,\text{Im}\left(\frac{zp'(z)}{p(z)} + \frac{1+z^2}{1-z^2}\right). \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{L(r)}{M(r)} \leq & \int_0^{2\pi} \left|\operatorname{Re}\left(\frac{1-a(z)}{1+a(z)}\left(\frac{zp'(z)}{p(z)}\right)\right)\right| d\theta \\ & + \int_0^{2\pi} \left|\operatorname{Re}\left(\frac{1-a(z)}{1+a(z)}\left(\frac{1+z^2}{1-z^2}\right)\right)\right| d\theta \\ & + \int_0^{2\pi} \left|\operatorname{Im} \frac{zp'(z)}{p(z)}\right| d\theta + \int_0^{2\pi} \left|\operatorname{Im} \frac{1+z^2}{1-z^2}\right| d\theta,\end{aligned}$$

that is,

$$L(r) \leq M(r)\left(I_1+I_2+I_3+I_4\right). \tag{5.9}$$

The function p is subordinate to $(1+z)/(1-z)$, and thus $zp'(z)/p(z) = 2zw'(z)/(1-w^2(z))$ for some analytic self-map w of $\mathbb{D}$ with $w(0)=0$. It also follows from the Schwarz-Pick inequality that [123, p.243]

$$\frac{|w'(z)|}{1-|w(z)|^2} \leq \frac{1}{1-|z|^2}.$$

Thus

$$\begin{aligned} I_1 &= \int_0^{2\pi} \left|\operatorname{Re}\left(\frac{1-a(z)}{1+a(z)}\left(\frac{zp'(z)}{p(z)}\right)\right)\right| d\theta \leq \int_0^{2\pi} \left|\frac{1+a(z)}{1-a(z)}\right| \left|\frac{zp'(z)}{p(z)}\right| d\theta \\ &\leq \int_0^{2\pi} \left|\frac{1+z}{1-z}\right| \left|\frac{2zw'(z)}{1-w^2(z)}\right| d\theta \leq \frac{1+|z|}{1-|z|} \int_0^{2\pi} \frac{2|z||w'(z)|}{1-|w(z)|^2} d\theta \\ &\leq \frac{1+|z|}{1-|z|} \int_0^{2\pi} \frac{2|z|}{1-|z|^2} d\theta = \frac{4\pi|z|(1+|z|)}{(1-|z|)(1-|z|^2)} = \frac{4\pi r}{(1-r)^2}. \end{aligned}$$

Since $[(1-a(z)/(1+a(z))][(1+z^2)/(1-z^2)]$ is subordinate to $((1+z)/(1-z))^2$, it follows from Parseval's theorem that

$$\begin{aligned} I_2 &= \int_0^{2\pi} \left|\operatorname{Re}\left(\frac{1-a(z)}{1+a(z)}\left(\frac{1+z^2}{1-z^2}\right)\right)\right| d\theta \\ &\leq \int_0^{2\pi} \left|\left(\frac{1+z}{1-z}\right)^2\right| d\theta \leq 2\pi\left(1+4\sum_{n=1}^{\infty} r^{2n}\right) \\ &= 2\pi\left(1+\frac{4r^2}{1-r^2}\right) = 2\pi\left(\frac{1+3r^2}{1-r^2}\right). \end{aligned}$$

Also,

$$I_3 = \int_0^{2\pi} \left| \operatorname{Im} \frac{zp'(z)}{p(z)} \right| d\theta \le \int_0^{2\pi} \left| \frac{zp'(z)}{p(z)} \right| d\theta$$
$$\le \int_0^{2\pi} \left| \frac{2zw'(z)}{1-w^2(z)} \right| d\theta \le \int_0^{2\pi} \frac{2|z|}{1-|z|^2} d\theta = \frac{4\pi r}{1-r^2}.$$

Finally,

$$I_4 = \int_0^{2\pi} \left| \operatorname{Im} \frac{1+z^2}{1-z^2} \right| d\theta \le \int_0^{2\pi} \left| \frac{1+z^2}{1-z^2} \right| d\theta$$
$$\le 2\pi \frac{1+|z|^2}{1-|z|^2} = 2\pi \frac{1+r^2}{1-r^2}.$$

Substituting the bounds for I_1, I_2, I_3 and I_4 into (5.9) yields

$$L(r) \le 2\pi M(r) \left(\frac{2r}{(1-r)^2} + \frac{2+2r+4r^2}{1-r^2} \right)$$
$$= 4\pi M(r) \frac{1+r+2r^2-2r^3}{(1-r)(1-r^2)}. \qquad \square$$

5.3 Univalent Logharmonic Mappings in The Class TLh

For an analytic univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, it is known that f is typically real if and only if the image $f(\mathbb{D})$ is a domain symmetric with respect to the real axis. However, it is not true that a univalent logharmonic mapping $F(z) = zh(z)\overline{g(z)} \in TLh$ if and only if the image of $F(\mathbb{D})$ is a symmetric domain with respect to the real axis. The following example shows a univalent logharmonic mapping $F(z) = zh(z)\overline{g(z)} \in TLh$ but $F(\mathbb{D})$ is not a symmetric domain.

Example 5.1. Let

$$F(z) = zh(z)\overline{g(z)} = z\left(1 + \frac{iz}{3}\right)\left(1 + \frac{i\overline{z}}{3}\right).$$

It is evident that $F(0) = 0$, and $h(0) = g(0) = 1$, where $h(z) = 1 + iz/3$ and $g(z) = 1 - iz/3$.

Since

$$\frac{F_z(z)}{F(z)} = \frac{3+2iz}{z(3+iz)}, \quad \text{and} \quad \overline{\left(\frac{F_{\overline{z}}(z)}{F(z)}\right)} = \overline{\left(\frac{i}{3+i\overline{z}}\right)} = \frac{-i}{3-iz},$$

it follows that

$$|a(z)| = \left| \frac{\overline{\left(\frac{F_{\overline{z}}(z)}{F(z)}\right)}}{\frac{F_z(z)}{F(z)}} \right| = \left| \frac{-iz(3+iz)}{(3-iz)(3+2iz)} \right|.$$

Furthermore,

$$|z|^2|3+iz|^2 < |3-iz|^2|3+2iz|^2,$$

and thus $|a(z)| < 1$. Hence F is logharmonic in $\mathbb{D}$ with respect to $a \in B_0$.

Let

$$\psi(z) = \frac{zh(z)}{g(z)} = \frac{z(3+iz)}{(3-iz)}.$$

Then $\psi(0) = 0$, $\psi'(0) = 1$, and

$$\frac{z\psi'(z)}{\psi(z)} = \frac{z^2+6iz+9}{z^2+9}.$$

Thus for $z = re^{i\theta} \in \mathbb{D}$,

$$\begin{aligned} \operatorname{Re}\left(\frac{z\psi'(z)}{\psi(z)}\right) &= \frac{(9-r^2)^2+36r^2\cos^2\theta - 6r(9-r^2)\sin\theta}{(9+r^2\cos 2\theta)^2 + r^4\sin^2 2\theta} \\ &> \frac{10}{(9+r^2\cos 2\theta)^2 + r^4\sin^2 2\theta} > 0. \end{aligned}$$

Hence $\psi \in \mathcal{ST}$. It follows from Lemma 5.3 that F is a univalent starlike logharmonic mapping.

Next, let $\varphi(z) = zh(z)g(z) = z\left(1+z^2/9\right)$. Evidently, for $z = x+iy$

$$\mathrm{Im}(\varphi(z)) = \frac{y}{9}\left(3x^2+9-y^2\right).$$

Then

$$\mathrm{Im}(z)\mathrm{Im}(\varphi(z)) = y^2\left(\frac{3x^2+(9-y^2)}{9}\right) > 0$$

whenever Im $(z) \neq 0$. Hence φ is typically real, and thus $F \in TLh$.

A simple calculation shows that

$$F(z) = z\left(1+\frac{2i}{3}\mathrm{Re}\, z - \frac{|z|^2}{9}\right).$$

With

$$I(t) = \mathrm{Im}\, F(e^{it}) = \frac{2\cos^2 t}{3} + \frac{8}{9}\sin t,$$

then

$$I'(t) = \frac{4}{3}\cos t\left(\frac{2}{3} - \sin t\right).$$

It follows that $I'(t) = 0$, whence $t = \pm\pi/2$ or $t = \sin^{-1}(2/3)$. Thus

$$M = \max_{|t|\leq\pi} I(t) = I\left(\sin^{-1}\frac{2}{3}\right) = \frac{2}{3}\left(\frac{\sqrt{5}}{3}\right)^2 + \frac{16}{27} = \frac{26}{27},$$

and

$$m = \min_{|t|\leq\pi} I(t) = I\left(-\frac{\pi}{2}\right) = \frac{-8}{9}.$$

Hence $F(\mathbb{D})$ is not symmetric with respect to the real axis.

Figure 5.1 shows the mapping $F(z) = z(1 + iz/3)(1 + i\bar{z}/3)$ which is not symmetric with respect to the real axis.

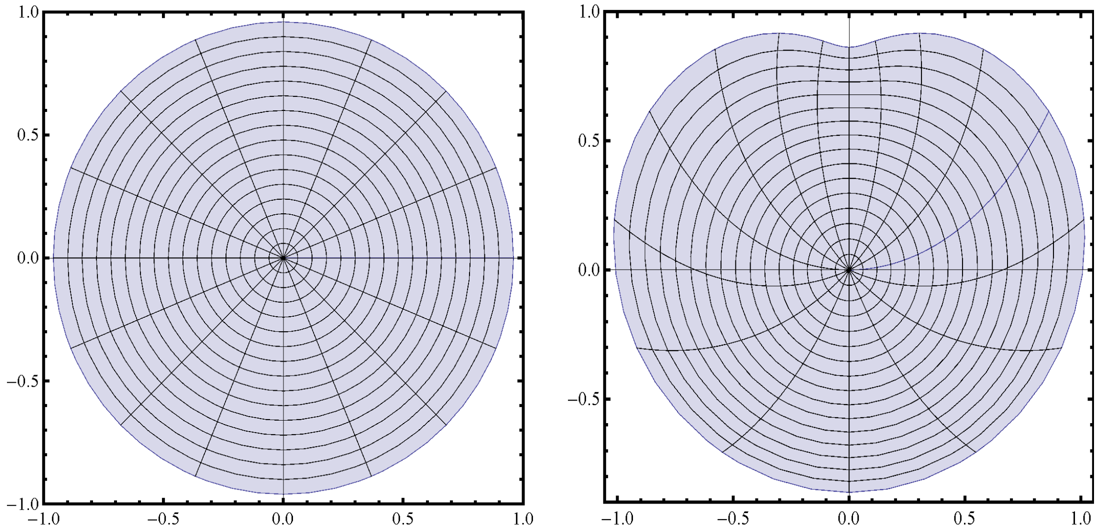

Figure 5.1: Graph of $F(z) = z(1 + \frac{iz}{3})(1 + \frac{i\bar{z}}{3})$.

Our next example illustrates a univalent logharmonic mappings from $\mathbb{D}$ onto a symmetric domain Ω which does not belong to the class TLh.

Example 5.2. Consider the function

$$F(z) = z\frac{1-\bar{z}}{1-z}\exp\left\{\mathrm{Re}\,\left(\frac{4z}{1-z}\right)\right\}.$$

Then $F(0) = 0$, and $h(0) = g(0) = 1$, where

$$h(z) = \frac{\exp\left\{\frac{2z}{1-z}\right\}}{1-z}, \quad \text{and} \quad g(z) = \exp\left\{\frac{2z}{1-z}\right\}(1-z).$$

Since

$$\frac{F_z(z)}{F(z)} = \frac{1+z}{z(1-z)^2}, \quad \text{and} \quad \overline{\left(\frac{F_{\bar{z}}(z)}{F(z)}\right)} = \overline{\left(\frac{1+\bar{z}}{(1-\bar{z})^2}\right)} = \frac{1+z}{(1-z)^2},$$

it follows that

$$|a(z)| = \left|\frac{\overline{\left(\frac{F_{\bar{z}}(z)}{F(z)}\right)}}{\frac{F_z(z)}{F(z)}}\right| = |z| < 1.$$

Thus F is logharmonic in $\mathbb{D}$ with respect to $a \in B_0$.

Let

$$\psi(z) = \frac{zh(z)}{g(z)} = \frac{z}{(1-z)^2}.$$

Then

$$\mathrm{Re}\left(\frac{z\psi'(z)}{\psi(z)}\right) = \mathrm{Re}\left(\frac{1+z}{1-z}\right) > 0.$$

Thus $\psi \in S^*$, and hence Lemma 5.3 shows that F is a univalent starlike logharmonic mapping.

It is evident that F has real coefficients , that is,

$$F(z) = z\frac{1-\overline{z}}{1-z}\exp\left\{\frac{2\overline{z}}{1-\overline{z}}\right\}\exp\left\{\frac{2z}{1-z}\right\} = \overline{F(\overline{z})}.$$

Then Lemma 5.1 shows that $F(\mathbb{D})$ is a symmetric with respect to the real axis.

Let $\varphi(z) = zh(z)g(z) = z\exp\{4z/(1-z)\}$. Then for $z_0 = (1-2/\pi) + 2i/\pi \in \mathbb{D}$,

$$\begin{aligned}
\mathrm{Re}\left(\frac{1-z_0^2}{z_0}\varphi(z_0)\right) &= \mathrm{Re}\left((1-z_0^2)\exp\left\{\frac{4z_0}{1-z_0}\right\}\right) \\
&= \mathrm{Re}\left(\frac{4}{\pi}\left(1-i\Big(1-\frac{2}{\pi}\Big)\right)\Big(-\exp\{\pi-4\}\Big)\right) \\
&= -\frac{4}{\pi}\exp\{(4-\pi)\}\,\mathrm{Re}\left(1-i\Big(1-\frac{2}{\pi}\Big)\right) \\
&= -\frac{4}{\pi}\exp\{(4-\pi)\} < 0.
\end{aligned}$$

Thus $(1-z^2)\varphi(z)/z \notin \mathcal{P}_{\mathbb{R}}$ for some $z \in \mathbb{D}$.. It follows from Lemma 5.5 that φ is not typically real, and hence $F \notin TLh$.

Figure 5.2 shows the mapping $F(z) = z\exp\{\text{Re }(4z/(1-z))\}(1-\bar{z})/(1-z)$ which does not belong to the class TLh, but yet maps $\mathbb{D}$ onto a symmetric domain with respect to the real axis $F(\mathbb{D})$.

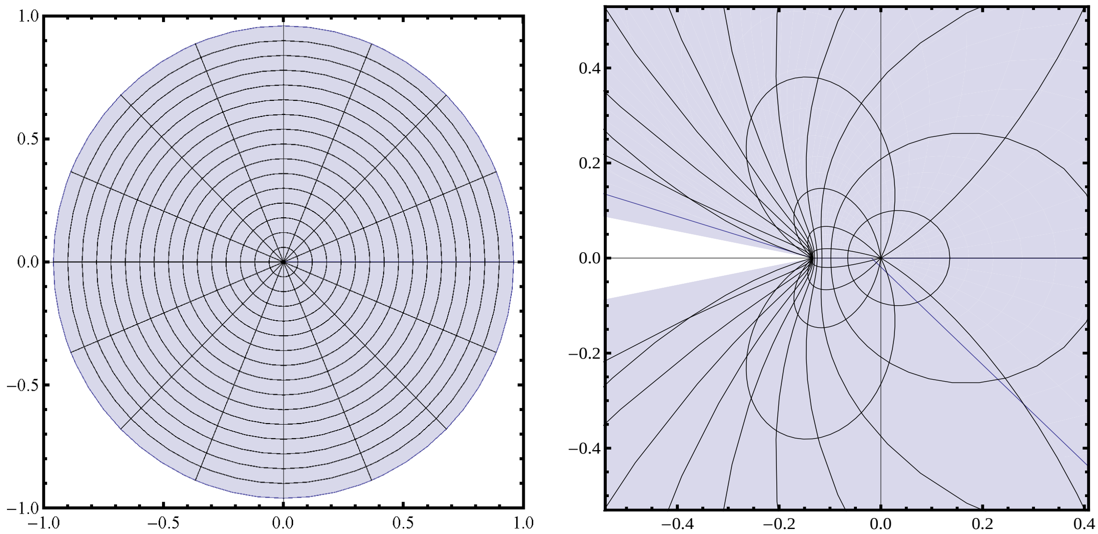

Figure 5.2: Graph of $F(z) = z\dfrac{1-\bar{z}}{1-z}\exp\left\{\text{Re }\left(\dfrac{4z}{1-z}\right)\right\}$.

The final result in this section describes when a univalent logharmonic function belongs to the class TLh.

Theorem 5.4. *Let $f(z) = zh(z)\overline{g(z)} \in \mathcal{S}_{Lh}$, and suppose that the second dilatation function a has real coefficients, that is, $\overline{a(\bar{z})} = a(z)$.*

(a) If $f \in TLh$, then $f(\mathbb{D})$ is symmetric with respect to the real axis.

(b) Suppose $f(\mathbb{D})$ is a strictly starlike Jordan domain symmetric with respect to the real axis, then $f \in TLh$ in the disk $|z| < \sqrt{2} - 1$.

Proof. (a) Let $\varphi(z) = zh(z)g(z)$ be analytically typically real. Then φ has real coefficients. It follows from (5.5) that

$$\frac{g'(z)}{g(z)} = \frac{a(z)}{1+a(z)}\frac{\varphi'(z)}{\varphi(z)},$$

which readily yields the solution g. Thus g has real coefficients. It is also evident that h has real coefficients since $h(z) = \varphi(z)/zg(z)$. Therefore, $f(z) = zh(z)\overline{g(z)}$ has real coefficients. It follows from Lemma 5.1 that $f(\mathbb{D})$ is symmetric with respect to the real axis.

(b) Suppose $F(z) = \overline{f(\overline{z})}$, and $f(z) = zh(z)\overline{g(z)}$ is a univalent logharmonic mapping with respect to $a \in B_0$. Then F is univalent in $\mathbb{D}$. Further, let $w \in F(\mathbb{D})$, that is, $w = \overline{f(\overline{z})} \in F(\mathbb{D})$. Since $f(\mathbb{D})$ is a symmetric domain with respect to the real axis, it follows that $\overline{f(z)} \in f(\mathbb{D})$ for all $f(z) \in f(\mathbb{D})$. Moreover, $\mathbb{D}$ is symmetric, and thus $\overline{f(\overline{z})} \in f(\mathbb{D})$ whenever $f(z) \in f(\mathbb{D})$. Hence $F(\mathbb{D}) \subset f(\mathbb{D})$.

Let $F(z) = zH(z)\overline{G(z)} = z\overline{h(\overline{z})}g(\overline{z})$ with $H(z) = \overline{h(\overline{z})}$ and $G(z) = \overline{g(\overline{z})}$. Then $F(0) = 0$, $H(0) = 1$, and $G(0) = 1$.

Let $a^*(z) = F\overline{F_{\overline{z}}}/F_z\overline{F}$. Then

$$a^*(z) = \frac{\overline{\left(\frac{F_{\overline{z}}(z)}{F(z)}\right)}}{\frac{F_z(z)}{F(z)}} = \frac{\overline{\left(\frac{(g(\overline{z}))_{\overline{z}}}{g(\overline{z})}\right)}}{\frac{1}{z} + \frac{\left(\overline{h(\overline{z})}\right)_z}{\overline{h(\overline{z})}}} = \frac{\frac{\left(\overline{g(\overline{z})}\right)_z}{\overline{g(\overline{z})}}}{\overline{\frac{1}{z} + \left(\frac{(h(\overline{z}))_{\overline{z}}}{h(\overline{z})}\right)}} = \overline{a(\overline{z})}.$$

Since a has real coefficients, it is evident that $a^*(z) = a(z)$. Therefore, F is a logharmonic mapping with respect to the same a. Also, $H(0) = h(0) = 1$. Then by Lemma 5.2, there is only one univalent logharmonic mapping from $\mathbb{D}$ onto $f(\mathbb{D})$ which is a solution of (5.1) normalized by $f(0) = 0$, $h(0) = 1$, and $g(0) = 1$. In other words, $f(z) = F(z) = \overline{f(\overline{z})}$, and thus f has real coefficients. Hence $\psi(z) = zh(z)/g(z) = f(z)/|g(z)|^2$ has real coefficients.

Direct calculations yield

$$\frac{\psi'(z)}{\psi(z)} = \frac{1}{z} + \frac{h'(z)}{h(z)} - \frac{g'(z)}{g(z)}. \tag{5.10}$$

Since f is a solution of (5.1), it follows that

$$\frac{g'(z)}{g(z)} = a(z)\left(\frac{1}{z} + \frac{h'(z)}{h(z)}\right). \tag{5.11}$$

Combining (5.10) and (5.11) we obtain

$$\frac{g'(z)}{g(z)} = \frac{a(z)}{1-a(z)} \frac{\psi'(z)}{\psi(z)},$$

which by integration leads to

$$g(z) = \exp \int_0^z \frac{a(t)}{1-a(t)} \frac{\psi'(t)}{\psi(t)} dt.$$

Then g, and so does h, have real coefficients, and thus $\varphi(z) = zh(z)g(z)$ also has real coefficients.

It is known in [6] that if $f_0(z) = zh_0(z)\overline{g_0(z)}$ is a starlike univalent logharmonic mapping in $\mathbb{D}$, then $\varphi_0(z) = zh_0(z)g_0(z)$ is a starlike univalent analytic in the disk $|z| < \rho = \sqrt{2}-1$. Thus $\varphi(\rho z) \in \mathcal{S}$. Furthermore, φ has real coefficients. It follows from Proposition 1.1 in Section 1.3 that φ is typically real in $|z| < \rho$. Therefore, $f \in TLh$ in $|z| < \sqrt{2}-1$. $\square$

5.4 On A Subclass of TLh

Let TLh_1 be the subclass of TLh consisting of all mappings $F(z) = z\overline{(\varphi(z)/z)}$, where $\varphi \in T$.

The following result determines necessary and sufficient conditions for a mapping $F(z) = z\overline{(\varphi(z)/z)}$ to be in the class TLh_1.

Lemma 5.7. *Let $F(z) = z\overline{(\varphi(z)/z)}$, where $\varphi \in T$. Then F is logharmonic mapping with respect to $a \in B_0$, that is, $F \in TLh_1$ if and only if $|z\varphi'(z)/\varphi(z) - 1| < 1$.*

Proof. A simple calculation shows that

$$\frac{F_z(z)}{F(z)} = \frac{1}{z}, \quad \text{and} \quad \overline{\left(\frac{F_{\bar{z}}(z)}{F(z)}\right)} = \overline{\left(\frac{\overline{(\varphi(z))_{\bar{z}}}}{\overline{\varphi(z)}} - \frac{1}{\bar{z}}\right)} = \frac{\varphi'(z)}{\varphi(z)} - \frac{1}{z}. \tag{5.12}$$

Then

$$a(z) = \frac{\overline{\left(\frac{F_{\bar{z}}(z)}{F(z)}\right)}}{\frac{F_z(z)}{F(z)}} = \frac{\dfrac{\varphi'(z)}{\varphi(z)} - \dfrac{1}{z}}{\dfrac{1}{z}} = \frac{z\varphi'(z)}{\varphi(z)} - 1. \tag{5.13}$$

and $a(0) = 0$, since $\varphi(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Hence the required result follows. $\square$

An integral representation for TLh_1 is established in the following result.

Theorem 5.5. *Let $F(z) = z\overline{(\varphi(z)/z)} \in TLh_1$ be a logharmonic with respect to $a \in B_0$. Then F admits the representation*

$$F(z) = z \exp \overline{\int_0^z \frac{a(s)}{s} ds}. \tag{5.14}$$

Proof. It is evident from (5.13) that

$$\frac{d}{dz} \log\left(\frac{\varphi(z)}{z}\right) = \frac{a(z)}{z}.$$

Thus

$$\frac{\varphi(z)}{z} = \exp \int_0^z \frac{a(s)}{s} ds,$$

yields the desired result, which completes the proof of this theorem. $\square$

Next, the radius of starlikeness of mappings in the class TLh_1 is determined.

Theorem 5.6. *Let $F(z) = z\overline{(\varphi(z)/z)} \in TLh_1$. Then F maps the disk $|z| < 1/3$ onto a starlike domain.*

Proof. It is follows from (5.12) that

$$\frac{zF_z(z)}{F(z)} = 1, \quad \text{and} \quad \frac{\bar{z}F_{\bar{z}}(z)}{F(z)} = \overline{\left(\frac{z\varphi'(z)}{\varphi(z)}\right)} - 1.$$

Therefore,

$$\operatorname{Re}\frac{zF_z(z) - \bar{z}F_{\bar{z}}(z)}{F(z)} = \operatorname{Re}\left(2 - \frac{z\varphi'(z)}{\varphi(z)}\right).$$

Since $\varphi \in T$, it follows from Lemma 5.5 that $\varphi(z) = zp(z)/(1-z^2)$, where $p \in \mathcal{P}_{\mathbb{R}}$.

Simple calculations give

$$\frac{z\varphi'(z)}{\varphi(z)} = \frac{1+z^2}{1-z^2} + \frac{zp'(z)}{p(z)}.$$

Now

$$\operatorname{Re}\left(\frac{1+z^2}{1-z^2}\right) \leq \frac{1+|z|^2}{1-|z|^2},$$

and Lemma 5.4 shows that

$$\operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right) \leq \frac{2|z|}{1-|z|^2}.$$

Therefore,

$$\operatorname{Re}\left(\frac{z\varphi'(z)}{\varphi(z)}\right) \leq \frac{1+|z|^2}{1-|z|^2} + \frac{2|z|}{1-|z|^2} = \frac{1+|z|}{1-|z|}.$$

Thus

$$\operatorname{Re}\frac{zF_z(z) - \bar{z}F_{\bar{z}}(z)}{F(z)} \geq 2 - \left(\frac{1+|z|}{1-|z|}\right) = \frac{1-3|z|}{1-|z|},$$

and hence $\operatorname{Re}\Big(\big(zF_z - \bar{z}F_{\bar{z}}\big)/F\Big) > 0$ provided $|z| < 1/3$. Thus F maps $\{z : |z| < 1/3\}$ onto a starlike domain. $\square$

CHAPTER 6

CONCLUSION

This chapter presents a summary of work that was done in this thesis. Four research problems were discussed which will motivate other researchers in this field for more intense research in the future.

The class $\mathcal{U}$ consists of all functions $f \in \mathcal{A}$ satisfying $|(z/f(z))^2 f'(z) - 1| < 1$ in the unit disk. One of the research problems investigated in this thesis is the radius problem. The sharp radius of the class $\mathcal{U}$ for several classes of functions is determined. These include the class of normalized analytic functions f satisfying the inequality $Re\, f(z)/g(z) > 0$ or $|f(z)/g(z) - 1| < 1$ in $\mathbb{D}$, where g belongs to a certain class of analytic functions. The estimation for the $\mathcal{U}$-radius of the class of functions f satisfying the inequality $|f'(z) - 1| < 1$ or $\operatorname{Re} f(z)/z > \alpha$, $0 \leq \alpha < 1$, in $\mathbb{D}$ is obtained. Furthermore, this thesis validates the conjecture of Obradović and Ponnusamy concerning the radius of univalence for product involving univalent functions. A good continuation to the work done here would be to consider the class $\mathcal{U}(\lambda)$ of all functions $f \in \mathcal{A}$ satisfying $|(z/f(z))^2 f'(z) - 1| < \lambda$ in the unit disk, where $\lambda > 0$, and to investigate the $\mathcal{U}(\lambda)$-radius for various classes of normalized analytic functions.

Various interesting properties including coefficient bounds and coefficient inequalities for several subclasses of analytic functions have been investigated. The technique used by Ma and Minda for the Fekete-Szego problem for subclasses of convex and starlike functions was used by many authors to solve the same problem for other classes. This technique was also used to solve Hankel determinant problem for subclasses of analytic functions. Bounds for the second Hankel determinant $H_2(2) = b_{k+1}b_{3k+1} - b_{2k+1}^2$ for the kth-root transform $F(z) = \left(f\left(z^k\right)\right)^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1}z^{kn+1}$ of Ma-Minda starlike and convex functions are determined in this thesis. Similar problems are also treated for related classes defined by subordination. The results are derived through several meticulous lengthy computations. Thus in several instances, these computations were validated by use of the computer algebra system Mathematica. These bounds are expressed in terms of the coefficients of the given function φ, and thus connect with earlier known results for particular choices of φ. Another possible area for research is to investigate the second Hankel determinant for the kth-root transform of the class $\mathscr{U}$.

This thesis also obtained sufficient conditions for the function $F(z) = f(z)|f(z)|^{2\gamma}$ to be $\alpha-$spirallike logharmonic mapping. By taking product combination of the two mappings possessing the given property, a new logharmoinc mapping with a specified property is constructed. In particular, if $f_1(z) = zh_1(z)\overline{g_1(z)}$, and $f_2(z) = zh_2(z)\overline{g_2(z)}$ are univalent starlike logharmonic with respect to the same $a \in B_0$, a new univalent starlike logharmonic mapping $F(z) = f_1^{\lambda}(z)f_2^{1-\lambda}(z),\ 0 \leq \lambda \leq 1$, with respect to the same a is established. In addition, if $f_1(z) = zh_1(z)\overline{g_1(z)}$ is a logharmonic with respect to $a_1 \in B_0$, and $f_2(z) = zh_2(z)\overline{g_2(z)}$ is a logharmonic with respect to $a_2 \in B_0$, sufficient conditions are obtained to ensure their product $F(z) = f_1^{\lambda}(z)f_2^{1-\lambda}(z),\ 0 \leq \lambda \leq 1$ is a univalent starlike logharmonic mapping with respect to $\mu \in B_0$. Several examples of univalent starlike logharmonic mapping constructed from the product are provided.

The thesis concludes by considering the class TLh of all normalized logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ satisfying $\varphi(z) = zh(z)g(z) \in HG$ is typically real analytic in the unit disk. An integral representation for such a mapping f is obtained. The connection between this class and the class of logharmonic mapping with positive real part is established. The radius of starlikeness for the class TLh, as well as an upper estimate for its arclength are determined. Moreover, sufficient and necessary geometric conditions for $\varphi(z) = zh(z)g(z)$ to be typically real are also investigated when $f(z) = zh(z)\overline{g(z)}$ has a dilatation with real coefficients. Furthermore, an integral representation and the radius of starlikeness for a subclass of TLh are determined.

Notably, the study of logharmonic mappings is a rich area of research. In this thesis, we tried to highlight and solve some of the problems related to logharmonic mappings. However, there are always some open problems which can be considered in further work. For example, finding the radius of starlikeness of mappings in the class $\mathcal{S}_{Lh}$ and the sharp radius of starlikeness of mappings in the class TLh.

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