BOHR'S INEQUALITY AND ITS EXTENSIONS

NG ZHEN CHUAN

UNIVERSITI SAINS MALAYSIA

2017

BOHR'S INEQUALITY AND ITS EXTENSIONS

by

NG ZHEN CHUAN

**Thesis submitted in fulfilment of the requirements
for the degree of
Doctor of Philosophy**

November 2017

ACKNOWLEDGEMENT

First and foremost I offer my sincerest gratitude to my supervisor, Prof. Dato' Indera Dr. Rosihan M. Ali who has supported me throughout my doctoral study with his patience and knowledge whilst allowing me the room to work in my own way. His thorough checking of my work including the correction of my grammatical mistakes are greatly appreciated.

I am also deeply grateful to my field supervisor, Prof. Dr. Yusuf Abu Muhanna, for his valuable suggestions, encouragement and guidance throughout my study. Being an expert in my research area, he is able to view things differently and provide new ideas towards my writing.

Thanks are due as well to the Complex Function Theory Group in School of Mathematical Sciences, Universiti Sains Malaysia. The seminars held in the school are very inspiring; in particular, those given by invited speakers which have always been the sources of my research motivation.

My sincere thanks also go to the entire staff of the School of Mathematical Sciences USM and the authorities of USM for providing excellent facilities and research environment to me. Moreover, they have always been a great help in handling matters related to my candidature.

The MyPhD scholarship under MyBrain15 programme awarded by the Ministry of Higher Education Malaysia is gratefully acknowledged.

Last but not least, I would like to thank my parents for giving birth to me in the first place and supporting me spiritually throughout my life.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

$\Delta_{x,y}$	Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, 7
λ	spherical chordal distance, 18
λ_U	density of hyperbolic metric on U, 13
λ_{U_0}	density of hyperbolic metric on U_0, 56
∂U	unit circle $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$, 15
∂U^+	$\left\{e^{it} : -\frac{\pi}{2} \le t \le \frac{\pi}{2}\right\} \cup \{iy : -1 \le y \le 1\}$, 64
$\frac{\partial}{\partial z}$	$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, 8
$\frac{\partial}{\partial \bar{z}}$	$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, 8
$\prec$	subordinate to, 6
arg	argument function, 19
$B(G)$	second Bohr radius on domain $G \subset \mathbb{C}^n$, 109
$\mathbb{C}$	complex plane, 1
$\overline{\mathbb{C}}$	extended complex plane $\mathbb{C} \cup \{\infty\}$, 114
$\mathbb{C}^n$	n-fold Cartesian product of $\mathbb{C}$, 107
cos	cosine function, 73
d	Euclidean distance, 15
d_U	hyperbolic distance on U, 13
$d_{U_{1/3}}$	hyperbolic distance on $U_{1/3}$, 58
d_{U_0}	hyperbolic distance on U_0, 57

$\overline{D}_{min}$ smallest closed disk containing the closure of D, 82

exp exponential function, 31

$f_{z\bar{z}}$ $\frac{\partial^2 f}{\partial z \partial \bar{z}}$, 8

$\|f\|_\infty$ $\sup_{|z|<1} |f(z)|$, 16

$\|f\|_G$ $\sup_{z\in G} |f(z)|$, 111

$f * g$ Hadamard product (or convolution) of f and g, 22

$\mathscr{H}$ right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, 6

$H(G)$ class of all analytic functions on some domain G, 113

$H^\infty(G)$ class of all bounded analytic functions on some domain G, 111

$H(U)$ class of all analytic functions on unit disk U, 1

$H(U_r)$ class of all analytic functions on disk U_r, 114

$H(G,D)$ class of of all analytic functions $f : G \to D$, 122

$H(U,U)$ class of of all analytic self-map on unit disk U, 6

$H(U,U_0)$ class of of all analytic functions $f : U \to U_0$, 36

$H(U,U^+)$ class of of all analytic functions $f : U \to U^+$, 61

$H(U,U^h)$ class of of all analytic functions $f : U \to U^h$, 69

$H(U,W_\alpha)$ class of of all analytic functions $f : U \to W_\alpha$, 30

$H(U^h,U^h)$ class of of all analytic self-map on U^h, 61

$H(U^h,U_q)$ class of of all analytic functions $f : U^h \to U_q$, 60

$H^\infty(U,X)$ class of of all bounded analytic functions $f : U \to X$, 116

inf infimum function, 14

J_f Jacobian of f, 9

$\mathcal{K}$ $\{f \in S : f(U) \text{ is a convex domain}\}$, 5

$K(G)$ first Bohr radius on domain $G \subset \mathbb{C}^n$, 107

$K_n(v,\lambda)$ λ-Bohr radius of v, 117

log logarithmic function, 11

min minimum function, 46

$\mathcal{M}$ majorant function, 16

$\mathbb{N}$ set of natural numbers, 117

$\|\cdot\|$ norm in Banach space, 108

P Poisson kernel, 87

$\mathscr{P}_{LH}$ $\left\{f(z) = h(z)\overline{g(z)} : f \text{ is logharmonic in } U \text{ and } \operatorname{Re} f(z) > 0 \text{ for } z \in U\right\}$, 10

$\mathscr{P}_{LH(M)}$ $\left\{f \in \mathscr{P}_{LH} : \left|\frac{h(z)}{g(z)} - M\right| < M, M \geq 1\right\}$, 10

$p_U(z,w)$ pseudo-hyperbolic distance between z and w, 14

$\mathbb{R}$ set of real numbers, 3

r_h $\tanh(1/2)$, 60

$R(\alpha,\gamma,h)$ $\left\{f \in H(U) : f(z) + \alpha z f'(z) + \gamma z^2 f''(z) \prec h(z), \quad z \in U\right\}$, 22

$R(\alpha,h)$ $\{f \in H(U) : f'(z) + \alpha z f''(z) \prec h(z), \quad z \in U\}$, 22

$s(x)$ $\sin x/x$, 88

sin sine function, 88

$\mathcal{S}$ $\{f \in H(U) : f \text{ is univalent}, f(0) = 0, f'(0) = 1\}$, 3

$\mathcal{S}^*$ $\{f \in S : f(U) \text{ is a starlike domain with respect to } 0\}$, 4

S_{Lh} $\left\{f = zh(z)\overline{g(z)} : f \text{ is univalent logharmonic in } U, h(0) = g(0) = 1\right\}$, 93

ST^0_{Lh} $\{f \in S_{Lh} : f(U) \text{ is starlike with respect to } 0\}$, 93

$\mathscr{S}$ the strip $\{z \in \mathbb{C} : |\mathrm{Re}\, z| < \rho\}$, 83

S_W class of all univalent, harmonic, orientation-preserving mappings $f : U \to W$ with normalization $f(0) = 1$, 87

$\overline{S_W}$ closure of S_W, 87

$S(f)$ $\{g \in H(U) : g \prec f\}$, 82

$S(h)$ $\{f \in H(U) : f \prec h\}$, 23

sup supremum function, 15

tanh hyperbolic tangent, 60

U unit disk $\{z \in \mathbb{C} : |z| < 1\}$, 1

U_0 unit disk punctured at the origin $U \backslash \{0\}$, 36

U^+ unit semi-disk $\{z \in U : \mathrm{Re}\, z > 0\}$, 61

U^h hyperbolic unit disk $\{z \in U : d_U(0, z) < 1\}$, 60

$\overline{U}$ closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$, 114

U_r $\{z \in \mathbb{C} : |z| < r\}$, 114

$\overline{U}_r$ $\{z \in \mathbb{C} : |z| \leq r\}$, 36

U_{r_h} $\{z \in \mathbb{C} : |z| < r_h\}$, 60

U^n $\{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_k| < 1\}$, 107

W $\left\{w \in \mathbb{C} : |\arg w| < \frac{\pi}{4}\right\}$, 87

W_α $\left\{w \in \mathbb{C} : |\arg w| < \frac{\alpha\pi}{2}\right\}$, 30

KETAKSAMAAN BOHR DAN PELANJUTAN

ABSTRAK

Jika $f(z) = \sum_{n=0}^{\infty} a_n z^n$ merupakan peta diri analisis pada unit cakera U, maka $d(\sum_{n=0}^{\infty} |a_n z^n|, |a_0|) \leq d(a_0, \partial U)$ bagi $|z| \leq 1/3$, dengan d menandakan jarak Euklidan dan ∂U bulatan unit. Pernyataan ini disebut sebagai Teorem Bohr, yang dibuktikan oleh Harald Bohr pada tahun 1914. Tesis ini memberi tumpuan kepada pengitlakan Teorem Bohr. Andaikan h sebagai fungsi univalen yang tertakrif pada U. Andaikan juga $R(\alpha, \gamma, h)$ sebagai kelas fungsi f analisis dalam U dengan $f(z) + \alpha z f'(z) + \gamma z^2 f''(z)$ yang tersubordinasi kepada $h(z)$. Teorem Bohr bagi kelas $R(\alpha, \gamma, h)$ diperoleh untuk h suatu fungsi cembung dan fungsi berbintang terhadap $h(0)$. Teorem Bohr untuk kelas fungsi analisis yang memeta U ke domain cekung dan juga ke domain cakera unit berliang diperoleh dalam bab yang seterusnya. Jejari klasik Bohr $1/3$ ditunjukkan tak berubah apabila jarak Euclidean digantikan sama ada dengan jarak sentuhan sfera atau dengan jarak model cakera Poincaré. Tambahan lagi, teorem Bohr untuk set cembung Euklidan ditunjukkan mempunyai analog dalam model cakera Poincaré. Akhirnya, Teorem Bohr diperoleh untuk beberapa subkelas pemetaan harmonik dan logharmonik yang tertakrif pada unit cakera U.

BOHR'S INEQUALITY AND ITS EXTENSIONS

ABSTRACT

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic self-map defined on the unit disk U, then $d(\sum_{n=0}^{\infty} |a_n z^n|, |a_0|) \leq d(a_0, \partial U)$ for $|z| \leq 1/3$, where d denote the Euclidean distance and ∂U the unit circle. The result is known as the Bohr's theorem which was proved by Harald Bohr in 1914. This thesis focuses on generalizing the Bohr's theorem. Let h be a univalent function defined on U. Also, let $R(\alpha, \gamma, h)$ be the class of functions f analytic in U such that the differential $f(z) + \alpha z f'(z) + \gamma z^2 f''(z)$ is subordinate to $h(z)$. The Bohr's theorems for the class $R(\alpha, \gamma, h)$ are proved for h being a convex function and a starlike function with respect to $h(0)$. The Bohr's theorems for the class of analytic functions mapping U into concave wedges and punctured unit disk are next obtained in the following chapter. The classical Bohr radius $1/3$ is shown to be invariant by replacing the Euclidean distance d with either the spherical chordal distance or the distance in Poincaré disk model. Also, the Bohr's theorem for any Euclidean convex set is shown to have its analogous version in the Poincaré disk model. Finally, the Bohr's theorems are obtained for some subclasses of harmonic and logharmonic mappings defined on the unit disk U.

CHAPTER 1

INTRODUCTION

1.1 Analytic Functions

Let $\mathbb{C}$ be the complex plane and $U := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk. Let f be a function on U and $z_0 \in U$. We say that f is differentiable at z_0 if the derivative of f at z_0 given by

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists. If f is differentiable at every point of U, then f is said to be analytic in U since U is an open set. Let $H(U)$ denote the class of all analytic functions defined on U. By using the Cauchy integral formula, it can be shown that if $f \in H(U)$, then f is represented by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in U, \tag{1.1}$$

where

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta, \quad n \geq 0,$$

for any fixed r, $0 < r < 1$.

Write $U = \cup_{n=0}^{\infty} K_n$ where $K_0 = \{0\}$ and $K_n = \{z : |z| \leq r_n < 1\}$ for $n \geq 1$ where $(r_n)_{n \geq 1}$ is a strictly increasing sequence of positive real numbers such that $r_n \to 1$ as $n \to \infty$. The space $H(U)$ can be made into a complete metric space by defining the metric on $H(U)$ as

$$\rho(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_n}{1+\|f-g\|_n}, \quad f,g \in H(U),$$

where $\|f-g\|_n = \sup_{z\in K_n} |f(z)-g(z)|$. The topology on $H(U)$ given by the metric ρ is then equivalent to the topology of uniform convergence on compact subsets of U (see [14, p. 221]). Finally, it follows from theorems of Weiestrass and Montel that this space is complete [76, p. 38].

1.2 Univalent Functions

An analytic function f is said to be univalent in a domain D if $f(z) \neq f(w)$ whenever $z \neq w$ for all $z, w \in D$. In particular, f is locally univalent at a point $z_0 \in D$ if it is univalent in some neighborhood of z_0. The existence of a unique analytic function which maps U conformally onto any simply connected domain strictly contained in $\mathbb{C}$ follows from the Riemann Mapping Theorem:

Theorem 1.1. *[14, p. 230] (see also [69, p. 11]) Given any simply connected domain D which is not the whole plane, and a point $z_0 \in D$, there exists a unique analytic function f in D, normalized by the conditions $f(z_0) = 0$ and $f'(z_0) > 0$, such that f defines a one-to-one mapping of D onto the unit disk U.*

As a consequence of this theorem, the study of analytic univalent functions on a simply connected domain D can now be reduced to the study of analytic univalent functions on the unit disk U.

The post-composition of a univalent function with the affine map $\alpha z + \beta$ defined on $\mathbb{C}$, $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$, is again a univalent function. Thus, the study of analytic univalent functions can be further restricted to the class $\mathcal{S}$ which consists of all analytic univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$. The *Koebe function*

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n, \quad z \in U$$

is a function in $\mathcal{S}$ which maps U conformally onto $\mathbb{C} \backslash (-\infty, -1/4]$. Indeed, the Koebe function and its rotations $e^{-it} k(e^{it} z)$, $t \in \mathbb{R}$, appear as extremal functions for various research problems arisen in exploring the class $\mathcal{S}$.

One such problem is to determine the maximum value of $|a_n|$ in $\mathcal{S}$ for $n \geq 2$. This is a well-defined problem as $\mathcal{S}$ is a compact subset of $H(U)$ (see [76, Theorem 4.1]) and the function $J(f) = a_n$ defined on $\mathcal{S}$ has a maximum modulus, that is, there exists a $f_0 \in \mathcal{S}$ such that $|J(f)| \leq |J(f_0)|$ for all f (see [76, Theorem 4.2]). In 1916, Bieberbarch[33] obtained the estimate for a_2:

Theorem 1.2. *(Bieberbarch Theorem)[69, Theorem 2.2] If $f \in \mathcal{S}$, then $|a_2| \leq 2$, with equality if and only if f is a rotation of the Koebe function.*

In the same paper, Bieberbarch made a conjecture:

Theorem 1.3. *(Bieberbarch Conjecture)[69, p. 37] If $f \in \mathcal{S}$, then $|a_n| \leq n$, with equality if and only if f is a rotation of the Koebe function.*

The Bieberbarch theorem is applied to prove theorems regarding the class $\mathcal{S}$ such as the Koebe one-quater theorem [69, Theorem 2.3], the distortion theorem [69, Theorem 2.5] and the growth theorem [69, Theorem 2.6]. Consequently, the researchers reckoned that the Bieberbarch conjecture is true because of the extremal role played by Koebe function (and its rotations) in those theorems. A proof of Bieberbarch conjecture was eventually given by Louis de Branges [48] in 1985.

1.2.1 Starlike and Convex Functions

In the effort of validating the Bieberbarch Conjecture, researchers considered certain subclasses of $\mathcal{S}$ which are determined by natural geometric conditions.

A domain D is called *a starlike domain with respect to* $w_0 \in D$ if $tw + (1-t)w_0 \in D$ whenever $w \in D$ for all $0 \leq t \leq 1$. A univalent function f in U is called *a starlike function with respect to* $w_0 \in f(U)$ if $f(U)$ is a starlike domain with respect to w_0. In particular, if $w_0 = 0$, then f is known as a starlike function. Let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ which consists of starlike functions. An analytic characterization of $\mathcal{S}^*$ is given as follows.

Theorem 1.4. *[76, Theorem 2.2] A function $f \in \mathcal{S}^*$ if and only if $f \in \mathcal{S}$ and*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in U.$$

Since $\mathcal{S}^*$ contains the Koebe function and it is a compact subset of $H(U)$ (see [76, Theorem 4.1]), it can be proved that the Bieberbarch's Conjecture is true for the subclass $\mathcal{S}^*$ (see [76, Theorem 2.4]).

Another kind of function which is closely related to the starlike function is the convex function. A univalent function f in U is called *a convex function* if $f(U)$ is a *convex domain*, that is, $tw_1 + (1-t)w_2 \in f(U)$ for all $w_1, w_2 \in f(U)$ and $0 \leq t \leq 1$. Let $\mathcal{K}$ denote the subclass of $\mathcal{S}$ which consists of convex functions. Similarly, an analytic characterization of $\mathcal{K}$ is given by

Theorem 1.5. *[76, Theorem 2.6] A function $f \in \mathcal{K}$ if and only if $f \in \mathcal{S}$ and*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in U.$$

A close connection between classes $\mathcal{S}^*$ and $\mathcal{K}$ is shown in Alexander's theorem [69, Theorem 2.12] which states that $f \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$. The relation is then applied to deduce the coefficient bounds from the previously known coefficient bounds of $\mathcal{S}^*$ giving $|a_n| \leq 1$, $n \geq 2$ for all $f \in \mathcal{K}$.

1.3 Differential Subordinations

The famous Noshiro-Warschawski theorem states that if f is analytic in a convex domain D and

$$\operatorname{Re} f'(z) > 0, \quad z \in U,$$

then f is univalent in D (see [69, Theorem 2.16]). This theorem suggests the characterization of an analytic function through its derivative which is a type of the differential implications [95, p. 1]. Another example is the lemma proved by Miller, Mocanu and Reade [96]: if α is real and $p \in H(U)$ such that

$$\operatorname{Re}\left[p(z) + \alpha \frac{zp'(z)}{p(z)}\right] > 0 \quad \text{for all } z \in U,$$

then $\operatorname{Re} p(z) > 0$. Let $\mathscr{H} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ denote the right half-plane. In other words, if

$$p(z) + \alpha \frac{zp'(z)}{p(z)} \in \mathscr{H} \quad \text{for all } z \in U,$$

then $p(U) \subseteq \mathscr{H}$.

Let $H(U,U)$ denote the class of of all analytic self-map on U. Before making any further progress, recall that for functions $f, g \in H(U)$, g is said to be subordinate to f, written $g \prec f$, if $g = f \circ \phi$ for some $\phi \in H(U,U)$ with $\phi(0) = 0$. Further, if f is univalent in U, then $g \prec f$ if $g(0) = f(0)$ and $g(U) \subseteq f(U)$. Miller and Mocanu [95, p. 3] introduced the notion of differential subordination, which is the complex analogue of differential inequality by replacing the real variable concept with the theory of subordination.

Let Ω and Δ be sets in $\mathbb{C}$, let $p \in H(U)$ with $p(0) = a$ for some constant $a \in \mathbb{C}$ and let $\psi(r,s,t;z) : \mathbb{C}^3 \times U \to \mathbb{C}$. Then the following relation

$$\left\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in U\right\} \subset \Omega \quad \Rightarrow \quad p(U) \subset \Delta, \tag{1.2}$$

is a general formulation of function characterization. There are three problems that can be stated based on the inclusion (1.2).

(i) Given Ω and Δ, find the condition on ψ so that (1.2) holds. Such a ψ is called an admissible function.

(ii) Given ψ and Ω, find the smallest Δ so that (1.2) holds.

(iii) Given ψ and Δ, find the largest Ω so that (1.2) holds.

If Ω is a simply connected domain and $\Omega \neq \mathbb{C}$, then the Riemann mapping theorem ensures the existence of a unique conformal mapping h of U onto Ω such that $h(0) = \psi(a,0,0;0)$. Further, if $\psi(p(z),zp'(z),z^2p''(z);z) \in H(U)$, then in terms of subordination, (1.2) can be rewritten as

$$\psi(p(z),zp'(z),z^2p''(z);z) \prec h(z) \Rightarrow p(U) \subset \Delta.$$

If p is analytic in U, then p is called a *solution* of the (second-order) differential subordination. Further, if q is conformal mapping of U onto Δ such that $q(0) = a$, then (1.2) becomes

$$\psi(p(z),zp'(z),z^2p''(z);z) \prec h(z) \Rightarrow p(z) \prec q(z),$$

and the univalent function q is called a *dominant* if $p \prec q$ for all solutions p. Also, the *best dominant* $\tilde{q}$ is the dominant such that $\tilde{q} \prec q$ for all dominants q (see [95, p. 16]). The monograph [61] by Milller and Mocanu and references therein are excellent resources for the study on differential subordination.

1.4 Harmonic Mappings

Recall that a real-valued function $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$, with continuous second partial derivatives, is (real) harmonic if it satisfies Laplace's equation:

$$\Delta_{x,y}\, u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A complex-valued function $f(x,y) = u(x,y) + iv(x,y)$ is harmonic if both u and v are (real) harmonic. Write $z = x + iy$. The Wirtinger derivatives (differential operators) are defined as follows:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then for a complex-valued function f, f is harmonic if

$$f_{z\bar{z}} = \frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{1}{4}\Delta_{x,y} f = 0.$$

If f is a complex-valued harmonic function defined on a simply connected domain $D \subset \mathbb{C}$, then f can be expressed as

$$f(z) = h(z) + \overline{g(z)} = a_0 + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{m=1}^{\infty} b_m z^m},$$

where h and g are analytic in D. If D is the unit disk U and $h(0) = f(0)$, then the representation is unique and is called the *canonical representation* of f (see [66, p. 7]). The Jacobian of f is given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2.$$

It is well known that (see [66, p. 2] or [92]) a complex-valued harmonic function f is locally one-to-one in D if and only if J_f is nonvanishing in D. Further, if $J_f > 0$ in D then f is said to be *locally univalent* in D, that is, locally one-to-one and sense preserving in D. A complex-valued harmonic function f is said to be *univalent* in D if f is one-to-one and sense preserving in D.

A complex-valued harmonic function can also be viewed as a solution to a partial differential equation as stated in the following result:

Theorem 1.6. *([81, Lemma 2.1]) A complex valued function f defined in a domain D is open, harmonic and sense preserving in D if and only if there is an $a \in H(U,U)$ such that f is a non-constant solution of*

$$\overline{\left(\frac{\partial f}{\partial \bar{z}}\right)} = a \frac{\partial f}{\partial z}.$$

The theory of complex-valued harmonic functions serves as an active research area which can be seen from [11, 46, 66, 67, 68, 80, 81] as such mappings are closely related to the theory of minimal surfaces (see [99, 100]).

Throughout this thesis, we shall use the term harmonic function to indicate a complex-valued harmonic function.

1.5 Logharmonic Mappings

A logharmonic mapping defined in U is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{f_{\bar{z}}}}{\overline{f}} = a \frac{f_z}{f},$$

where $a \in H(U,U)$ is called the second dilatation function. Thus the Jacobian

$$J_f = |f_z|^2 \left(1 - |a|^2\right)$$

is positive and all non-constant logharmonic mappings are therefore sense-preserving and open in U. In [44], the class of locally univalent logharmonic mappings is shown to play an instrumental role in validating the Iwaniec conjecture involving the Beurling-Ahlfors operator.

When f is a nonvanishing logharmonic mapping in U, it is known that f can be expressed as

$$f(z) = h(z)\overline{g(z)}, \tag{1.3}$$

where h and g are in $H(U)$. In [94], Mao *et al.* introduced the Schwarzian derivative for these nonvanishing logharmonic mappings. They established the Schwarz lemma for this class and obtained two versions of Landau's theorem. Denote by $\mathscr{P}_{LH}$ the class consisting of logharmonic mappings f in U of the form (1.3) satisfying $\operatorname{Re} f(z) > 0$ for all $z \in U$. The subclass $\mathscr{P}_{LH(M)}$ defined by

$$\mathscr{P}_{LH(M)} = \left\{ f : f = h(z)\overline{g(z)} \in \mathscr{P}_{LH}, \left| \frac{h(z)}{g(z)} - M \right| < M, M \geq 1 \right\}$$

was recently investigated in [101].

If f is a non-constant logharmonic mapping of U which vanishes only at $z = 0$, then [2] f admits the representation

$$f(z) = z^m |z|^{2\beta m} h(z)\overline{g(z)}, \tag{1.4}$$

where m is a nonnegative integer, $\operatorname{Re}\beta > -1/2$, and h and g are analytic functions on U satisfying $g(0) = 1$ and $h(0) \neq 0$. The exponent β in (1.4) depends only on $a(0)$ and can be expressed by

$$\beta = \overline{a(0)}\frac{1+a(0)}{1-|a(0)|^2}.$$

Note that $f(0) \neq 0$ if and only if $m = 0$, and that a univalent logharmonic mapping in U vanishes at the origin if and only if $m = 1$, that is, f has the form

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)}, \quad z \in U,$$

where $\operatorname{Re}\beta > -1/2$, $0 \notin (hg)(U)$ and $g(0) = 1$. This class has been widely studied in the works of [1, 2, 3, 4, 5]. In this case, it follows that $F(\zeta) = \log f(e^{\zeta})$ are univalent harmonic mappings of the half-plane $\{\zeta : \operatorname{Re}\zeta < 0\}$.

1.6 Spherical Chordal Distance

Let $\mathbb{S}$ denote the unit sphere $\{Z = (Z_1, Z_2, Z_3) \in \mathbb{R}^3 : |Z|^2 = 1\}$ and $N = (0,0,1)$ be its north pole. Then every point in the complex plane $\mathbb{C}$ corresponds to an unique point on $\mathbb{S}\backslash\{N\}$ via stereographic projection from N. Let $L_z(t) = (tx, ty, 1-t)$ be the line segment connecting N and $z = x+iy \in \mathbb{C}$ with coordinate (x,y) in the xy-plane. Note that L_z intersects $\mathbb{S}$ at a unique point Z indicating t satisfies the equation

$$(tx)^2 + (ty)^2 + (1-t)^2 = 1.$$

Thus $t = 2/(1+|z|^2)$ giving

$$Z = \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2}\right) = \left(\frac{z+\bar{z}}{1+|z|^2}, \frac{z-\bar{z}}{i(1+|z|^2)}, \frac{|z|^2-1}{1+|z|^2}\right).$$

Discussions on conformality and circles preserving properties of the stereographic projection can be found in [64, Problem 75]. The Euclidean distance between points Z and W on $\mathbb{S}$ is known as the *spherical chordal distance* between z and w, denoted by $\lambda(w,z)$, where

$$\lambda^2(Z,W) = (Z_1-W_1)^2+(Z_2-W_2)^2+(Z_3-W_3)^2 = 2-2(Z_1W_1+Z_2W_2+Z_3W_3).$$

If Z and W are the stereographic projections of z and w in $\mathbb{C}$ respectively, then

$$\begin{aligned}
Z_1W_1+Z_2W_2+Z_3W_3 =& \frac{(z+\bar{z})(w+\overline{w})-(z-\bar{z})(w-\overline{w})+(|z|^2-1)(|w|^2-1)}{(1+|z|^2)(1+|w|^2)}\\
=& \frac{2(z\overline{w}+\bar{z}w)+|zw|^2-|z|^2-|w|^2+1}{(1+|z|^2)(1+|w|^2)}\\
=& \frac{2(z\overline{w}+\bar{z}w)+(1+|z|^2)(1+|w|^2)-2|z|^2-2|w|^2}{(1+|z|^2)(1+|w|^2)}\\
=& \frac{(1+|z|^2)(1+|w|^2)-2|z-w|^2}{(1+|z|^2)(1+|w|^2)}.
\end{aligned}$$

Thus

$$\lambda(z,w) = \frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}.$$

1.7 Poincaré Disk Model

Recall the classical Schwarz's Lemma:

Theorem 1.7. *[60, p. 4](see also [85, Theorem 2.1]) Let f be an analytic self-map of U. If $f(0)=0$, then $|f(z)|\leq|z|$ for all $z\in U$ and $|f'(0)|\leq 1$. Further, if $|f(z_0)|=|z_0|$ for some $z_0\in U\backslash\{0\}$, or if $|f'(0)|=1$, then $f(z)=e^{i\theta}z$ for some constant $\theta\in\mathbb{R}$.*

A generalization of Schwarz's Lemma was presented by Pick [106], which is known as the Schwarz-Pick Lemma:

Theorem 1.8. *[60, p. 5](see also [85, Theorem 2.3]) If f is an analytic self-map of U, then*

(i)

$$\left|\frac{f(z)-f(w)}{1-f(z)\overline{f(w)}}\right| \leq \left|\frac{z-w}{1-z\overline{w}}\right| \quad \textit{for all } z,w \in U;$$

(ii)

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} \quad \textit{for all } z,w \in U.$$

Equality occurs in both (i) and (ii) if f is an conformal automorphism of U. If equality holds in (i) for one pair of points $z \neq w$ or if equality holds in (ii) at one point z, then f is a conformal automorphism of U.

The unit disk U with the hyperbolic metric (see [31])

$$\lambda_U(z)|dz| = \frac{2|dz|}{1-|z|^2},$$

is known as the Poincaré disk model. By (ii), the metric $\lambda_U(z)|dz|$ is invariant under conformal automorphism of U and induces a distance function d_U on U by

$$d_U(z,w) = \inf_{\gamma} \int_{\gamma} \lambda_U(z)\,|dz|$$

over all smooth curves γ in U joining z to w. Similar to the invariance of Euclidean distance under rotation and translation in $\mathbb{C}$, d_U is invariant under conformal automorphism of U. It was shown in [31, Theorem 2.2] that

$$d_U(z,w) = \log \frac{1+p_U(z,w)}{1-p_U(z,w)} = 2\tanh^{-1} p_U(z,w),$$

where the pseudo-hyperbolic distance $p_U(z,w)$ is given by

$$p_U(z,w) = \left| \frac{z-w}{1-z\overline{w}} \right|.$$

1.8 Bohr's inequality

A series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$ is an ordinary Dirichlet series, where $a_n, s \in \mathbb{C}$. Now, if the series converges for some $s_0 = \sigma_0 + it_0$, then it is convergent for all $s = \sigma + it$ with $\sigma > \sigma_0$ (see [79, Theorem 1]). Thus, the maximal domain of convergence is exactly the half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma_c\}$ where

$$\sigma_c = \inf_{s\in\mathbb{C}} \left\{ \operatorname{Re} s : \sum_{n=1}^{\infty} \frac{a_n}{n^s} < \infty \right\}.$$

The term σ_c is then known as the *the abscissa of convergence* for $\sum_{n=1}^{\infty} a_n n^{-s}$. Similarly, the quantity

$$\sigma_a = \inf_{\sigma} \left\{ \sigma \text{ is real} : \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} < \infty \right\},$$

is called the *the abscissa of absolute convergence* for $\sum_{n=1}^{\infty} a_n n^{-s}$. Finally, *the abscissa of uniform convergence* for $\sum_{n=1}^{\infty} a_n n^{-s}$ is defined to be the unique real number σ_u such that the Dirichlet series converges uniformly in the half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma_u\}$.

In 1913, Harald Bohr published the *absolute convergence problem* [39] which asked for the value of

$$S_0 := \sup(\sigma_a - \sigma_u),$$

where the supremum is taken over all ordinary Dirichlet series. In fact, this problem can be reduced to a problem on power series in an infinite number of complex variables [39, 38], which allowed Bohr to obtain the inequality $S_0 \leq 1/2$ [39, Satz X]. While attempting the absolute convergence problem, Bohr returned to the one dimensional case and proved the Bohr's inequality (or Bohr's theorem):

Theorem 1.9. *([40]) If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(U,U)$, then*

$$\sum_{n=0}^{\infty} |a_n z^n| \leq 1 \tag{1.5}$$

for $|z| \leq 1/6$.

The value $1/6$ is further improved independently by Riesz, Schur and Wiener to $1/3$ which is optimal. Other proofs can also be found in [102, 112, 115]. Thus $1/3$ is then known as the *Bohr radius* of $H(U,U)$, and the class $H(U,U)$ is said to have Bohr phenomenon. The notion of the Bohr phenomenon was first introduced by Bénéteau, Dahlner and Khavinson [32] for a Banach space X of analytic functions on the disk U. The Bohr's inequality (1.5) can also be put in the form

$$d\left(\sum_{n=0}^{\infty} |a_n z^n|, |f(0)|\right) \leq d(f(0), \partial U), \tag{1.6}$$

where d is the Euclidean distance and ∂U the unit circle. Further, the Bohr's inequality can be paraphrased in terms of the supremum norm, $\|f\|_\infty$:

$$\sum_{n=0}^{\infty} |a_n z^n| \leq \|f\|_\infty = \sup_{|z|<1} |f(z)|. \tag{1.7}$$

1.9 About the thesis

1.9.1 Background - Bohr and distances

For an analytic function f defined in U of the form (1.1), define its associated *majorant function* [36] by

$$\mathcal{M}f(z) := \sum_{n=0}^{\infty} |a_n| z^n.$$

If $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is another analytic function on U, then

$$\begin{aligned} \mathcal{M}(f+g)(|z|) \leq& \mathcal{M}f(|z|) + \mathcal{M}g(|z|); \\ \mathcal{M}(fg)(|z|) \leq& \mathcal{M}f(|z|)\mathcal{M}g(|z|). \end{aligned} \tag{1.8}$$

Recall the classical Bohr's theorem with Bohr's inequality of the form in (1.6):

Theorem 1.10. *If $f \in H(U,U)$, then*

$$d\left(\mathcal{M}f(|z|), |f(0)|\right) \leq d(f(0), \partial U)$$

for $|z| \leq 1/3$, where d is the Euclidean distance, and ∂U is the boundary of U. The radius $1/3$ is sharp.

The research on investigating the Bohr's theorem in distance form was initiated by Aizenberg. He proved that

Theorem 1.11. *[17, Theorem 2.1] Let f be an analytic function from U into a domain $G \subset \mathbb{C}$. Further suppose the convex hull $\tilde{G}$ of G satisfies $\tilde{G} \neq \mathbb{C}$. Then*

$$d(\mathcal{M}f(|z|), |f(0)|) \leq d(f(0), \partial\tilde{G})$$

for $|z| \leq 1/3$. The value $1/3$ is the best, provided there exists a point $p \in \mathbb{C}$ satisfying $p \in \partial\tilde{G} \cap \partial G \cap \partial D$ for some disk $D \subset G$.

The result covered the case where G is a convex domain and so extended the classical Bohr's theorem where $G = U$. The domain G was further extended by Abu-Muhanna [6] by using the technique of subordination. He applied both the Koebe one-quarter theorem and de Branges's theorem, or the Bieberbarch's conjecture, to prove

Theorem 1.12. *[6, Theorem 1] Let f be a univalent (analytic and injective) function on U. If $g \prec f$, then*

$$d\left(\mathcal{M}g(|z|), |g(0)|\right) \leq d(f(0), \partial f(U))$$

for $|z| \leq 3 - 2\sqrt{2} \approx 0.17157$. The sharp radius $3 - 2\sqrt{2}$ is attained by the Koebe function $z/(1-z)^2$.

Recently, Abu-Muhanna and Ali [7] studied the class $H(U, \Omega)$ where Ω is a domain exterior to a compact convex set and proved

Theorem 1.13. *Suppose that the universal covering map from U into Ω has a univalent logarithmic branch that maps U into the complement of a convex set. If $0 \notin \Omega$, $1 \in \partial\Omega$*

and $f \in H(U,\Omega)$ with $f(0) > 1$, then for $|z| < 3-2\sqrt{2} \approx 0.17157$,

$$\lambda\left(\mathcal{M}f(|z|),|f(0)|\right) \leq \lambda\left(f(0),\partial\Omega\right),$$

where λ is the spherical chordal distance. In particular, if $\overline{G}$ is the closed unit disk, then the sharp radius $1/3$ is obtained.

Meanwhile, a link was established between the Bohr's inequality for classes of analytic functions $H(U,G)$ and the hyperbolic metric done by Abu-Muhanna and Ali [8] in the following year. That paper discussed the case where G is the right half-plane, the slit region and the exterior of U.

We end this subsection by stating the Bohr's inequality for bounded harmonic mappings as proved by Abu-Muhanna [6].

Theorem 1.14. *[6, Theorem 2] Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ be a complex-valued harmonic function on U. If $|f(z)| < 1$ for all $z \in U$, then*

$$\sum_{n=1}^{\infty} |e^{i\mu}a_n + e^{-i\mu}b_n||z|^n \leq d(|\operatorname{Re} e^{i\mu}a_0|,\partial U), \quad \textit{for any } \mu \in \mathbb{R},$$

for $|z| \leq 1/3$. The radius $1/3$ is sharp.

1.9.2 Scope of thesis

The aim of the research work is to extend Theorem 1.10 by

(a) establishing the Bohr's theorem for the class of analytic functions mapping U into some non-convex domain D,

(b) replacing the Euclidean distance with other distances, and

(c) extending the Bohr's theorem to some subclasses of analytic functions as well as classes of non-analytic functions.

The thesis is divided into six chapters. Briefly, Chapter 2 discusses the Bohr's theorem for the class $R(\alpha,\gamma,h)$ consisting of functions f which are analytic in U and satisfying the differential subordination relation

$$f(z)+\alpha zf'(z)+\gamma z^2 f''(z)\prec h(z), \quad z\in U,\, \alpha\geq\gamma\geq 0.$$

The Bohr's theorems are developed for the case when h is a convex function in U as well as the case when h is starlike with respect to $h(0)$. The results are proved by applying the Koebe one-quarter theorem and the theory of differential subordination. Simply note that if $\alpha=\gamma=0$, then the Bohr radii $1/3$ (convex h) and $3-\sqrt{2}$ (starlike h) are the known radii in Theorem 1.11 and Theorem 1.12, respectively.

Chapter 3 consists of two sections. The first section studies the Bohr's theorem for the class of analytic functions mapping the unit disk U to concave-wedge domains

$$W_\alpha=\left\{w\in\mathbb{C}:|\arg w|<\frac{\alpha\pi}{2}\right\}, \quad 1\leq\alpha\leq 2.$$

The Bohr radius is obtained by using the technique of subordination and has the value $(2^{\frac{1}{\alpha}}-1)/(2^{\frac{1}{\alpha}}+1)$. In particular, if $\alpha=1$, then the Bohr radius is $1/3$ as stated in Theorem 1.11 and $3-2\sqrt{2}$ for $\alpha=2$ as stated in Theorem 1.12. The next section focuses on the class of analytic functions f that maps the unit disk U to the punctured unit disk $U_0 = U \backslash \{0\}$. The development of the Bohr's theorem depends heavily on the coefficient estimate obtained by Koepf and Schmersau [86, p. 248] as well as the Herglotz representation theorem for analytic functions [76, Corollary 3.6].

Chapter 4 focuses on developing the Bohr's theorem in non-Euclidean geometry. The classical Bohr's theorem with respect to the spherical chordal distance λ defined by

$$\lambda(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}, \quad z_1, z_2 \in U,$$

is shown to have value $1/3$. The first section also shows that by replacing the Euclidean distance d with λ, it is possible to slightly improve the constraint in a Bohr's theorem obtained in earlier chapter. The hyperbolic Bohr's theorem is presented in the following section. By defining the hyperbolic unit disk U^h in the Poincaré disk model, an analogous Bohr's theorem for the class of analytic self-maps of U^h is obtained and the (hyperbolic) Bohr radius has the value $\tanh(1/2)/3$. Further, Theorem 1.11 has its hyperbolic version in the Poincaré disk model and the Bohr radius is shown to be $\tanh(1/2)/3$, implying the invariance of Bohr radius in hyperbolic geometry. Additionally, the main theorem is applied to obtain the Bohr-type theorem for other hyperbolic regions.

Chapter 5 is devoted to studying the Bohr's theorem in the class of non-analytic functions. The Bohr's theorem for the class of harmonic functions mapping U into a bounded domain in $\mathbb{C}$ can be found in the first section. In particular, if the bounded domain is taken to be U itself, then the Bohr's theorem is reduced to Theorem 2 in [6]. The Bohr's theorem for the class of univalent, harmonic, orientation-preserving mappings of U into the convex wedge

$$W = \{w \in \mathbb{C} : |\arg w| < \pi/4\}.$$

is established as well. Both the Bohr's theorems are shown to have the same Bohr radius $1/3$. The final section deals with the construction of Bohr-type inequality for the class of univalent logharmonic functions f of the form $f(z) = zh(z)\overline{g(z)}$ mapping U onto a domain which is starlike with respect to the origin. The distortion theorem for this class of functions can also be found in this section.

Chapter 6 serves as a survey of the work on developing Bohr's theorem. There are several directions in extending the classical Bohr's theorem. Among those researches, the n-dimensional Bohr radii study is very much well developed and the *first Bohr radius* (see Chapter 6, Section 6.1) has its asymptotic value proved to be $\sqrt{\log n/n}$ in [30] recently.

CHAPTER 2

BOHR AND DIFFERENTIAL SUBORDINATIONS

In this chapter, we shall investigate a special class of differential subordination $R(\alpha,\gamma,h)$. For $\alpha \geq \gamma \geq 0$, and for a given univalent function $h \in H(U)$, let

$$R(\alpha,\gamma,h) := \left\{ f \in H(U) : f(z) + \alpha z f'(z) + \gamma z^2 f''(z) \prec h(z), \quad z \in U \right\}.$$

The investigation of such functions f can be seen as an extension to the study of the class

$$R(\alpha,h) = \left\{ f \in H(U) : f'(z) + \alpha z f''(z) \prec h(z), \quad z \in U \right\}$$

or its variations for an appropriate function h. This class has been investigated in several works, and more recently in [114, 116]. It was shown in Ali *et. al* [23] that $f(z) \prec h(z)$ whenever $f \in R(\alpha,\gamma,h)$. The notion of convolution will be needed to deduce the latter assertion.

For two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $H(U)$, the Hadamard product (or convolution) of f and g is the function $f * g$ defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

The following auxiliary function will be useful: let

$$\phi_\lambda(z) = \int_0^1 \frac{dt}{1 - zt^\lambda} = \sum_{n=0}^\infty \frac{z^n}{1 + \lambda n}.$$

From [110] it is known that ϕ_λ is convex in U provided $\operatorname{Re}\lambda \geq 0$.

Now for $\alpha \geq \gamma \geq 0$, let

$$\nu + \mu = \alpha - \gamma, \quad \mu\nu = \gamma,$$

and

$$q(z) = \int_0^1 \int_0^1 h(zt^\mu s^\nu)dtds = (\phi_\nu * \phi_\mu) * h(z) \in R(\alpha, \gamma, h). \tag{2.1}$$

Let $S(h) := \{f \in H(U) : f \prec h\}$ denote the class of analytic functions on U subordinate to h. In [23], Ali *et. al* showed that

$$f(z) \prec q(z) \prec h(z)$$

for every $f \in R(\alpha, \gamma, h)$. Thus $R(\alpha, \gamma, h) \subset S(h)$.

2.1 $R(\alpha, \gamma, h)$ with convex h

The following result gives the Bohr radius for $R(\alpha, \gamma, h)$ with convex function h.

Theorem 2.1. *Let $f(z) = \sum_{n=0}^\infty a_n z^n \in R(\alpha, \gamma, h)$, and $h \in \mathcal{S}$ be convex. Then*

$$d(\mathcal{M}f(|z|), |f(0)|) = \sum_{n=1}^\infty |a_n z^n| \leq d(h(0), \partial h(U))$$

for all $|z| \leq r_{CV}(\alpha,\gamma)$*, where* $r_{CV}(\alpha,\gamma)$ *is the smallest positive root of the equation*

$$(\phi_\mu * \phi_\nu)(r) - 1 = \sum_{n=1}^{\infty} \frac{1}{(1+\mu n)(1+\nu n)} r^n = \frac{1}{2}.$$

Further, this bound is sharp. An extremal case occurs when $f(z) := q(z)$ *as defined in* (2.1) *and* $h(z) := z/(1-z)$.

Proof. Let $F(z) = f(z) + \alpha z f'(z) + \gamma z^2 f''(z) \prec h(z)$. Then

$$F(z) = \sum_{n=0}^{\infty} [1 + \alpha n + \gamma n(n-1)] a_n z^n,$$

and

$$\frac{1}{h'(0)} \sum_{n=1}^{\infty} [1 + \alpha n + \gamma n(n-1)] a_n z^n = \frac{F(z) - F(0)}{h'(0)} \prec \frac{h(z) - h(0)}{h'(0)}.$$

It follows from [69, Theorem 6.4(i)] that

$$\left| \frac{1 + \alpha n + \gamma n(n-1)}{h'(0)} \right| |a_n| \leq 1, \quad n \geq 1.$$

Hence

$$|a_n| \leq \frac{|h'(0)|}{1 + (\mu + \nu)n + \mu\nu n^2}, \quad n \geq 1,$$

which readily yields

$$\sum_{n=1}^{\infty} |a_n| r^n \leq \sum_{n=1}^{\infty} \frac{|h'(0)|}{1 + (\mu + \nu)n + \mu\nu n^2} r^n.$$

Since $H(z) = \frac{h(z)-h(0)}{h'(0)}$ is a normalized convex function on U, it follows from [69, Theorem 2.15] that

$$d(0, \partial\Omega) \geq 1/2 \quad \text{where } \Omega = H(U),$$

implying

$$d(h(0), \partial h(U)) = \inf_{\zeta \in \partial U} |h(\zeta) - h(0)| \geq \frac{|h'(0)|}{2}, \quad z \in U.$$

Thus

$$\sum_{n=1}^{\infty} |a_n| r^n \leq 2d(h(0), \partial h(U)) \left(\sum_{n=1}^{\infty} \frac{1}{(1+\mu n)(1+\nu n)} r^n \right),$$

and the Bohr radius $r_{CV}(\alpha, \gamma)$ is the smallest positive root of the equation

$$\sum_{n=1}^{\infty} \frac{1}{(1+\mu n)(1+\nu n)} r^n = \frac{1}{2}. \tag{2.2}$$

□

If h is convex, then q as given in (2.1) is convex (see [111, (0.1)]). Figure 2.1 describes the extremal case. With $h(z) := l(z) = z/(1-z)$, then $d(h(0), \partial h(U)) = 1/2$, and $q(z) = (\phi_\mu * \phi_\nu) * l(z)$ maps the Bohr circle of radius r_{CV} into $\{w : |w| \leq 1/2\}$. Here the image of the Bohr circle is depicted by a bold closed curve.

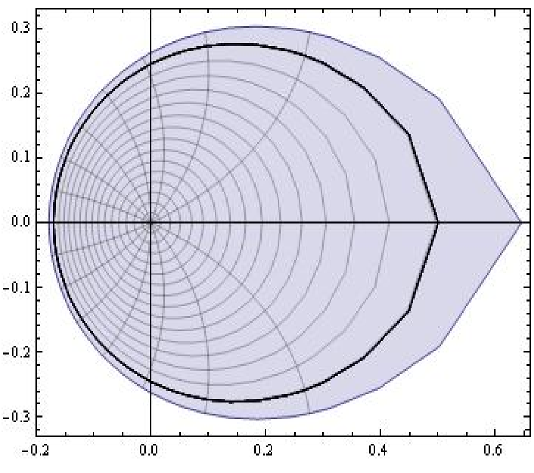

Figure 2.1: Image of the Bohr circle under $q(z) = (\phi_\mu * \phi_\nu) * \frac{z}{1-z}$ for $\alpha = 3, \gamma = 1$.

Remark 2.1. *The Bohr radius* $r_{CV}(0,0) = 1/3$ *was obtained in [17, Theorem 2.1].*

From (2.2), it is known that for any $f \in R(\alpha,\gamma,h)$ and h convex, the Bohr radius $r_{CV}(\alpha,\gamma)$ can be found by solving the equation

$$\sum_{n=1}^{\infty} \frac{1}{1+\alpha n+\gamma n(n-1)} r^n = \sum_{n=1}^{\infty} \frac{1}{(1+\mu n)(1+\nu n)} r^n = \frac{1}{2}$$

for the smallest positive root. Table 2.1 gives the values of the Bohr radius for different choices of the parameters α and γ. Note that $r_{CV}(\alpha,\gamma)$ approaches 1 for increasing α and γ.

Table 2.1: The Bohr radius $r_{CV}(\alpha,\gamma)$ for different α and γ

α	$r_{CV}(\alpha,0)$
0	0.333333
0.1	0.365245
1	0.582812
10	0.994200
20	0.999958
28	0.999999

α	γ	$r_{CV}(\alpha,\gamma)$
0	0	0.333333
1	0.5	0.649755
1	0.9	0.684027
4	0.9	0.981325
4	1	0.986793
4	4/3	0.999999

2.2 $R(\alpha,\gamma,h)$ with starlike h

The following theorem deals with subordination to a starlike function with respect to certain fixed point.

Theorem 2.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in R(\alpha,\gamma,h)$*, and* $h \in \mathcal{S}$ *be starlike with respect to* $h(0)$*. Then*

$$d(\mathcal{M}f(|z|),|f(0)|) = \sum_{n=1}^{\infty} |a_n z^n| \le d(h(0),\partial h(U))$$

for all $|z| \leq r_{ST}(\alpha,\gamma)$*, where* $r_{ST}(\alpha,\gamma)$ *is the smallest positive root of the equation*

$$(\phi_\mu * \phi_\nu)(r) - 1 = \sum_{n=1}^{\infty} \frac{n}{(1+\mu n)(1+\nu n)} r^n = \frac{1}{4}.$$

This bound is sharp. An extremal case occurs when $f(z) := q(z)$ *as defined in* (2.1) *and* $h(z) := k(z) = z/(1-z)^2$.

Proof. Since

$$F(z) = f(z) + \alpha z f'(z) + \gamma z^2 f''(z) = \sum_{n=0}^{\infty} [1+\alpha n + \gamma n(n-1)] a_n z^n \prec h(z),$$

it follows that

$$\frac{1}{h'(0)} \sum_{n=1}^{\infty} [1+\alpha n + \gamma n(n-1)] a_n z^n = \frac{F(z)-F(0)}{h'(0)} \prec \frac{h(z)-h(0)}{h'(0)} = H(z).$$

Note that H is starlike with respect to the origin. Thus [69, Theorem 6.4(ii)]

$$\left| \frac{1+\alpha n + \gamma n(n-1)}{h'(0)} \right| |a_n| \leq n, \quad n \geq 1,$$

which yields

$$\sum_{n=1}^{\infty} |a_n| r^n \leq \sum_{n=1}^{\infty} \frac{n|h'(0)|}{1+(\mu+\nu)n + \mu\nu n^2} r^n.$$

Since $H(z)$ is a normalized starlike function on U, it follows from Koebe one-quater theorem that

$$|H(z)| \geq 1/4, \quad z \in \partial U,$$

implying

$$d(h(0), \partial h(U)) = \inf_{\zeta \in \partial U} |h(\zeta) - h(0)| \geq \frac{|h'(0)|}{4}, \quad z \in U.$$

Thus

$$\sum_{n=1}^{\infty} |a_n| r^n \leq 4d(h(0), \partial h(U)) \left(\sum_{n=1}^{\infty} \frac{n}{(1+\mu n)(1+\nu n)} r^n \right),$$

and the Bohr radius $r_{ST}(\alpha, \gamma)$ is the smallest positive root of the equation

$$\sum_{n=1}^{\infty} \frac{n}{(1+\mu n)(1+\nu n)} r^n = \frac{1}{4}. \tag{2.3}$$

$\square$

If h is starlike, then q as given in (2.1) is starlike (see [111, (0.1)]). Figure 2.2 describes an extremal case. Here $d(h(0), \partial h(U)) = 1/4$ for $h(z) := k(z) = z/(1-z)^2$, and $q(z) = (\phi_\mu * \phi_\nu) * k(z)$ maps the Bohr circle, depicted as the bold closed curve, into $\{w : |w| \leq 1/4\}$.

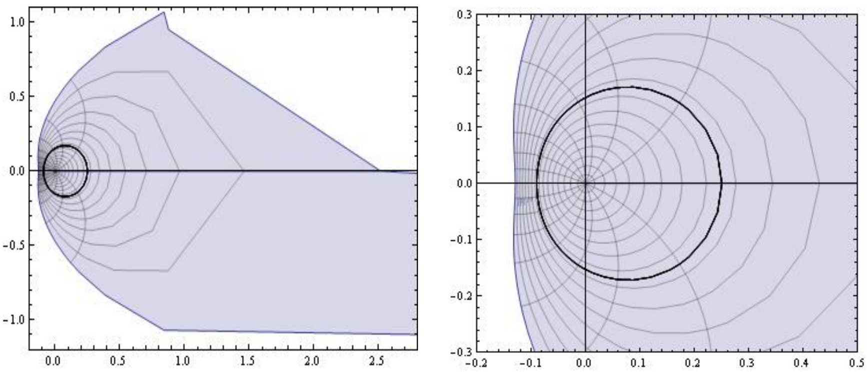

Figure 2.2: Image of the Bohr circle under $q(z) = (\phi_\mu * \phi_\nu) * \frac{z}{(1-z)^2}$ for $\alpha = 3, \gamma = 1$.

Remark 2.2. *The Bohr radius $r_{ST}(0,0) = 3 - 2\sqrt{2}$ is equal to the Bohr radius for the class of analytic functions subordinated to a univalent function, see [6, Theorem 1].*

From (2.3), the Bohr radius r_{ST} can be found by solving the equation

$$\sum_{n=1}^{\infty} \frac{n}{1+\alpha n+\gamma n(n-1)} r^n = \sum_{n=1}^{\infty} \frac{n}{(1+\mu n)(1+\nu n)} r^n = \frac{1}{4}$$

for a positive real root. Several values of $r_{ST}(\alpha, \gamma)$ are listed in Table 2.2.

Table 2.2: The Bohr radius $r_{ST}(\alpha, \gamma)$ for various α and γ

α	$r_{ST}(\alpha, 0)$
0	0.171573
0.1	0.188154
1	0.308210
10	0.723763
100	0.961586
1000000	0.999996

α	γ	$r_{ST}(\alpha, \gamma)$
0	0	0.171573
1	0.1	0.315797
2	1	0.459619
10	1	0.765923
100	10	0.994215
100	35	0.999963

CHAPTER 3

BOHR AND CODOMAINS

3.1 Unit disk to concave wedges

A link to the earlier results of Aizenberg [17] and Abu Muhanna [6] could be established by considering the concave-wedge domains

$$W_\alpha := \left\{ w \in \mathbb{C} : |\arg w| < \frac{\alpha\pi}{2} \right\}, \quad 1 \leq \alpha \leq 2. \tag{3.1}$$

In this instance, the conformal map of U onto W_α is given by

$$F_{\alpha,t}(z) = t\left(\frac{1+z}{1-z}\right)^\alpha = t\left(1 + \sum_{n=1}^{\infty} A_{n,\alpha} z^n\right), \quad t > 0. \tag{3.2}$$

When $\alpha = 1$, the domain reduces to a convex half-plane, while the case $\alpha = 2$ yields a slit domain. Denote by $H(U, W_\alpha)$ the class consisting of analytic functions f mapping the unit disk U into the wedge domain W_α given by (3.1).

The following result of [10] will be needed.

Proposition 3.1. *If F is an analytic univalent function mapping U onto $\Omega \subseteq \mathbb{C}$, where the complement of Ω is convex and $F(z) \neq 0$, then any analytic function f subordinate to $F^n, n = 1, 2, \ldots$, can be expressed as*

$$f(z) = \int_{|x|=1} F^n(xz)\, d\mu(x)$$

for some probability measure μ on the unit circle $|x| = 1$. In particular, if f subordinate to F^n for all n, then

$$f(z) = \int_{|x|=1} \exp(F(xz))\, d\mu(x)$$

for every f subordinate to $\exp(F)$.

The following result will also be helpful.

Lemma 3.1. *Let $F_{\alpha,t}(z) = t\left((1+z)/(1-z)\right)^{\alpha} = t\left(1+\sum_{n=1}^{\infty} A_{n,\alpha} z^n\right)$ be given by* (3.2)*, $\alpha \in [1,2]$. Then $A_{n,\alpha} > 0$ for $n = 1, 2, \ldots$.*

Proof. Evidently

$$F'_{\alpha,t}(z) = \frac{2\alpha}{1-z^2} F_{\alpha,t}(z). \tag{3.3}$$

Expanding (3.3) leads to

$$\begin{aligned}
\sum_{n=1}^{\infty} n A_{n,\alpha} z^{n-1} = & 2\alpha \left(1 + \sum_{n=1}^{\infty} z^{2n}\right)\left(1 + \sum_{n=1}^{\infty} A_{n,\alpha} z^n\right) \\
= & 2\alpha \left(1 + \sum_{n=1}^{\infty} z^{2n} + \sum_{n=1}^{\infty} A_{n,\alpha} z^n \right. \\
& \left. + z^2 \sum_{n=1}^{\infty} A_{n,\alpha} z^n + \cdots + z^{2m} \sum_{n=1}^{\infty} A_{n,\alpha} z^n + \cdots \right) \\
= & 2\alpha \left(1 + A_{1,\alpha} z + (1 + A_{2,\alpha}) z^2 + (A_{3,\alpha} + A_{1,\alpha}) z^3 \right. \\
& \left. + (1 + A_{4,\alpha} + A_{2,\alpha}) z^4 + (A_{5,\alpha} + A_{3,\alpha} + A_{1,\alpha}) z^5 + \cdots \right).
\end{aligned}$$

Thus, by defining $A_{0,\alpha} = 1$, it follows that

$$A_{n+1,\alpha} = \frac{2\alpha}{n+1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} A_{n-2k,\alpha} \tag{3.4}$$

for all $n \geq 1$, where $\lfloor \; \rfloor$ is the greatest integer function.

It follows by induction that

$$A_{n,\alpha} = p_n(\alpha), \quad n = 1, 2, \ldots \tag{3.5}$$

is a polynomial of degree n with positive coefficients. Indeed it holds for $n = 1$ since $A_{1,\alpha} = 2\alpha > 0$. Assuming that (3.5) holds for $n = m$, then (3.4) yields

$$A_{m+1,\alpha} = \frac{2\alpha}{m+1} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} A_{m-2k,\alpha} = \frac{2\alpha}{m+1} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} p_{m-2k}(\alpha) = p_{m+1}(\alpha)$$

where

$$p_{m+1}(\alpha) = \frac{2\alpha}{m+1} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} p_{m-2k}(\alpha).$$

Since p_{m+1} is a polynomial of degree $m+1$ with positive coefficient in each term, evidently (3.5) is true for all $n \geq 1$. Consequently, $A_{n,\alpha} = p_n(\alpha) > 0$ for all $n \geq 1$. $\square$

The following are the main results for this section.

Theorem 3.1. *Let $\alpha \in [1,2]$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(U, W_\alpha)$ with $a_0 > 0$, then*

$$d(\mathcal{M}f(|z|), f(0)) = \sum_{n=1}^{\infty} |a_n z^n| \leq d(f(0), \partial W_\alpha)$$

for $|z| \leq r_\alpha = (2^{1/\alpha} - 1)/(2^{1/\alpha} + 1)$. The function $f = F_{\alpha,a_0}$ in (3.2) shows that the Bohr radius r_α is sharp.

Proof. Since f is subordinate to F_{α,a_0}, it follows from Proposition 3.1 and (3.2) that

$$\sum_{n=0}^{\infty} a_n z^n = \int_{|x|=1} a_0 \left(1+\sum_{n=1}^{\infty} A_{n,\alpha} x^n z^n\right) \, d\mu(x)$$

for some probability measure μ on the unit circle $|x|=1$. Note that the uniqueness of Taylor series gives

$$a_n = \int_{|x|=1} a_0 A_{n,\alpha} x^n \, d\mu(x),$$

which then implies

$$|a_n| \le \int_{|x|=1} a_0 \left|A_{n,\alpha}\right| |x^n| \, d\mu(x) = |A_n|.$$

Hence, by Lemma 3.1,

$$\begin{aligned}\mathcal{M}f(r) - a_0 \le & a_0 \sum_{n=1}^{\infty} A_n r^n = a_0 \left[\left(\frac{1+r}{1-r}\right)^{\alpha} - 1\right] \\ = & d(a_0, \partial W_\alpha) \left[\left(\frac{1+r}{1-r}\right)^{\alpha} - 1\right] \le d(a_0, \partial W_\alpha)\end{aligned}$$

for $|z| = r \le r_\alpha$, where r_α is the smallest positive root of the equation

$$\left(\frac{1+r}{1-r}\right)^{\alpha} - 1 = 1.$$

Thus $r_\alpha = (2^{\frac{1}{\alpha}} - 1)/(2^{\frac{1}{\alpha}} + 1)$. $\square$

Theorem 3.2. *Let* $\alpha \in [1,2]$. *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(U, W_\alpha)$, *then*

$$\sum_{n=0}^{\infty} |a_n z^n| - |a_0|^* \le d(|a_0|^*, \partial W_\alpha)$$

for $|z| \le r_\alpha = c_0(2^{1/\alpha}-1)/(2^{1/\alpha}+1)$, *where* $|a_0|^* = F_{\alpha,1}\left(\left|F_{\alpha,1}^{-1}(a_0)\right|\right)$, $c_0 = 1 - 2\left|F_{\alpha,1}^{-1}(a_0)\right|/3$ *and* $F_{\alpha,1}$ *is given by* (3.2). *The function* $f = F_{\alpha,1}$ *shows that the Bohr radius* r_α *is sharp.*

Proof. Since $f \in H(U, W_\alpha)$, there exists $b \in U$ such that $F_{\alpha,1}(b) = a_0$. Let

$$\varphi(z) = \sum_{n=0}^{\infty} b_n z^n = \frac{z+b}{1+\bar{b}z} = b + \left(1-|b|^2\right)\sum_{n=1}^{\infty}(-\bar{b})^{n-1}z^n.$$

Then $(F_{\alpha,1}\circ\varphi)(0) = F_{\alpha,1}(\varphi(0)) = a_0 = f(0)$ and the fact that $F_{\alpha,1}\circ\varphi$ maps U conformally onto W_α yield

$$f \prec F_{\alpha,1}\circ\varphi. \tag{3.6}$$

Next, let $|a_0|^* = F_{\alpha,1}(|b|)$. Then

$$|a_0|^* = F_{\alpha,1}\left(\left|F_{\alpha,1}^{-1}(a_0)\right|\right) \ge \left|F_{\alpha,1}(F_{\alpha,1}^{-1}(a_0))\right| = |a_0|.$$

Now the function

$$\mathcal{M}\varphi(z) = \sum_{n=0}^{\infty}|b_n|z^n = \frac{|b| + (1-2|b|^2)z}{1-|b|z}$$

maps the disk $|z| < 1/3$ into U. Let $G = F_{\alpha,1}\circ\mathcal{M}\varphi$ and write $|z| = r$. Then $G(z) \in W_\alpha$ for $|z| < 1/3$ and

$$G(r) \le |a_0|^*\left(\frac{1+r/c_0}{1-r/c_0}\right)^{\alpha} = F_{\alpha,1}\left(\frac{|b|+r/c_0}{1+|b|r/c_0}\right), \tag{3.7}$$

for $r < 1/3$ where $c_0 = 1 - 2|b|/3$, and equality holds when $r = 1/3$. Further (1.8) gives

$$\mathcal{M}(F_{\alpha,1}\circ\varphi)(r) \leq G(r). \tag{3.8}$$

Hence using (3.6), (3.7) and (3.8), it follows that

$$\mathcal{M}f(r) \leq \mathcal{M}(F_{\alpha,1}\circ\varphi)(r) \leq G(r) \leq |a_0|^* \left(\frac{1+r/c_0}{1-r/c_0}\right)^{\alpha}$$

for $r \leq 1/3$. Consequently, $\mathcal{M}f(r) - |a_0|^* \leq |a_0|^* = d(|a_0|^*, \partial W_\alpha)$ provided $r \leq r_\alpha$, where r_α is the smallest positive root of

$$\left(\frac{1+r/c_0}{1-r/c_0}\right)^{\alpha} - 1 = 1,$$

that is, $r_\alpha = c_0(2^{\frac{1}{\alpha}} - 1)/(2^{\frac{1}{\alpha}} + 1)$. □

Remark 3.1. *Since $\alpha \in [1,2]$, it follows that $0.17157 \approx 3-2\sqrt{2} \leq r_\alpha/c_0 \leq 1/3$.*

Remark 3.2. *If $a_0 \geq 1$, then $|a_0|^* = a_0$ and Theorem 3.2 is equivalent to Theorem 3.1. However the case $0 < a_0 < 1$ gives $|a_0|^* = 1/a_0$. A generalization of Theorem 3.2 and Theorem 3.1 can be found in [24].*

Remark 3.3. *The Bohr radius for the half-plane is $r_1 = 1/3$, and $r_2 = 3-2\sqrt{2}$ for the slit-map. Since every convex domain lies in a half-plane, it readily follows from Theorem 3.1 that the Bohr radius for convex domains is $1/3$. When the class of functions is subordinate to an analytic univalent function, it follows from de Brange's Theorem [72] that the moduli of its Taylor's coefficients are bounded by the coefficients of the slit-map, which from Theorem 3.1, readily yields the Bohr radius $3-2\sqrt{2}$ for this class [6].*

3.2 Unit disk to punctured unit disk

This section is devoted to the development of Bohr's theorem for the class of analytic functions mapping U to the simplest doubly connected domain, that is, the punctured unit disk. Denote by U_0 the unit disk punctured at the origin, and $\overline{U}_r$ the closed disk $\{z : |z| \leq r\}$.

The following theorem shows that the Bohr radius $1/3$ also holds for the subclass $H(U, U_0)$ of $H(U, U)$.

Theorem 3.3. *If $f \in H(U, U_0)$, then*

$$\mathcal{M}f(\overline{U}_{1/3}) \subseteq U, \tag{3.9}$$

or equivalently,

$$d\left(\mathcal{M}f(|z|), |f(0)|\right) \leq d(f(0), \partial U) \tag{3.10}$$

for $|z| \leq 1/3$. The radius $1/3$ is the best.

Proof. Since $f \in H(U, U_0) \subset H(U, U)$, the inclusion (3.9) follows immediately from the classical Bohr's theorem. To show the value $1/3$ is the best, consider the function

$$f_t(z) = \exp\left(-t\frac{1+z}{1-z}\right), \quad t > 0,\ z \in U. \tag{3.11}$$

Note that

$$\begin{aligned}
\exp\left(-t\frac{1+z}{1-z}\right) &= \exp\left(-t-2t\sum_{n=1}^{\infty} z^n\right) = \frac{1}{e^t}\exp\left(-2t\sum_{n=1}^{\infty} z^n\right) \\
&= \frac{1}{e^t}+\frac{1}{e^t}\sum_{m=1}^{\infty}\frac{(-2t)^m}{m!}\left(\sum_{n=1}^{\infty} z^n\right)^m \\
&= \frac{1}{e^t}+\frac{1}{e^t}\sum_{m=1}^{\infty}\frac{(-2t)^m}{m!}\sum_{n=m}^{\infty} c_n z^n \\
&= \frac{1}{e^t}+\frac{1}{e^t}\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{(-2t)^m}{m!} c_n z^n \\
&= \frac{1}{e^t}+\frac{1}{e^t}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(-2t)^m}{m!} c_n z^n,
\end{aligned}$$

where

$$c_n = \sum_{p_1+\cdots+p_m=n} 1 = \binom{n-1}{m-1}$$

and the summation is taken over all m-tuple $(p_1,\ldots,p_m)$ of positive integers satisfying $p_1+\cdots+p_m=n$. Thus the Taylor series expansion of $f_t(z)$ about the origin is given by

$$f_t(z) = \frac{1}{e^t}+\frac{1}{e^t}\sum_{n=1}^{\infty}\left[\sum_{m=1}^{n}\frac{(-2t)^m}{m!}\binom{n-1}{m-1}\right] z^n.$$

Since

$$\left|\sum_{m=1}^{n}\frac{(-2t)^m}{m!}\binom{n-1}{m-1}\right| \geq -\sum_{m=1}^{n}\frac{(-2t)^m}{m!}\binom{n-1}{m-1},$$

it follows that

$$\begin{aligned}
\mathcal{M}f_t(|z|) &\geq \frac{1}{e^t}-\frac{1}{e^t}\sum_{n=1}^{\infty}\left[\sum_{m=1}^{n}\frac{(-2t)^m}{m!}\binom{n-1}{m-1}\right]|z|^n \\
&= \frac{2}{e^t}-f_t(|z|). \qquad (3.12)
\end{aligned}$$

Let $a_0 = f_t(0)$. Since $t = -\log a_0 = -\log|a_0|$, f_t can be written as

$$f_t(z) = \exp\left(\log|a_0|\frac{1+z}{1-z}\right) = |a_0|\exp\left(\log|a_0|\frac{2z}{1-z}\right)$$
$$= |a_0||a_0|^{\frac{2z}{1-z}}. \tag{3.13}$$

Hence, by letting $|z| = r$, (3.12) and (3.13) imply

$$\mathcal{M}f_t(|z|) \geq 2|a_0| - f_t(|z|) = |a_0|(2-|a_0|^{\frac{2r}{1-r}}) > 1 \tag{3.14}$$

as $a_0 \longrightarrow 1$ and $r > 1/3$. To be precise, consider the real-valued function

$$g(x) = -\frac{\log(1-x)}{\log(1+x)}, \quad x \in (0,1).$$

Then

$$\begin{aligned} g'(x) &= \frac{\frac{\log(1+x)}{1-x} + \frac{\log(1-x)}{1+x}}{(\log(1+x))^2} = \frac{(1+x)\log(1+x)+(1-x)\log(1-x)}{(1-x^2)\,(\log(1+x))^2} \\ &= \frac{[\log(1+x)+\log(1-x)]+x[\log(1+x)-\log(1-x)]}{(1-x^2)\,(\log(1+x))^2} \\ &= \frac{2\left[\left(1-\frac{1}{2}\right)x^2+\left(\frac{1}{3}-\frac{1}{4}\right)x^4+\left(\frac{1}{5}-\frac{1}{6}\right)x^6+\cdots+\right]}{(1-x^2)\,(\log(1+x))^2} > 0, \end{aligned}$$

indicates that g is continuous and strictly increasing in $(0,1)$. Further, $\lim_{x\to 0} g(x) = 1$ and $\lim_{x\to 1} g(x) = \infty$ implies that for any $r_0 > 1/3$, there exists $x_0 \in (0,1)$ such that

$$1 < -\frac{\log(1-x_0)}{\log(1+x_0)} < \frac{2r_0}{1-r_0}.$$

Equivalently,

$$1 - x_0 > (1 + x_0)^{-\frac{2r_0}{1-r_0}}.$$

Hence by selecting a function f_t with $|a_0| = 1/(1+x_0)$, it follows that

$$|a_0|(2 - |a_0|^{\frac{2r_0}{1-r_0}}) > 1,$$

which gives $\mathcal{M}f_t(r_0) > 1$. On the other hand, the inequality

$$|a_0|(2 - |a_0|^{\frac{2r}{1-r}}) \leq 2|a_0| - |a_0|^2 \leq 1$$

holds for all functions f_t with $|a_0| < 1$ and $0 \leq r \leq 1/3$. Hence the radius $1/3$ is the best. $\square$

Since $f(U) \subseteq U_0$, the Bohr's theorem for the class $H(U,U_0)$ suggests replacing the domain U in both (3.9) and (3.10) by U_0. To this end, we first examine the case for functions $f_t \in H(U,U_0)$ given by (3.11). For such functions f_t, Koepf and Schmersau [86, p. 248] obtained the estimate

$$\left| \frac{1}{e^t} \sum_{m=1}^{n} \frac{(-2t)^m}{m!} \binom{n-1}{m-1} \right| < \sqrt{\frac{2t}{n}}, \quad t \in (0,2n), \quad n > 0. \tag{3.15}$$

Also, it would soon become evident that the number

$$\alpha_0 := \frac{1}{3e} - \frac{1}{9\sqrt{6}} \approx 0.07727$$

plays a prominent role in the sequel.

Lemma 3.2. *Let f_t be given by* (3.11) *with* $0 < t \le 1$. *Then* $\mathcal{M}f_t(\overline{U}_{1/3}) \subseteq U_0$ *and*

$$\left|\mathcal{M}f_t(z) - \frac{1}{e^t}\right| < \frac{1}{e^t} - \alpha_0, \quad z \in \overline{U}_{1/3}.$$

In particular,

$$\mathcal{M}f_t(|z|) - \frac{1}{e^t} < \frac{1}{e^t} - \alpha_0, \quad |z| \le 1/3.$$

Proof. Write

$$\begin{aligned}
f_t(z) =& \exp\left(-t\frac{1+z}{1-z}\right) = \exp\left(-t - 2t\sum_{n=1}^{\infty} z^n\right) \\
=& \frac{1}{e^t} + \sum_{m=1}^{\infty} \frac{1}{e^t m!}\left(-2t\sum_{n=1}^{\infty} z^n\right)^m \\
=& \frac{1}{e^t} - \frac{2t}{e^t}z - \frac{2t(1-t)}{e^t}z^2 + a_3 z^3 + \cdots .
\end{aligned}$$

Thus for $|z| \le 1/3$, (3.15) gives

$$\begin{aligned}
\left|\mathcal{M}f_t(z) - \frac{1}{e^t}\right| <& \frac{2t}{3e^t} + \frac{2t(1-t)}{9e^t} + \sum_{n=3}^{\infty} \frac{\sqrt{2t}}{3^n\sqrt{n}} \\
<& \frac{2t}{3e^t} + \frac{2t(1-t)}{9e^t} + \sqrt{\frac{2t}{3}}\sum_{n=3}^{\infty} \frac{1}{3^n} \\
=& \frac{2t}{3e^t} + \frac{2t(1-t)}{9e^t} + \frac{1}{9}\sqrt{\frac{t}{6}} \\
=& \frac{1}{e^t} - y_1(t) < \frac{1}{e^t} - \alpha_0,
\end{aligned} \tag{3.16}$$

where

$$y_1(t) := \frac{1}{e^t} - \frac{2t}{3e^t} - \frac{2t(1-t)}{9e^t} - \frac{1}{9}\sqrt{\frac{t}{6}}$$

is strictly decreasing in $[0,1]$. Thus $y_1(t) > \alpha_0$ in $[0,1)$ and $y_1(1) = \alpha_0$ as shown in Figure 3.1. It follows that $|\mathcal{M}f_t(z)| > 0$, which along with Theorem 3.3 give $\mathcal{M}f_t(\overline{U}_{1/3}) \subseteq U_0$. $\square$

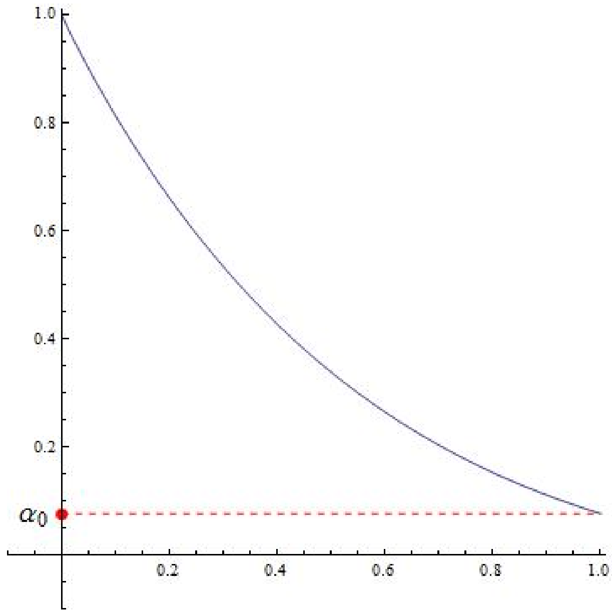

Figure 3.1: Graph of function $y_1(t)$ over the interval [0,1]

Remark 3.4. *Equation y_1 in Lemma 3.2 has a root at $t_0 \approx 1.35299$, and indeed y_1 is strictly decreasing in $[0,t_0]$. We shall however be only interested in the interval $t \in (0,1]$.*

Lemma 3.3. *Let $f_{a,N} \in H(U,U_0)$ be of the form*

$$f_{a,N}(z) = \exp\left(-\sum_{k=1}^{N} t_k a \frac{1+x_k z}{1-x_k z}\right), \quad z \in U, \tag{3.17}$$

with $0 < a \leq 1$, $|x_k| = 1$ for each k, and $t_k > 0$ satisfies $\sum_{k=1}^{N} t_k = 1$. Then $\mathcal{M}f_{a,N}(\overline{U}_{1/3}) \subseteq U_0$ and

$$\left|\mathcal{M}f_{a,N}(z) - \frac{1}{e^a}\right| < \frac{1}{e^a} - \alpha_0, \quad z \in \overline{U}_{1/3}.$$

Proof. Since $f_{a,N}$ is analytic in U, it can be expressed in its Taylor series

$$\begin{aligned}
\exp\left(-\sum_{k=1}^{N} t_k a \frac{1+x_k z}{1-x_k z}\right) &= \exp\left(-a-2a\sum_{n=1}^{\infty}\left(\sum_{k=1}^{N} t_k x_k^n\right) z^n\right) \\
&= \frac{1}{e^a}\exp\left(-2a\sum_{n=1}^{\infty}\left(\sum_{k=1}^{N} t_k x_k^n\right) z^n\right) \\
&= \frac{1}{e^a}+\frac{1}{e^a}\sum_{m=1}^{\infty}\frac{(-2a)^m}{m!}\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{N} t_k x_k^n\right) z^n\right)^m \\
&= \frac{1}{e^a}+\frac{1}{e^a}\sum_{m=1}^{\infty}\frac{(-2a)^m}{m!}\sum_{n=m}^{\infty} d_n z^n \\
&= \frac{1}{e^a}+\frac{1}{e^a}\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{(-2a)^m}{m!} d_n z^n \\
&= \frac{1}{e^a}+\frac{1}{e^a}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(-2a)^m}{m!} d_n z^n,
\end{aligned}$$

where

$$d_n = \sum_{s_1+\cdots+s_m=n}\left(\sum_{k=1}^{N} t_k x_k^{s_1}\right)\cdots\left(\sum_{k=1}^{N} t_k x_k^{s_m}\right)$$

and the outer sum is taken over all m-tuples $(s_1,\ldots,s_m)$ of positive integers satisfying $s_1+\cdots+s_m=n$. Note that

$$|d_n| \le \sum_{s_1+\cdots+s_m=n}\left(\sum_{k=1}^{N} t_k\right)\cdots\left(\sum_{k=1}^{N} t_k\right) = \sum_{s_1+\cdots+s_m=n} 1 = \binom{n-1}{m-1}.$$

Next let

$$f_a(z) = \exp\left(-a\frac{1+z}{1-z}\right) = \frac{1}{e^a}+\frac{1}{e^a}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(-2a)^m}{m!}\binom{n-1}{m-1} z^n.$$

Thus for $|z| \leq 1/3$,

$$\left|\mathcal{M}f_{a,N}(z) - \frac{1}{e^a}\right| \leq \frac{1}{e^a} \sum_{n=1}^{\infty} \left|\sum_{m=1}^{n} \frac{(-2a)^m}{m!}\right| |d_n||z|^n$$
$$\leq \mathcal{M}f_a(|z|) - \frac{1}{e^a} < \frac{1}{e^a} - \alpha_0, \tag{3.18}$$

where the last inequality follows from Lemma 3.2. Hence $|\mathcal{M}f_{a,N}(z)| > 0$ on $\overline{U}_{1/3}$, which together with Theorem 3.3 yield $\mathcal{M}f_{a,N}(\overline{U}_{1/3}) \subseteq U_0$. $\square$

Theorem 3.4. *Let $f \in H(U, U_0)$ with $1/e \leq |f(0)| < 1$. Then $\mathcal{M}f(\overline{U}_{1/3}) \subseteq U_0$ and*

$$|\mathcal{M}f(z) - \mathcal{M}f(0)| < \mathcal{M}f(0), \quad z \in \overline{U}_{1/3}.$$

In particular,

$$\mathcal{M}f(|z|) - |f(0)| < |f(0)|, \quad |z| \leq 1/3.$$

Proof. It suffices to consider the case $f(0) > 0$. Since $0 < |f(z)| < 1$, it follows that $-\operatorname{Re}\log f(z) > 0$ in U. Thus,

$$\log f(z) = \log f(0) \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x),$$

or

$$f(z) = \exp\left(-\int_{|x|=1} a\frac{1+xz}{1-xz} d\mu(x)\right)$$

for some probability measure μ on ∂U, and $0 < a = -\log f(0) \leq 1$. If f has the form (3.17), then the results evidently follow from Lemma 3.3.

Consider the compact disk $\overline{U}_\rho$ with $1/3 < \rho < 1$. By Corollary 3.7 in [76], if f

does not have the form (3.17), then there exists a sequence of functions $\{g_n\}$ of the form (3.17) satisfying $g_n(0) = f(0)$ for each n, and g_n converges uniformly to f on $\overline{U}_\rho$. Thus for a given $\varepsilon > 0$, there exists a positive integer N such that

$$|g_n(z) - f(z)| < \frac{\varepsilon}{M} \quad \text{for all } z \in \overline{U}_\rho,$$

and $n > N$, where $M = \max_{z \in \overline{U}_{1/3}}\{|z|/(\rho - |z|)\}$. The Cauchy Integral formula yields

$$\begin{aligned} \left|g_n^{(k)}(0) - f^{(k)}(0)\right| &= \left|\frac{k!}{2\pi i}\oint_{\partial \overline{U_\rho}} \frac{g_n(\zeta) - f(\zeta)}{\zeta^{k+1}} d\zeta\right| \\ &\leq \frac{k!}{2\pi}\int_0^{2\pi} \frac{|g_n(\zeta(t)) - f(\zeta(t))|}{\rho^k} dt < \frac{\varepsilon k!}{M\rho^k}. \end{aligned}$$

Hence, for all $|z| \leq 1/3$ and $n > N$,

$$\begin{aligned} \left|\mathcal{M}g_n(z) - \mathcal{M}f(z)\right| &\leq \left|\mathcal{M}(g_n - f)(|z|)\right| \\ &= \sum_{k=1}^{\infty} \left|\frac{g_n^{(k)}(0) - f^{(k)}(0)}{k!}\right| |z|^k \\ &< \frac{\varepsilon}{M}\sum_{k=1}^{\infty}\left(\frac{|z|}{\rho}\right)^k = \frac{\varepsilon |z|}{M(\rho - |z|)} \leq \varepsilon, \end{aligned}$$

implying $\mathcal{M}g_n \to \mathcal{M}f$ uniformly on $\overline{U}_{1/3}$.

Now, for any $\varepsilon > 0$, there exists a corresponding positive integer N such that

$$\sup_{z \in \overline{U}_{1/3}} |\mathcal{M}g_n(z) - \mathcal{M}f(z)| < \varepsilon \quad \text{for all } n > N.$$

Lemma 3.3 and the inequality above imply that

$$\sup_{z\in\overline{U}_{1/3}} |\mathcal{M}f(z)-f(0)| \leq \sup_{z\in\overline{U}_{1/3}} |\mathcal{M}g_n(z)-f(0)| + \sup_{z\in\overline{U}_{1/3}} |\mathcal{M}g_n(z)-\mathcal{M}f(z)|$$

$$< f(0)-\alpha_0+\varepsilon.$$

Hence $|\mathcal{M}f(z)-f(0)| \leq f(0)-\alpha_0 < f(0)$ for all $z\in\overline{U}_{1/3}$, and so $|\mathcal{M}f(z)|>0$ on $\overline{U}_{1/3}$. Further Theorem 3.3 gives $\mathcal{M}f(\overline{U}_{1/3})\subseteq U_0$. □

The following result yields the Bohr radius for the class $\{f\in H(U,U_0): 1/e \leq |f(0)|<1\}$.

Theorem 3.5. *If $f\in H(U,U_0)$ with $1/e\leq|f(0)|<1$, then*

$$\mathcal{M}f(\overline{U}_{1/3})\subseteq U_0 \tag{3.19}$$

and

$$d(\mathcal{M}f(|z|),|f(0)|)\leq d(f(0),\partial U_0) \tag{3.20}$$

for $|z|\leq 1/3$. The radius $1/3$ is best possible.

Proof. The inclusion (3.19) follows from Theorem 3.4. Now, assume that $r=|z|\leq 1/3$. The inequality in Theorem 3.4 implies that

$$d(\mathcal{M}f(r),|f(0)|)=\mathcal{M}f(r)-|f(0)|<|f(0)|.$$

On the other hand, since $f \in H(U,U_0)$, Theorem 3.3 gives $\mathcal{M}f(r) < 1$ and so

$$d(\mathcal{M}f(r), |f(0)|) = \mathcal{M}f(r) - |f(0)| < 1 - |f(0)|.$$

Then the inclusion (3.20) follows from the two inequalities above since

$$d(f(0), \partial U_0) = \min\{|f(0)|, 1 - |f(0)|\}.$$

That the value $1/3$ is the best follows from the proof of Theorem 3.3. □

Remark 3.5. *Relations* (3.9) *and* (3.10) *in Theorem 3.3 are equivalent. However* (3.19) *and* (3.20) *in Theorem 3.5 are not since* $d(f(0), \partial U_0) = |f(0)| \neq 1 - |f(0)|$ *for* $|f(0)| < 1/2$.

Next, we look at removing the constraint on $|f(0)|$ in Theorem 3.5. Denote by $\lceil\ \rceil$ the least integer function, that is, $\lceil a \rceil$ is the smallest integer greater than or equal to a.

Lemma 3.4. *Suppose* $a > 0$, *and*

$$f_{a,N}(z) = \exp\left(-\sum_{k=1}^{N} t_k a \frac{1 + x_k z}{1 - x_k z}\right) \in H(U,U_0), \tag{3.21}$$

where $|x_k| = 1$ *for each* k, *and* $t_k > 0$ *satisfies* $\sum_{k=1}^{N} t_k = 1$. *Then*

$$\left(\mathcal{M}f_{a,N}(|z|)\right)^{1/\lceil a \rceil} - \frac{1}{e^{a/\lceil a \rceil}} < \frac{1}{e^{a/\lceil a \rceil}} - \alpha_0, \quad |z| \leq 1/3.$$

Proof. If $a \in (0,1]$, then $\lceil a \rceil = 1$ and the result follows from Lemma 3.3. Assume now that $a > 1$. It follows from the proof of Lemma 3.3 that

$$\left| \mathcal{M} f_{a,N}(z) - \frac{1}{e^a} \right| \leq \mathcal{M} f_a(|z|) - \frac{1}{e^a},$$

which gives

$$|\mathcal{M} f_{a,N}(z)| \leq \mathcal{M} f_a(|z|). \tag{3.22}$$

Since $\mathcal{M}(fg)(|z|) \leq \mathcal{M}(f)(|z|)\mathcal{M}(g)(|z|)$, Lemma 3.2 yields

$$\mathcal{M} f_a(|z|) = \mathcal{M} \left(f_{a/\lceil a \rceil} \right)^{\lceil a \rceil} (|z|) \leq \left(\mathcal{M} f_{a/\lceil a \rceil}(|z|) \right)^{\lceil a \rceil} < \left(\frac{2}{e^{a/\lceil a \rceil}} - \alpha_0 \right)^{\lceil a \rceil} \tag{3.23}$$

for $|z| \leq 1/3$. Thus

$$\left(\mathcal{M} f_{a,N}(|z|) \right)^{1/\lceil a \rceil} - \frac{1}{e^{a/\lceil a \rceil}} < \frac{1}{e^{a/\lceil a \rceil}} - \alpha_0. \qquad \square$$

Theorem 3.6. *Let $f \in H(U, U_0)$ and $a = -\log|f(0)|$. Then*

$$(\mathcal{M} f(|z|))^{1/\lceil a \rceil} - |f(0)|^{1/\lceil a \rceil} < |f(0)|^{1/\lceil a \rceil}, \quad |z| \leq 1/3.$$

Proof. It suffices to consider the case $f(0) > 0$. Let $a = -\log f(0)$. Then

$$f(z) = \exp\left(-\int_{|x|=1} a \frac{1+xz}{1-xz} d\mu(x) \right)$$

for some probability measure μ on ∂U. If f has the form (3.21), then the result follows from Lemma 3.4.

Consider the compact disk $\overline{U}_\rho$ with $1/3 < \rho < 1$. If f does not have the form (3.21), then there exists a sequence of functions $\{g_n\}$ of the form (3.21) satisfying $g_n(0) = f(0)$ for each n, and g_n converges uniformly to f on $\overline{U}_\rho$. Applying the same argument as in the proof of Theorem 3.4, it can be shown that $\mathcal{M}g_n$ converges to $\mathcal{M}f$ uniformly on $\overline{U}_{1/3}$.

Thus, for any $\varepsilon > 0$, there exists a corresponding positive integer N such that for all $n \geq N$ and $z \in \overline{U}_{1/3}$,

$$|\mathcal{M}g_n(z) - \mathcal{M}f(z)| < \varepsilon,$$

and thus

$$|\mathcal{M}f(z)| < |\mathcal{M}g_n(z)| + \varepsilon.$$

Further (3.22) and (3.23) imply that

$$|\mathcal{M}f(z)| < \left(2\,(f(0))^{1/\lceil a\rceil} - \alpha_0\right)^{\lceil a\rceil} + \varepsilon.$$

Since ε is arbitrary, it follows that

$$|\mathcal{M}f(z)| \leq \left(2\,(f(0))^{1/\lceil a\rceil} - \alpha_0\right)^{\lceil a\rceil} < \left(2\,(f(0))^{1/\lceil a\rceil}\right)^{\lceil a\rceil},$$

and consequently

$$(\mathcal{M}f(|z|))^{1/\lceil a\rceil} - (f(0))^{1/\lceil a\rceil} < (f(0))^{1/\lceil a\rceil}$$

for $|z| \leq 1/3$. $\square$

Theorem 3.7. *If $f \in H(U,U_0)$ and $a = -\log|f(0)|$, then*

$$d\left((\mathcal{M}f(|z|))^{1/\lceil a \rceil}, |f(0)|^{1/\lceil a \rceil}\right) \leq d\left((f(0))^{1/\lceil a \rceil}, \partial U_0\right) \tag{3.24}$$

for $|z| \leq 1/3$. The radius $1/3$ is best possible.

Proof. The inclusion (3.24) follows from Theorem 3.6. Now, assume that $r = |z| \leq 1/3$. The inequality in Theorem 3.6 implies that

$$d\left((\mathcal{M}f(r))^{1/\lceil a \rceil}, |f(0)|^{1/\lceil a \rceil}\right) = (\mathcal{M}f(r))^{1/\lceil a \rceil} - |f(0)|^{1/\lceil a \rceil} < |f(0)|^{1/\lceil a \rceil}.$$

On the other hand, since $f \in H(U,U_0)$, Theorem 3.3 gives $\mathcal{M}f(r) < 1$ implying $(\mathcal{M}f(r))^{1/\lceil a \rceil} < 1$. Thus

$$d\left((\mathcal{M}f(r))^{1/\lceil a \rceil}, |f(0)|^{1/\lceil a \rceil}\right) = (\mathcal{M}f(r))^{1/\lceil a \rceil} - |f(0)|^{1/\lceil a \rceil} < 1 - |f(0)|^{1/\lceil a \rceil}.$$

Then the inclusion (3.24) follows from the two inequalities above since

$$d\left((f(0))^{1/\lceil a \rceil}, \partial U_0\right) = \min\{|f(0)|^{1/\lceil a \rceil}, 1 - |f(0)|^{1/\lceil a \rceil}\}.$$

To show the value $1/3$ is the best, consider the function $f_t \in H(U,U_0)$ given by (3.11) with $1/2 \leq f_{t/\lceil t \rceil}(0) < 1$. Then it suffices to show that

$$d((\mathcal{M}f_t(|z|))^{1/\lceil t \rceil}, (f_t(0))^{1/\lceil t \rceil}) > d((f_t(0))^{1/\lceil t \rceil}, \partial U_0), \quad |z| > 1/3. \tag{3.25}$$

Since

$$d\left((\mathcal{M}f_t(|z|))^{1/\lceil t\rceil}, (f_t(0))^{1/\lceil t\rceil}\right) = (\mathcal{M}f_t(|z|))^{1/\lceil t\rceil} - (f_t(0))^{1/\lceil t\rceil},$$

and

$$d\left((f_t(0))^{1/\lceil t\rceil}, \partial U_0\right) = 1 - (f_t(0))^{1/\lceil t\rceil},$$

it follows that (3.25) can be reduced to

$$(\mathcal{M}f_t(|z|))^{1/\lceil t\rceil} > 1 \quad \text{or} \quad \mathcal{M}f_t(|z|) > 1, \quad |z| > 1/3.$$

Indeed, the inequality holds as is shown in the proof of Theorem 3.3. □

CHAPTER 4

BOHR AND NON-EUCLIDEAN GEOMETRY

4.1 Bohr's theorems in non-Euclidean distances

4.1.1 Classical Bohr's theorem

We show that the classical Bohr radius $1/3$ remains invariant after replacing the Euclidean distance d with the spherical chordal distance λ:

$$\lambda(z_1,z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}, \quad z_1, z_2 \in U.$$

Theorem 4.1. *If $f \in H(U,U)$, then*

$$\lambda(\mathcal{M}f(|z|), |f(0)|) \leq \lambda(f(0), \partial U)$$

for $|z| \leq 1/3$. The value $1/3$ is the best.

Proof. Since $f \in H(U,U)$, the classical Bohr's theorem implies that $|f(0)| \leq \mathcal{M}f(|z|) < 1$ when $|z| \leq 1/3$. Hence for $|z| \leq 1/3$,

$$\lambda(|f(0)|, \mathcal{M}f(|z|)) < \lambda(|f(0)|, 1) = \lambda(f(0), \partial U).$$

To show the sharpness, consider the Möbius transformation

$$\varphi(z) = \frac{z+a}{1+az}, \quad 0 < a < 1, \; z \in U.$$

Then

$$\varphi(z) = a + \sum_{n=1}^{\infty} (1-a^2)(-a)^{n-1} z^n$$

yields

$$\mathcal{M}\varphi(z) = a + \sum_{n=1}^{\infty} (1-a^2) a^{n-1} z^n = 2a + \frac{z-a}{1-az}.$$

Since $\mathcal{M}\varphi(|z|)$ is increasing for $|z|$, it follows that for $|z| > 1/3$,

$$\mathcal{M}\varphi(|z|) > 2a + \frac{1-3a}{3-a} = \frac{1+3a-2a^2}{3-a} > a.$$

Thus, $\lambda(\mathcal{M}\varphi(|z|), a) > \lambda((1+3a-2a^2)/(3-a), a)$. Since

$$\frac{\lambda\big((1+3a-2a^2)/(3-a), a\big)}{\lambda(a,1)} = \frac{\sqrt{2}(1+a)}{\sqrt{(3-a)^2+(1+3a-2a^2)^2}},$$

it follows that

$$\lambda(\mathcal{M}\varphi(|z|), a) > \lambda(a,1) \frac{\sqrt{2}(1+a)}{\sqrt{(3-a)^2+(1+3a-2a^2)^2}} \to \lambda(a,1)$$

whenever $a \to 1$. □

4.1.2 Punctured disk and non-Euclidean geometry

In this subsection, we show that it is possible to slightly improve the constraint in Theorem 3.5 by replacing d with λ. Let U_0 be the punctured unit disk and $a \in U_0$.

Then $\lambda(a,\partial U_0) = \min\{\lambda(|a|,0),\lambda(|a|,1)\}$. Since

$$\frac{\lambda(|a|,0)}{\lambda(|a|,1)} = \frac{\sqrt{2}|a|}{1-|a|}\begin{cases} >1, & \text{if } |a| > \sqrt{2}-1, \\ <1, & \text{if } |a| < \sqrt{2}-1, \\ =1, & \text{if } |a| = \sqrt{2}-1, \end{cases}$$

it follows that

$$\lambda(a,\partial U_0) = \begin{cases} \lambda(|a|,1), & \text{if } |a| > \sqrt{2}-1, \\ \lambda(|a|,0), & \text{if } |a| < \sqrt{2}-1, \\ \lambda(|a|,0) = \lambda(|a|,1), & \text{if } |a| = \sqrt{2}-1. \end{cases}$$

Theorem 4.2. *If $f \in H(U,U_0)$ with $1/\sqrt{e^3} \leq |f(0)| < 1$, then*

$$\lambda(\mathcal{M}f(|z|),|f(0)|) \leq \lambda(f(0),\partial U_0)$$

for $|z| \leq 1/3$. The value $1/3$ is the best.

Proof. Consider f in the form (1.1) with $a_0 > 0$ and assume that $|z| \leq 1/3$. If $1/\sqrt{e}^3 \leq a_0 \leq \sqrt{2}-1$, then

$$\lambda(a_0,\partial U_0) = \lambda(a_0,0) = \frac{a_0}{\sqrt{1+a_0^2}}.$$

Thus

$$\lambda(a_0,\mathcal{M}f(|z|)) \leq \lambda(a_0,\partial U_0)$$

becomes

$$\frac{\mathcal{M}f(|z|)-a_0}{\sqrt{1+a_0^2}\sqrt{1+(\mathcal{M}f(|z|))^2}} \leq \frac{a_0}{\sqrt{1+a_0^2}},$$

which is equivalent to

$$(1-a_0^2)\big((\mathcal{M}f(|z|))^2-2a_0\mathcal{M}f(|z|)\big)\leq 0.$$

Hence it suffices to prove that

$$\mathcal{M}f(|z|)\leq \frac{2a_0}{1-a_0^2}.$$

Now, if f is of the form (3.17), then it follows from (3.16) and (3.18) that

$$\mathcal{M}f(|z|)<\frac{1}{e^t}+\frac{2t}{3e^t}+\frac{2t(1-t)}{9e^t}+\frac{1}{9}\sqrt{\frac{t}{6}},$$

where $t=-\log a_0$. Let

$$y_2(t)=\frac{1}{e^t}+\frac{2t}{3e^t}+\frac{2t(1-t)}{9e^t}+\frac{1}{9}\sqrt{\frac{t}{6}}-\frac{2e^{-t}}{1-e^{-2t}}.$$

Then $y_2(t_0)=0$ for some $t_0\approx 1.532$ and $y_2(t)<0$ for $0<t\leq 1.5$, as shown in Figure 4.1. Hence for $0<t\leq 1.5$,

$$\mathcal{M}f(|z|)<y_2(t)+\frac{2e^{-t}}{1-e^{-2t}}<\frac{2e^{-t}}{1-e^{-2t}}=\frac{2a_0}{1-a_0^2}.$$

On the other hand, if f is not of the form (3.17), then on the disk $\overline{U}_\rho=\{z:|z|\leq\rho\}$ with $1/3<\rho<1$, there exists a sequence of functions $\{g_k\}$ of the form (3.17) such

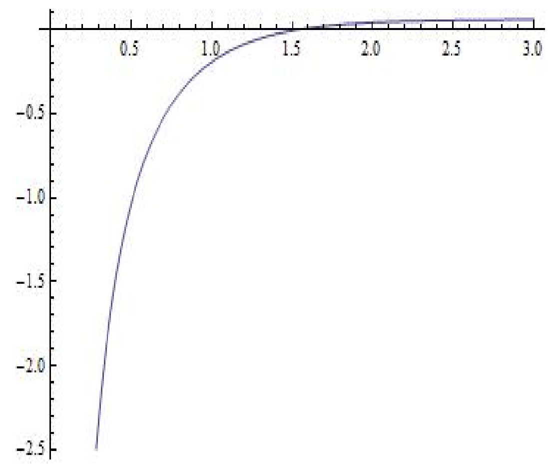

Figure 4.1: Graph of function $y_2(t)$ over the interval [0,3]

that $g_k(0) = a_0$ for each k, and g_k converges uniformly to f on $\overline{U}_\rho$. Since

$$\lambda(a_0, \mathcal{M}g_k(|z|)) < \lambda(a_0, 0)$$

and $\mathcal{M}g_k$ converges uniformly to $\mathcal{M}f$ on $\overline{U}_{1/3}$ (see the proof of Theorem 3.4), it follows that for $|z| \leq 1/3$,

$$\lambda(a_0, \mathcal{M}f(|z|)) \leq \lambda(a_0, 0).$$

Finally, if $\sqrt{2} - 1 \leq a_0 < 1$, then

$$\lambda(a_0, \partial U_0) = \lambda(a_0, 1).$$

Since $f \in H(U, U_0) \subset H(U, U)$, the classical Bohr's theorem implies that $a_0 \leq \mathcal{M}f(|z|) < 1$ when $|z| \leq 1/3$. Hence for $|z| \leq 1/3$,

$$\lambda(a_0, \mathcal{M}f(|z|)) < \lambda(a_0, 1).$$

To show the sharpness, consider the function f_t given by (3.11) with $\sqrt{2}-1 \le a_0 = e^{-t} < 1$. From (3.14), it is known that

$$\mathcal{M} f_t(|z|) \ge 2a_0 - a_0^{(1+|z|)/(1-|z|)} \ge a_0, \quad z \in U.$$

Hence

$$\lambda(a_0, \mathcal{M} f(|z|)) \ge \lambda(a_0, 2a_0 - a_0^{(1+|z|)/(1-|z|)}). \tag{4.1}$$

Then, by applying L'Hospital's rule,

$$\begin{aligned}\frac{\lambda(a_0, 2a_0 - a_0^{(1+|z|)/(1-|z|)})}{\lambda(a_0,1)} &= \frac{\sqrt{2}a_0}{\sqrt{1+\left(2a_0 - a_0^{(1+|z|)/(1-|z|)}\right)^2}} \cdot \frac{1-a_0^{2|z|/(1-|z|)}}{1-a_0} \\ &\to \frac{2|z|}{1-|z|} > 1 \end{aligned} \tag{4.2}$$

if and only if $|z| > 1/3$ as $a_0 \to 1$. Consequently, (4.1) and (4.2) give

$$\lambda(a_0, \mathcal{M} f(|z|)) > \lambda(a_0, 1) = \lambda(a_0, \partial U_0)$$

for $|z| > 1/3$ as $a_0 \to 1$. Thus the value $1/3$ is best possible. $\square$

We end this subsection by presenting a Bohr-type inequality in hyperbolic distance on U_0. The density of the hyperbolic metric [31] on U_0 is given by

$$\lambda_{U_0}(z) = \frac{1}{|z| \log(1/|z|)}.$$

If $d_{U_0}(a,b)$ denote the hyperbolic distance between a and b, then

$$d_{U_0}(a,b) = \int_a^b \frac{|dz|}{|z|\log(1/|z|)} = \log\left|\frac{\log 1/|b|}{\log 1/|a|}\right|.$$

Theorem 4.3. *Let $f \in H(U,U_0)$ with $1/e \leq |f(0)| < 1$. Then*

$$d_{U_0}(\mathcal{M}f(|z|),|f(0)|) \leq \log\frac{1+3|z|}{1-3|z|}$$

for $|z| < 1/3$. In particular,

(a) when $|z| < 1/9$,

$$d_{U_0}(\mathcal{M}f(|z|),|f(0)|) < \log 2 = \frac{2}{\lambda_{U_0}(\frac{1}{2})};$$

(b) when $|z| < (e-1)/3(1+e) \approx 0.15404$,

$$d_{U_0}(\mathcal{M}f(|z|),|f(0)|) < 1 = \frac{e}{\lambda_{U_0}(\frac{1}{e})};$$

(c) when $|z| < (1-|f(0)|)/3(1+|f(0)|)$,

$$d_{U_0}(\mathcal{M}f(|z|),|f(0)|) < \frac{1/|f(0)|}{\lambda_{U_0}(|f(0)|)}.$$

Proof. By Theorem 3.4, $\mathcal{M}f(U_{1/3}) \subseteq U_0$. Define a covering map $F : U \to U_0$ by

$$F(z) = \exp\left(\log(|f(0)|)\frac{1+z}{1-z}\right).$$

Also, the conformal map $\psi(z) = 3z$ sends $U_{1/3}$ onto U. Note that $F \circ \psi : U_{1/3} \to U_0$ is also a covering map. Thus by [31, Theorem 10.5],

$$\begin{aligned} d_{U_0}(|f(0)|, \mathcal{M}f(|z|)) \leq & d_{U_0}\big((F \circ \psi)(0), (F \circ \psi)(|z|)\big) \\ = & d_{U_{1/3}}(0, |z|) = d_U(\psi(0), \psi(|z|)) \\ = & d_U(0, 3|z|) = \log \frac{1+3|z|}{1-3|z|} \end{aligned}$$

in $U_{1/3}$, where $d_{U_{1/3}}$ and d_U denote the hyperbolic distance on $U_{1/3}$ and U, respectively. Parts (a) and (b) are evident. For part (c), an upper bound for $|z|$ is obtained by solving the inequality

$$\log \frac{1+3|z|}{1-3|z|} < \frac{1/|f(0)|}{\lambda_{U_0}(|f(0)|)} = \log \left(\frac{1}{|f(0)|} \right). \qquad \square$$

4.2 Bohr and Poincaré Disk Model

4.2.1 Hyperbolic Disk to Hyperbolic Disk

In [71], Fournier and Ruscheweyh studied the Bohr's theorem for the class of analytic functions in the disk

$$U_\gamma = \left\{ z \in \mathbb{C} : \left| z + \frac{\gamma}{1-\gamma} \right| < \frac{1}{1-\gamma} \right\}, \quad 0 \leq \gamma < 1.$$

The Bohr radius is shown to be $(1+\gamma)/(3+\gamma)$. Generally there is no assurance that the Bohr phenomenon will occur for every given class of functions. For example, Aizenberg [18] showed that the Bohr phenomenon does not exist for the space of analytic functions defined in the annulus $\{z \in \mathbb{C} : t < |z| < 1\}, 0 < t < 1$.

The following result is an immediate extension of the classical Bohr's inequality for arbitrary disks centered at the origin. We state it for the sake of completeness.

Theorem 4.4. *If $f \in H(U_p, U_q)$, then*

$$\mathcal{M}f(|z|) \leq q$$

for $|z| \leq r_{U_p,U_q} = p/3$. Equivalently,

$$d(\mathcal{M}f(|z|), |f(0)|) \leq d(f(0), \partial U_q)$$

for $|z| \leq r_{U_p,U_q} = p/3$. The radius r_{U_p,U_q} is sharp.

Proof. For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(U_p, U_q)$, the function $g(z) = (1/q)f(pz)$ lies in $H(U,U)$. The classical Bohr's theorem gives

$$\frac{1}{q}\sum_{n=0}^{\infty} |a_n||pz|^n \leq 1 \quad \text{for } |z| \leq 1/3,$$

that is,

$$\mathcal{M}f(|z|) \leq q \quad \text{for } |z| \leq p/3.$$

The sharpness is demonstrated by the Möbius transformation

$$\phi(z) = q\frac{z/p - a}{1 - az/p} = q\frac{z - ap}{p - az}, \quad 0 < a < 1, \quad z \in U_p. \qquad \square$$

Theorem 4.4 readily leads to the Bohr's theorems in the Poincaré disk model. Consider now the hyperbolic unit disk

$$U^h := \{z \in U : d_U(0,z) < 1\}.$$

It is readily shown that U^h is the Euclidean disk U_{r_h} with

$$r_h := \tanh(1/2).$$

Theorem 4.5. *If $f \in H(U^h, U_q)$ with $0 < q < 1$, then*

$$d_U(\mathcal{M}f(|z|), |f(0)|) \leq d_U(f(0), \partial U_q) \tag{4.3}$$

for $|z| \leq r_{U^h,U_q} = \tanh(1/2)/3 \approx 0.15404$. The radius r_{U^h,U_q} is sharp.

Proof. Since d_U is additive along the line $(-1,1)$ [31, Theorem 2.2], it follows that

$$d_U(f(0), \partial U_q) = d_U(|f(0)|, q) = d_U(0,q) - d_U(0, |f(0)|),$$

and

$$d_U(\mathcal{M}f(|z|), |f(0)|) = d_U(0, \mathcal{M}f(|z|)) - d_U(0, |f(0)|).$$

Thus inequality (4.3) holds when

$$d_U(0, \mathcal{M}f(|z|)) \leq d_U(0,q),$$

that is, for

$$\mathcal{M}f(|z|) \leq q.$$

So it suffices to find the radius r such that

$$d(\mathcal{M}f(|z|),|f(0)|) \leq d(f(0),\partial U_q)$$

for $|z| \leq r$. This problem is a special case of Theorem 4.4 with $p = r_h$. The sharp radius $r_{U^h,U_q} = r_h/3$ is thus obtained. □

The preceding theorem yields the hyperbolic Bohr's theorem for analytic self-maps of the hyperbolic unit disk.

Corollary 4.1. *If $f \in H(U^h,U^h)$, then*

$$d_U(\mathcal{M}f(|z|),|f(0)|) \leq d_U(f(0),\partial U^h)$$

for $|z| \leq r_{U^h,U^h} = \tanh(1/2)/3 \approx 0.15404$. The radius r_{U^h,U^h} is sharp.

4.2.2 Hyperbolic Disk to Hyperbolic Convex Set

In this subsection, a similar Bohr's theorem as proved by Aizenberg (see Theorem 1.11) is obtained in the Poincaré disk model of the hyperbolic plane. Let

$$U^+ := \{z \in U : \mathrm{Re}\, z > 0\}$$

be the semi-disk, which acts as the right half-plane in the Poincaré disk model. We first obtain sharp coefficient bounds for functions in $H(U,U^+)$.

Lemma 4.1. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(U, U^+)$, then*

$$|a_n| \le \frac{2(\operatorname{Re} a_0)(1-|a_0|^2)}{\sqrt{(1+|a_0|^2)^2 - 4(\operatorname{Im} a_0)^2}}$$

for $n = 1, 2, \ldots$.

Proof. Let $g(z) = (1+z)/(1-z)$ be an analytic function mapping U conformally onto the right half-plane $\mathscr{H}$. Then for $b > 0$ and c real, the function $i(bg+ic)$ maps U conformally onto the upper half-plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, implying $\sqrt{i(bg+ic)}$ is a conformal map of U onto the quadrant $\{z \in \mathscr{H} : \arg z > 0\}$. Thus, define a map $F_{b,c} : U \to U^+$ by

$$F_{b,c}(z) := -i \frac{\sqrt{i\left(b\frac{1+z}{1-z} + ic\right)} - 1}{\sqrt{i\left(b\frac{1+z}{1-z} + ic\right)} + 1}, \quad z \in U. \tag{4.4}$$

Then $F_{b,c} = -ig^{-1} \circ i(bg+ic)$ maps U conformally onto U^+. Write $f(0) = a_0$. Note that $\operatorname{Re} a_0 > 0$.

Now $F_{b,c}(0) = f(0)$ provided

$$\frac{\sqrt{-c+ib} - 1}{\sqrt{-c+ib} + 1} = ia_0,$$

that is,

$$\sqrt{-c+ib} = \frac{1+ia_0}{1-ia_0} \cdot \frac{1+i\bar{a}_0}{1+i\bar{a}_0} = \frac{1-|a_0|^2 + 2i(\operatorname{Re} a_0)}{1+|a_0|^2 + 2(\operatorname{Im} a_0)}.$$

Thus

$$-c+ib = \frac{(1-|a_0|^2)^2 - 4(\operatorname{Re} a_0)^2}{(1+|a_0|^2 + 2(\operatorname{Im} a_0))^2} + i \frac{4(\operatorname{Re} a_0)(1-|a_0|^2)}{(1+|a_0|^2 + 2(\operatorname{Im} a_0))^2},$$

and so

$$b = \frac{4(\mathrm{Re}\, a_0)(1-|a_0|^2)}{(1+|a_0|^2+2(\mathrm{Im}\, a_0))^2} > 0, \quad c = \frac{4(\mathrm{Re}\, a_0)^2-(1-|a_0|^2)^2}{(1+|a_0|^2+2(\mathrm{Im}\, a_0))^2}.$$

Hence f is subordinate to $F_{b,c}$. Since $F_{b,c}$ is convex, it follows from [6, Lemma 3] (see also [69, Theorem 6.4]) that

$$|a_n| \leq |F'_{b,c}(0)|, \quad n = 1,2,\ldots.$$

Calculation shows that

$$\begin{aligned} F'_{b,c}(z) =& \frac{d}{dz}\left\{ -i\frac{\sqrt{i\left(b\frac{1+z}{1-z}+ic\right)}-1}{\sqrt{i\left(b\frac{1+z}{1-z}+ic\right)}+1} \right\} \\ =& \frac{2b}{(1-z)^2\sqrt{i\left(b\frac{1+z}{1-z}+ic\right)}\left(\sqrt{i\left(b\frac{1+z}{1-z}+ic\right)}+1\right)^2}. \end{aligned}$$

Now

$$b = \frac{4(\mathrm{Re}\, a_0)(1-|a_0|^2)}{(1+|a_0|^2+2(\mathrm{Im}\, a_0))^2} \quad \text{and} \quad \sqrt{i(b+ic)} = \frac{1+ia_0}{1-ia_0},$$

and so

$$\begin{aligned} F'_{b,c}(0) =& \frac{2b}{\sqrt{i(b+ic)}\left(\sqrt{i(b+ic)}+1\right)^2} \\ =& \frac{8(\mathrm{Re}\, a_0)(1-|a_0|^2)}{(1+|a_0|^2+2(\mathrm{Im}\, a_0))^2}\cdot\frac{1-ia_0}{1+ia_0}\cdot\frac{(1-ia_0)^2}{4} \\ =& \frac{2(\mathrm{Re}\, a_0)(1-|a_0|^2)(1-ia_0)}{(1+i\overline{a}_0)^2(1+ia_0)}. \end{aligned}$$

Thus

$$
\begin{aligned}
|F_{b,c}'(0)|^2 &= \frac{2(\operatorname{Re} a_0)(1-|a_0|^2)(1-ia_0)}{(1+i\overline{a}_0)^2(1+ia_0)} \cdot \frac{2(\operatorname{Re} a_0)(1-|a_0|^2)(1+i\overline{a}_0)}{(1-ia_0)^2(1-i\overline{a}_0)} \\
&= \frac{4(\operatorname{Re} a_0)^2(1-|a_0|^2)^2}{(1+|a_0|^2+2(\operatorname{Im} a_0))(1+|a_0|^2-2(\operatorname{Im} a_0))} \\
&= \frac{4(\operatorname{Re} a_0)^2(1-|a_0|^2)^2}{(1+|a_0|^2)^2-4(\operatorname{Im} a_0)^2}, \qquad (4.5)
\end{aligned}
$$

which establishes the result. □

Another important result is that for every $a \in U$, there exists a $w_0 \in \partial U^+$ satisfying $d_U(a, \partial U^+) = d_U(a, w_0)$. Since

$$
\partial U^+ = \left\{ e^{it} : -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\} \cup \{ iy : -1 \leq y \leq 1 \},
$$

it follows that

$$
d_U(a, \partial U^+) = \inf_{-1<y<1} d_U(a, iy) = \inf_{-1<y<1} 2 \tanh^{-1} \left| \frac{a-iy}{1+iay} \right|,
$$

where $a = a_1 + ia_2 \in U$.

Since $\tanh^{-1} x$ is increasing on $-1 < x < 1$, it suffices to find the minimum point of

$$
\begin{aligned}
h(y) := \left| \frac{a-iy}{1+iay} \right|^2 &= \left| \frac{a_1 + i(a_2-y)}{1-a_2y+ia_1y} \right|^2 \\
&= \frac{a_1^2 + (a_2-y)^2}{(1-a_2y)^2 + a_1^2y^2} = \frac{|a|^2 - 2ya_2 + y^2}{1-2ya_2+|a|^2y^2}
\end{aligned}
$$

in $(-1,1)$. Evidently

$$
\begin{aligned}
h'(y) &= 2(|a|^2-1)\frac{(1+y^2)a_2-(1+|a|^2)y}{(1-2ya_2+|a|^2y^2)^2} \\
&= \frac{2(y-a_2)\left((1-a_2y)^2+a_1^2y^2\right)}{\left((1-a_2y)^2+a_1^2y^2\right)^2} \\
&\quad -\frac{2\left(a_1^2y-a_2(1-a_2y)\right)\left(a_1^2+(a_2-y)^2\right)}{\left((1-a_2y)^2+a_1^2y^2\right)^2} \\
&= \frac{2\left(a_1^2+a_2^2-1\right)\left(a_2(1+y^2)-(1+a_1^2+a_2^2)y\right)}{\left((1-a_2y)^2+a_1^2y^2\right)^2},
\end{aligned}
$$

and so $h'(y)=0$ if and only if

$$
g(y)=\left(a_1^2+a_2^2-1\right)\left(a_2(1+y^2)-(1+a_1^2+a_2^2)y\right)=0. \tag{4.6}
$$

If $a_2=0$, then (4.6) gives $g(y)=(1-a_1^4)y$, which vanishes when $y=0$. Further note that $g(y)>0$ if $y>0$, and $g(y)<0$ if $y<0$. Thus $h(0)$ is the minimum point, and

$$
d_U(a,\partial U^+)=d_U(a,0).
$$

Assume now that $a_2\neq 0$. It follows from (4.6) that

$$
y_1=\frac{1+a_1^2+a_2^2+\sqrt{(1+a_1^2+a_2^2)^2-4a_2^2}}{2a_2}
$$

and

$$
y_2=\frac{1+a_1^2+a_2^2-\sqrt{(1+a_1^2+a_2^2)^2-4a_2^2}}{2a_2} \tag{4.7}
$$

are the critical points of h. Let $y_3=(1+a_1^2+a_2^2)/2a_2$ be the midpoint of y_1 and y_2.

Since

$$a_2(1+y^2)-(1+a_1^2+a_2^2)y = a_2\left[\left(y-\frac{1+a_1^2+a_2^2}{2a_2}\right)^2+\frac{4a_2^2-(1+a_1^2+a_2^2)^2}{4a_2^2}\right],$$

it follows from (4.6) that

$$\begin{aligned}
g(y_3) &= (a_1^2+a_2^2-1)\left(4a_2^2-(1+a_1^2+a_2^2)^2\right)/4a_2 \\
&= \left(1-(a_1^2+a_2^2)\right)\left((1+a_1^2+a_2^2)^2-(2a_2)^2\right)/4a_2 \\
&= \left(1-(a_1^2+a_2^2)\right)\left((1+a_1^2+a_2^2)-2a_2\right)\left((1+a_1^2+a_2^2)+2a_2\right)/4a_2 \\
&= \left(1-(a_1^2+a_2^2)\right)\left(a_1^2+(a_2-1)^2\right)\left(a_1^2+(a_2+1)^2\right)/4a_2.
\end{aligned}$$

Therefore, if $a_2 > 0$, then $y_2 < y_1$ and $g(y_3) > 0$. Since $h'(y_3) > 0$, the function h increases on (y_2, y_1). Hence $h(y_2)$ is the minimum point, and

$$d_U(a, \partial U^+) = d_U(a, iy_2).$$

On the other hand, if $a_2 < 0$, then $y_1 < y_2$ and $g(y_3) < 0$, and because $h'(y_3) < 0$, h decreases on (y_1, y_2). Hence $h(y_2)$ is the minimum point, and

$$d_U(a, \partial U^+) = d_U(a, iy_2).$$

The following result is thus established.

Lemma 4.2. *If $a \in U$, then $d_U(a, \partial U^+) = d_U(a, w_0)$ such that*

$$w_0 = \begin{cases} 0 & \text{if } \operatorname{Im} a = 0; \\ iy_2 & \text{if } \operatorname{Im} a \neq 0, \end{cases}$$

where y_2 is given by (4.7).

For $0 \leq t < 1$, define a Möbius transformation $\phi_t : U \to U$ by

$$\phi_t(z) = \frac{z+t}{1+tz}, \quad z \in U.$$

Then ϕ_t (and its inverse ϕ_{-t}) maps $(-1,1)$ to $(-1,1)$, and ϕ_t is an isometry of the hyperbolic distance d_U [31, Theorem 2.1]. The following is a generalization of the previous lemma (Lemma 4.2).

Lemma 4.3. *If $a \in U$, then $d_U(a, \partial\phi_t(U^+)) = d_U(a, \phi_t(w_0))$, where*

$$w_0 = \begin{cases} 0 & \text{if } \operatorname{Im} a = 0; \\ iy_2 & \text{if } \operatorname{Im} a \neq 0, \end{cases}$$

and y_2 is given by (4.7).

A path γ joining z to w in U is called a *hyperbolic geodesic* (or h-geodesic) [97] if

$$d_U(z,w) = \int_\gamma \lambda_U(z)|dz|,$$

where $\lambda_U(z)|dz|$ is the hyperbolic metric given in section 1.7. Indeed, the h-geodesic through z and w is $C \cap U$ where C is the unique Euclidean circle (or straight line) that

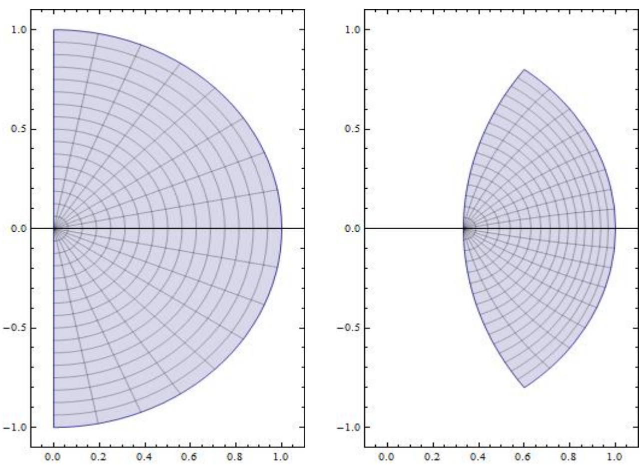

Figure 4.2: Graphs of U^+ (left) and $\phi_{1/3}(U^+)$ (right) on the complex plane.

passes through z and w and is orthogonal to the unit circle ∂U [31].

A set $G \subset U$ is called *hyperbolic convex* (or h-convex) [97] if for any pair z, w of distinct points in G, every h-geodesic joining z and w also lies in G. Analogous to [25, Defintion 2.29], a component of the complement of a h-geodesic in U is called an *open half-plane in U*. It is readily seen that every open half-plane is h-convex.

Since $\phi_t(iy)$, $-1 < y < 1$, is a h-geodesic, it follows that the open half-plane $\phi_t(U^+)$ is h-convex. The following result gives the Bohr radius for the h-convex domain $\phi_t(U^+)$.

Lemma 4.4. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(U^h, \phi_t(U^+))$, then*

$$d_U(\mathcal{M}f(|z|), |f(0)|) \le d_U(f(0), \partial\phi_t(U^+)) \tag{4.8}$$

for $|z| \le r_h/3 \approx 0.15404$, where $r_h = \tanh(1/2)$.

Proof. Recall that $U^h = U_{r_h}$. Define the function

$$\psi(w) := r_h w = z \in H(U, U^h)$$

which maps U conformally onto U^h. Then the function $f \circ \psi \in H(U, U^+)$ is subordinate to the function F_{b^h,c^h} of the form (4.4) for appropriate values b^h, c^h. So by Lemma 4.1,

$$\begin{aligned}\mathcal{M}(f \circ \psi)(|w|) =& |a_0| + \sum_{n=1}^{\infty} |a_n r_h^n||w^n| \\ \leq & |a_0| + |F'_{b^h,c^h}(0)| \frac{|z|/r_h}{1 - |z|/r_h} := F(|z|).\end{aligned}$$

By Lemma 4.3, inequality (4.8) holds when

$$d_U\left(|a_0|, F(|z|)\right) \leq d_U(a_0, \phi_t(w_0)).$$

Indeed, inequality above can be written as

$$d_U\left(0, F(|z|)\right) \leq d_U(a_0, \phi_t(w_0)) + d_U\left(0, a_0\right)$$

which is reduced to

$$\frac{1 + F(|z|)}{1 - F(|z|)} \leq \frac{1 + p_U(a_0, \phi_t(w_0))}{1 - p_U(a_0, \phi_t(w_0))} \left(\frac{1 + |a_0|}{1 - |a_0|}\right),$$

where

$$p_U(a_0, \phi_t(w_0)) = \left|\frac{\phi_t(w_0) - a_0}{1 - \phi_t(w_0)\overline{a}_0}\right|.$$

Note that for $z, w \in \mathbb{C}$,

$$\frac{1+z}{1-z}\left(\frac{1+w}{1-w}\right) = \frac{1+\frac{z+w}{1+zw}}{1-\frac{z+w}{1+zw}}.$$

Further, the real function $(1+r)/(1-r)$ increases with r. It follows that

$$F(|z|) \leq \frac{|a_0| + p_U(a_0, \phi_t(w_0))}{1 + |a_0| p_U(a_0, \phi_t(w_0))}.$$

Equivalently,

$$\begin{aligned}
\left|F'_{b^h,c^h}(0)\right| \frac{|z|/r_h}{1-|z|/r_h} \leq & \frac{|a_0| + p_U(a_0, \phi_t(w_0))}{1 + |a_0| p_U(a_0, \phi_t(w_0))} - |a_0| \\
= & \frac{(1-|a_0|^2) p_U(a_0, \phi_t(w_0))}{1 + |a_0| p_U(a_0, \phi_t(w_0))} \\
= & \frac{(1-|a_0|)(1+|a_0|) p_U(a_0, \phi_t(w_0))}{1 + |a_0| p_U(a_0, \phi_t(w_0))} \\
< & 1 - |a_0|
\end{aligned}$$

because $0 \leq p_U(a_0, \phi_t(w_0)) < 1$. Since (4.5) gives

$$\left|F'_{b,c}(0)\right| = (1-|a_0|)\sqrt{\frac{4(\operatorname{Re} a_0)^2(1+|a_0|)^2}{(1+|a_0|^2)^2 - 4(\operatorname{Im} a_0)^2}},$$

it follows that inequality (4.8) holds for

$$\frac{|z|/r_h}{1-|z|/r_h} \leq \sqrt{\inf_{a_0 \in U^+} \left\{ \frac{(1+|a_0|^2)^2 - 4(\operatorname{Im} a_0)^2}{4(\operatorname{Re} a_0)^2(1+|a_0|)^2} \right\}}.$$

Thus the Bohr radius $\tilde{r}$ satisfies the equation

$$\frac{\tilde{r}/r_h}{1-\tilde{r}/r_h} = \sqrt{\beta},$$

where β is given by

$$\beta = \inf_{a_0 \in U^+} \left\{ \frac{(1+|a_0|^2)^2 - 4(\operatorname{Im} a_0)^2}{4(\operatorname{Re} a_0)^2(1+|a_0|)^2} \right\} = \frac{1}{4}.$$

Thus

$$\tilde{r} = r_h \frac{1/2}{1+1/2} = \frac{r_h}{3}. \qquad \square$$

Denote by $e^{i\theta}\phi_t(U^+)$ the rotation of the half-planes $\phi_t(U^+)$ about the origin by an angle θ, $0 \le \theta < 2\pi$. If G is a set in an open half-plane, define the h-convex hull of G with respect to ϕ_t, denoted by $\tilde{G}^h$, to be the intersection of all $e^{i\theta}\phi_t(U^+)$ containing G. Note that $\tilde{G}^h$ is also Euclidean convex and lie in the minor component of the complement of some h-geodesic in U.

Theorem 4.6. *Let G be an open connected set in an open half-plane such that $\tilde{G}^h$ is not an empty set. If $f \in H(U^h, G)$, then*

$$d_U(\mathcal{M}f(|z|), |f(0)|) \le d_U(f(0), \partial\tilde{G}^h) \tag{4.9}$$

for $|z| \le \tanh(1/2)/3 \approx 0.15404$. *The value* $\tanh(1/2)/3$ *is the best if there exists a point $p \in \mathbb{C}$ such that $p \in \partial\tilde{G} \cap \partial G \cap \partial D$ for some Euclidean disk $D \subset G$.*

Before proving the main theorem, we shall first prove its special case.

Lemma 4.5. *The function* $\phi^h_{a,s,t}(z) \in H(U^h, \phi_t(U^+))$ *given by*

$$\phi^h_{a,s,t}(z) = s\left(1 + \frac{z/r_h - a}{1 - az/r_h}\right) + t, \quad 0 \le a < 1,\ 0 \le t < 1,$$

where $0 < s \le (1-t)/2$ *satisfies*

$$d_U(\mathcal{M}\phi^h_{a,s,t}(|z|), |\phi^h_{a,s,t}(0)|) \le d_U(\phi^h_{a,s,t}(0), \partial\phi^h_{a,s,t}(U^h)) \tag{4.10}$$

for $|z| \le \tanh(1/2)/3 \approx 0.15404$*. The value* $\tanh(1/2)/3$ *is the best.*

Proof. It follows from Lemma 4.4 that

$$d_U(\mathcal{M}\phi^h_{a,s,t}(|z|), |\phi^h_{a,s,t}(0)|) \le d_U(\phi^h_{a,s,t}(0), \partial\phi_t(U^+))$$

for $|z| \le \tanh(1/2)/3$. Since $\phi^h_{a,s,t}(0) > 0$, Lemma 4.3 shows that the preceding inequality is equivalent to

$$d_U(\mathcal{M}\phi^h_{a,s,t}(|z|), |\phi^h_{a,s,t}(0)|) \le d_U(\phi^h_{a,s,t}(0), t).$$

Next, to obtain (4.10), it is sufficient to prove

$$d_U(\phi^h_{a,s,t}(0), t) = d_U(\phi^h_{a,s,t}(0), \partial\phi^h_{a,s,t}(U^h)). \tag{4.11}$$

Let $a_0 = \phi^h_{a,s,t}(0) = s(1-a) + t > 0$. Now,

$$d_U(a_0, \partial\phi^h_{a,s,t}(U^h)) = \inf_{\zeta\in\partial U^h} d_U(a_0, \phi^h_{a,s,t}(\zeta))$$

$$
\begin{aligned}
&= \inf_{0\le\theta<2\pi} d_U(a_0, s(1+e^{i\theta})+t) \\
&= \inf_{0\le\theta<2\pi} \log \frac{1+p_U(a_0, s(1+e^{i\theta})+t)}{1-p_U(a_0, s(1+e^{i\theta})+t)},
\end{aligned}
$$

where

$$
p_U(a_0, s(1+e^{i\theta})+t) = \left| \frac{a_0 - s(1+e^{i\theta}) - t}{1-a_0(s(1+e^{i\theta})+t)} \right|.
$$

Thus, it suffices to minimize $p_U(a_0, s(1+e^{i\theta})+t)$ with respect to θ, or equivalently, minimizing

$$
\begin{aligned}
\left| \frac{a_0 - s(1+e^{i\theta}) - t}{1-a_0(s(1+e^{i\theta})+t)} \right|^2 &= \left| \frac{a_0 - t - s - se^{i\theta}}{1 - a_0 s - a_0 t - a_0 s e^{i\theta}} \right|^2 \\
&= \frac{(a_0 - t - s - s\cos\theta)^2 + s^2\sin^2\theta}{(1 - a_0 s - a_0 t - a_0 s\cos\theta)^2 + a_0^2 s^2 \sin^2\theta} \\
&= \frac{(a_0 - t - s)^2 - 2s(a_0 - t - s)\cos\theta + s^2}{(1 - a_0 s - a_0 t)^2 - 2a_0 s(1 - a_0 s - a_0 t)\cos\theta + a_0^2 s^2}.
\end{aligned}
$$

A critical point θ_0 occurs when

$$
\frac{-2s(1-a_0^2)\big(s+t+a_0^2(s+t) - a_0(1+2st+t^2)\big)\sin\theta_0}{((1 - a_0 s - a_0 t)^2 - 2a_0 s(1 - a_0 s - a_0 t)\cos\theta_0 + a_0^2 s^2)^2} = 0,
$$

or

$$
\big(s+t+(s(1-a)+t)^2(s+t) - (s(1-a)+t)(1+2st+t^2)\big)\sin\theta_0 = 0.
$$

Note that the coefficient $s+t+(s(1-a)+t)^2(s+t) - (s(1-a)+t)(1+2st+t^2) \neq 0$. Indeed, if

$$
s+t+(s(1-a)+t)^2(s+t) - (s(1-a)+t)(1+2st+t^2) = 0,
$$

then

$$a = \frac{-1+2s^2+2st+t^2 \pm \sqrt{(1-t^2)(1-4s^2-4st-t^2)}}{2(s^2+st)}.$$

Since $1-4s^2-4st-t^2 \geq 0$ for $0 < s \leq (1-t)/2$, it follows that a is real. Now,

$$\begin{aligned} 1-2s^2-2st-t^2 \geq & 1-2\left(\frac{1-t}{2}\right)^2 - 2t\left(\frac{1-t}{2}\right) - t^2 \\ = & 1-\frac{1-2t+t^2}{2} - t + t^2 - t^2 \\ = & \frac{1-t^2}{2} > 0, \end{aligned}$$

and by the fact that

$$(1-t^2)(1-4s^2-4st-t^2) = (1-2s^2-2st-t^2)^2 - 4(s^2+st)^2,$$

it is evident that $a < 0$ and so contradicting the positivity of a.

Hence the critical point occurs when $\sin\theta_0 = 0$, that is, at $\theta_0 = 0$ or $\theta_0 = \pi$, implying

$$d_U(a_0, \partial\phi^h_{a,s,t}(U^h)) = d_U(a_0, 2s+t) \quad \text{or} \quad d_U(a_0, \partial\phi^h_{a,s,t}(U^h)) = d_U(a_0, t).$$

It remains to check the changes of sign of

$$\frac{d}{d\theta}\big(p_U(a_0, s(1+e^{i\theta})+t)\big)^2 = \frac{-2s(1-a_0^2)\big(s+t+a_0^2(s+t)-a_0(1+2st+t^2)\big)\sin\theta}{((1-a_0s-a_0t)^2-2a_0s(1-a_0s-a_0t)\cos\theta+a_0^2s^2)^2}$$

for a given range of θ. Indeed, it suffices to check the changes of sign of

$$-\big(s+t+(s(1-a)+t)^2(s+t)-(s(1-a)+t)(1+2st+t^2)\big)\sin\theta.$$

Earlier it has been shown that

$$s+t+(s(1-a)+t)^2(s+t)-(s(1-a)+t)(1+2st+t^2) \tag{4.12}$$

is nonzero for all $0 \le a < 1$ and $0 < s \le (1-t)/2$. In particular, if $s = (1-t)/2$, then (4.12) becomes $(1/8)(1+a)^2(-1+t)^2(1+t) > 0$, which is always positive for all a, t. Thus, it suffices to determine only the sign changes of $-\sin\theta$ when θ varies, which then gives

$$d_U(a_0, \partial\phi^h_{a,s,t}(U^h)) = d_U(a_0, t).$$

To show the sharpness, it suffices to show that for every $|z| > r_h/3$ there exists a corresponding $0 < a < 1$ such that

$$d_U(\mathcal{M}\phi^h_{a,s,t}(|z|), a_0) > d_U(a_0, t).$$

Since

$$\phi^h_{a,s,t}(z) = s\left(1-a+(1-a^2)\sum_{n=1}^{\infty} a^{n-1}\left(\frac{z}{r_h}\right)^n\right)+t,$$

it follows that

$$\mathcal{M}\phi^h_{a,s,t}(z) = s\left(1-a+(1-a^2)\sum_{n=1}^{\infty} a^{n-1}\left(\frac{z}{r_h}\right)^n\right)+t$$

$$
\begin{aligned}
&= s\left(1+\frac{z/r_h-a}{1-az/r_h}\right)+t \\
&= \phi^h_{a,s,t}(0)+s\left(a+\frac{z/r_h-a}{1-az/r_h}\right).
\end{aligned}
$$

Now

$$d_U(\mathcal{M}\phi^h_{a,s,t}(|z|),a_0) > d_U(a_0,t)$$

gives

$$\frac{\mathcal{M}\phi^h_{a,s,t}(|z|)-a_0}{1-a_0\mathcal{M}\phi^h_{a,s,t}(|z|)} > \frac{a_0-t}{1-a_0t},$$

since $a_0=\phi^h_{a,s,t}(0)=s(1-a)+t>t$, which holds if and only if

$$s(1-a_0t)\left(a+\frac{|z|/r_h-a}{1-a|z|/r_h}\right) > s(1-a)\left[1-a_0^2-sa_0\left(a+\frac{|z|/r_h-a}{1-a|z|/r_h}\right)\right].$$

The latter is equivalent to

$$(1-a_0t+sa_0(1-a))\left(\frac{(1-a^2)|z|/r_h}{1-a|z|/r_h}\right) > (1-a)(1-a_0^2)$$

which holds provided

$$
\begin{aligned}
\frac{(1+a)|z|/r_h}{1-a|z|/r_h} &> \frac{1-a_0^2}{1-a_0t+a_0(a_0-t)} = \frac{1-a_0^2}{1+a_0^2-2ta_0} \\
&= \frac{1-(s(1-a)+t)^2}{1+(s(1-a)+t)^2-2t(s(1-a)+t)} \\
&= \frac{1-s^2(1-a)^2-t^2-2st(1-a)}{1+s^2(1-a)^2-t^2}.
\end{aligned}
$$

If $|z| = r_h/(3-\varepsilon)$ for some arbitrary small $\varepsilon > 0$, then

$$\frac{1+a}{3-\varepsilon-a} > \frac{1-s^2(1-a)^2-t^2-2st(1-a)}{1+s^2(1-a)^2-t^2}$$

which reduces to

$$\begin{aligned} a^2s((4-\varepsilon)s+2t)+2a(1+(-4+\varepsilon)s^2+(-4+\varepsilon)st-t^2) \\ -e(-1+s^2+2st+t^2)+2(-1+2s^2+3st+t^2) > 0. \end{aligned}$$

Hence $a < A_1$ or $a > A_2$, where

$$\begin{aligned} A_1 =& \frac{-1+(4-\varepsilon)s^2+(4-\varepsilon)st+t^2-\sqrt{-4\varepsilon s^2+\varepsilon^2s^2+(-1+2st+t^2)^2}}{(4-\varepsilon)s^2+2st}, \\ A_2 =& \frac{-1+(4-\varepsilon)s^2+(4-\varepsilon)st+t^2+\sqrt{-4\varepsilon s^2+\varepsilon^2s^2+(-1+2st+t^2)^2}}{(4-\varepsilon)s^2+2st}. \end{aligned}$$

Since

$$-1+(4-\varepsilon)s^2+(4-\varepsilon)st+t^2 \leq -\varepsilon(1-t^2)/4,$$

it follows that $a < A_1 < 0$, which contradicts $a > 0$. On the other hand, $A_2 < 1$ if and only if $-\varepsilon s((4-\varepsilon)s+2t)(1-t^2) < 0$, which always holds true. Hence for $|z| = r_h/(3-\varepsilon) > r_h/3$, there exists a real a with $A_2 < a < 1$, such that (4.9) fails to hold. Hence the value $r_h/3$ is best. □

Proof of Theorem 4.6. Since rotation about the origin is an isometry of the hyperbolic distance d_U [31, Theorem 2.1], it follows that Lemma 4.4 remains valid for the class $H(U^h, e^{i\theta}\phi_t(U^+))$ for all θ, which then implies (4.9).

Write p in the form $|p|e^{i\theta_p}$ for some $\theta_p \in [0, 2\pi)$. Then

$$p \in \partial \left(e^{i\theta_p} \phi^h_{a,s,|p|}(U^h) \right)$$

for all $0 < s \leq (1-|p|)/2$. Further, it is possible to choose some $0 < s_p \leq (1-|p|)/2$ such that $e^{i\theta_p} \phi^h_{a,s_p,|p|} \in H(U^h, D) \subset H(U^h, G)$ and so

$$p \in (\partial \tilde{G} \cap \partial G \cap \partial D) \cap \partial \left(e^{i\theta_p} \phi^h_{a,s_p,|p|}(U^h) \right).$$

By equation (4.11),

$$d_U \left(e^{i\theta_p} \phi^h_{a,s_p,|p|}(0), \partial \left(e^{i\theta_p} \phi^h_{a,s_p,|p|}(U^h) \right) \right) = d_U \left(e^{i\theta_p} \phi^h_{a,s_p,|p|}(0), p \right),$$

which then implies

$$d_U \left(e^{i\theta_p} \phi^h_{a,s_p,|p|}(0), \partial \tilde{G} \right) = d_U \left(e^{i\theta_p} \phi^h_{a,s_p,|p|}(0), p \right)$$

because $e^{i\theta_p} \phi^h_{a,s_p,|p|}(U^h) \subset G$. Thus

$$d_U \left(e^{i\theta_p} \phi^h_{a,s_p,|p|}(0), \partial \tilde{G} \right) = d_U \left(e^{i\theta_p} \phi^h_{a,s_p,|p|}(0), \partial \left(e^{i\theta_p} \phi^h_{a,s_p,|p|}(U^h) \right) \right)$$

implying the equivalence of (4.9) and (4.10) for the function $e^{i\theta_p} \phi^h_{a,s_p,|p|}$. Hence the radius $r_h/3$ is sharp. $\square$

Theorem 4.6 is applied to obtain the Bohr-type theorem for other hyperbolic regions. First, a region Ω is said to be a *hyperbolic region* if $\mathbb{C} \cup \{\infty\} \setminus \Omega$ contains at least three points. The Planar Uniformization Theorem [31, Theorem 10.2] states that each hyperbolic region Ω corresponds to a universal covering map $F \in H(U, \Omega)$.

The hyperbolic metric λ_Ω in the hyperbolic region Ω satisfies the relation [31, Theorem 10.5]

$$\lambda_\Omega(F(z))|F'(z)| = \lambda_U(z)$$

for all $z \in U$. The hyperbolic distance d_Ω is defined by

$$d_\Omega(w_1, w_2) = \inf_\gamma \int_\gamma \lambda_\Omega(w)\,|dw|$$

over all smooth curves γ in Ω joining w_1 to w_2. If Ω is simply connected, then

$$d_\Omega(F(z), F(w)) = d_U(z, w), \quad z, w \in U.$$

On the other hand, if Ω is multiply connected, then F is not injective giving

$$d_\Omega(F(z), F(w)) \leq d_U(z, w), \quad z, w \in U.$$

For a set G in a hyperbolic region Ω with a corresponding covering map $F \in H(U, \Omega)$, define the h-convex hull of G with respect to ϕ_t, denoted by $\tilde{G}^h$, to be the intersection of all $F(e^{i\theta}\phi_t(U^+))$ containing G.

Theorem 4.7. *Suppose $f \in H(U^h, G)$ with $f(0) \geq 0$ and $G \subseteq \Omega$ for some hyperbolic region $\Omega \subset \mathbb{C} \cup \{\infty\}$. Let $F \in H(U, \Omega)$ be a universal covering map of $\Omega \backslash \{f(0)\}$ by $U \backslash \{0\}$ with $F(0) = f(0)$. If $F(z) = \sum_{n=0}^{\infty} A_n z^n$ such that $A_n \geq 0$ for all $n \geq 1$ and*

$F^{-1}(\tilde{G}^h)$ is non-empty in U, then

$$d_\Omega(\mathcal{M}f(|z|),|f(0)|) \le d_\Omega(f(0),\partial\tilde{G}^h) \tag{4.13}$$

for $|z| \le r_h/3 = \tanh(1/2)/3 \approx 0.15404$.

Proof. Since F is a covering of Ω and $f \in H(U^h, G) \subset H(U^h, \Omega)$ with $f(0) = F(0)$, it follows that the function $F^{-1} \circ f$ has a branch at 0. The monodromy theorem now shows that $F^{-1} \circ f$ can be continued holomorphically to all of U^h. Consequently, Theorem 4.6 gives

$$d_U(\mathcal{M}(F^{-1} \circ f)(|z|), |F^{-1}(f(0))|) \le d_U(F^{-1}(f(0)), \partial F^{-1}(\tilde{G}^h)),$$

or because

$$d_U(0, \partial F^{-1}(\tilde{G}^h)) = d_\Omega(f(0), \partial\tilde{G}^h),$$

that

$$d_U(\mathcal{M}(F^{-1} \circ f)(|z|), 0) \le d_\Omega(f(0), \partial\tilde{G}^h)$$

for $|z| \le r_h/3 = \tanh(1/2)/3$. Thus, inequality (4.13) holds for $|z| \le r_h/3$ when

$$d_\Omega(\mathcal{M}f(|z|), |f(0)|) \le d_U(\mathcal{M}(F^{-1} \circ f)(|z|), 0).$$

Now by definition, $F(0) = f(0) \ge 0$ if and only if $z = 0$. It follows then that

$$d_\Omega(\mathcal{M}f(|z|), |f(0)|) = d_U(F^{-1}(\mathcal{M}f(|z|)), F^{-1}(|f(0)|))$$

$$=d_U(F^{-1}(\mathcal{M}f(|z|)),0).$$

So it suffices to show

$$d_U(F^{-1}(\mathcal{M}f(|z|)),0) \leq d_U(\mathcal{M}(F^{-1}\circ f)(|z|),0),$$

which reduces to

$$F^{-1}(\mathcal{M}f(|z|)) \leq \mathcal{M}(F^{-1}\circ f)(|z|).$$

Since F is increasing on $[0,1)$, the latter is equivalent to

$$\mathcal{M}f(|z|) \leq F(\mathcal{M}(F^{-1}\circ f)(|z|)).$$

Finally with $\phi = F^{-1}\circ f \in H(U^h,U)$, the above inequality becomes

$$\mathcal{M}(F\circ\phi)(|z|) \leq F(\mathcal{M}\phi(|z|)),$$

which holds true by applying the triangle inequality. $\square$

Observe that the preceding theorem holds for $\Omega = \mathbb{C}\backslash\{0,-1\}$ where its corresponding covering map is given by the modular function [98]

$$J(z) = 16z\prod_{n=1}^{\infty}\left(\frac{1+z^{2n}}{1-z^{2n-1}}\right)^8, \quad z\in U.$$

The theorem also holds for $\Omega = \mathscr{H}$, where $\mathscr{H} = \{z\in\mathbb{C} : \operatorname{Re} z > 0\}$ is the right half-plane in $\mathbb{C}$ and its covering map is given by $\phi_H(z) = (1+z)/(1-z) \in H(U,\mathscr{H})$.

CHAPTER 5

BOHR AND TYPES OF FUNCTIONS

5.1 Harmonic Mappings

5.1.1 Harmonic mappings into a bounded domain

In this subsection, we shall find the Bohr radius for bounded harmonic functions on the disk. Let D be a bounded set and denote by $\overline{D}$ the closure of D. Let $\overline{D}_{min}$ be the smallest closed disk containing the closure of D. Thus

$$\overline{D} \subseteq \overline{D}_{min} \subseteq \overline{E}$$

for any closed disk $\overline{E}$ containing $\overline{D}$. The following two lemmas are required to deduce the main theorem in this section.

Lemma 5.1. *(See [6]) If $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$, and $f(U)$ is a convex set, then*

$$|b_n| \leq |f'(0)| \leq 2d(f(0), \partial f(U)).$$

Lemma 5.2. *(See [6]) Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}$ be a complex-valued harmonic function on U. If f maps U into a bounded domain D, then*

$$|a_n| + |b_n| \leq \frac{4}{\pi}\rho, \tag{5.1}$$

$$|e^{i\mu} a_n + e^{-i\mu} b_n| \leq 2\left(\rho - |\mathrm{Re}\, e^{i\mu}(a_0 - w_0)|\right), \tag{5.2}$$

for any real μ and any $n \geq 1$, where ρ and w_0 are respectively the radius and center *of* $\overline{D}_{min}$.

Proof. Since $f(U)$ is contained in a disk with radius ρ and center w_0, it follows that

$$\rho = |f(z) - w_0| + d(f(z), \partial\overline{D}_{min}), \quad \text{or} \quad |f(z) - w_0| = \rho - d(f(z), \partial\overline{D}_{min});$$

that is,

$$|f(z) - w_0| \leq \rho$$

for all $z \in U$. Consequently

$$\left|\mathrm{Re}\,[e^{i\mu}[f(z) - w_0]]\right| \leq \left|e^{i\mu}[f(z) - w_0]\right| \leq \rho$$

for any real μ and any $z \in D$. Let

$$W_\mu(z) = e^{i\mu}(h(z) - w_0) + e^{-i\mu}g(z).$$

Then W_μ is analytic, $W_\mu(0) = e^{i\mu}(a_0 - w_0)$ and $\left|\mathrm{Re}\, W_\mu(z)\right| = \left|\mathrm{Re}\,[e^{i\mu}[f(z) - w_0]]\right| < \rho$.

The function

$$F(z) = \frac{2i}{\pi}\rho \log \frac{1+z}{1-z}$$

maps U conformally onto the strip $\mathscr{S} = \{z \in \mathbb{C} : |\mathrm{Re}\, z| < \rho\}$. As $e^{i\mu}(a_0 - w_0) \in \mathscr{S}$, choose $b \in U$ so that $F(b) = e^{i\mu}(a_0 - w_0)$ and let

$$\varphi(z) = \frac{z+b}{1+\bar{b}z}.$$

Then $F(\varphi(0)) = e^{i\mu}(a_0 - w_0) = W_\mu(0)$, and hence W_μ is subordinate to $F \circ \varphi$.

Simple calculations give

$$(F \circ \varphi)'(0) = \frac{4i(1-|b|^2)}{\pi(1-b^2)}\rho \quad \text{and so} \quad |(F \circ \varphi)'(0)| \le \frac{4}{\pi}\rho. \tag{5.3}$$

As $F \circ \varphi$ is convex and

$$d(F(\varphi(0)), \partial P) = \rho - |\operatorname{Re} W_\mu(0)| = \rho - |\operatorname{Re} e^{i\mu}(a_0 - w_0)|,$$

Lemma 5.1 implies (5.2). In addition, Lemma 5.1 and (5.3) imply

$$|e^{i\mu} a_n + e^{-i\mu} b_n| \le \frac{4}{\pi}\rho.$$

If $a_n = 0$, then inequality (5.1) is evident. If $a_n \ne 0$, then

$$|e^{i\mu} a_n + e^{-i\mu} b_n| = |a_n| \left| 1 + e^{-2i\mu}\left(\frac{b_n}{a_n}\right) \right|,$$

and μ can be chosen so that $e^{-2i\mu}(b_n/a_n) = |b_n/a_n|$, which gives (5.1). $\square$

Remark 5.1. *When $D = U$, Lemma 5.2 reduces to Lemma 4 in [6].*

The following is the main result in this subsection.

Theorem 5.1. *Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}$ be a complex-valued harmonic function on U. If $f : U \to D$ for some bounded domain D, then, for $|z| \le 1/3$,*

$$\sum_{n=1}^{\infty} |a_n z^n| + \sum_{n=1}^{\infty} |b_n z^n| \le \frac{2}{\pi}\rho \tag{5.4}$$

and

$$\sum_{n=1}^{\infty} |e^{i\mu} a_n + e^{-i\mu} b_n||z|^n + |\mathrm{Re}\, e^{i\mu}(a_0 - w_0)| \leq \rho, \quad \mu \in \mathbb{R}, \tag{5.5}$$

where ρ *and* w_0 *are respectively the radius and center of* $\overline{D}_{min}$.

The bound $1/3$ *is sharp as demonstrated by an analytic univalent mapping* f *from* U *onto* D. *In particular, if* D *is an open disk with radius* $\rho > 0$ *centered at* ρw_0, *then the sharpness is shown by the Möbius transformation*

$$\varphi(z) = e^{i\mu_0} \rho \left(\frac{z+a}{1+az} + |w_0| \right), \quad z \in U,$$

for some $0 < a < 1$ *and* μ_0 *satisfying* $w_0 = |w_0| e^{i\mu_0}$.

Proof. If $|z| = 1/3$, then it follows from (5.2) that

$$\sum_{n=1}^{\infty} |e^{i\mu} a_n + e^{-i\mu} b_n||z^n| \leq \rho - |\mathrm{Re}\, e^{i\mu}(a_0 - w_0)|,$$

and (5.5) is evident. On the other hand, (5.1) yields

$$|a_n| + |b_n| \leq \frac{4}{\pi} \rho,$$

and with $|z| = 1/3$ gives

$$\sum_{n=1}^{\infty} |a_n z^n| + \sum_{n=1}^{\infty} |b_n z^n| \leq \frac{1}{2} \left(\frac{4}{\pi} \rho \right) = \frac{2}{\pi} \rho,$$

which is (5.4).

For the sharpness, consider the Möbius transformation

$$\varphi(z) = e^{i\mu_0}\rho\left(\frac{z+a}{1+az} + |w_0|\right), \quad 0 < a < 1,\ z \in U.$$

Then

$$\varphi(z) = e^{i\mu_0}\rho(a+|w_0|) + e^{i\mu_0}\rho(1-a^2)\sum_{n=1}^{\infty}(-a)^{n-1}z^n$$

yields

$$\mathcal{M}\varphi(z) = \rho(a+|w_0|) + \sum_{n=1}^{\infty}\rho(1-a^2)a^{n-1}z^n = \rho(2a+|w_0|) + \rho\left(\frac{z-a}{1-az}\right).$$

For any fixed r_0, a brief computation shows that

$$\frac{1}{\rho}\mathcal{M}\varphi(r_0) - |w_0| = \frac{a+(1-2a^2)r_0}{1-ar_0} \geq 1$$

provided $r_0 \geq 1/(1+2a)$, or equivalently $a \geq (1/2)(1/r_0 - 1)$. Hence for any $r_0 > 1/3$, there exists an a satisfying $1 > a \geq (1/2)(1/r_0 - 1)$, where (5.5) does not hold in the open disk D with radius $\rho > 0$ centered at ρw_0. Hence the bound $1/3$ is best possible. □

Remark 5.2. *In the case D is the unit disk U, Theorem 5.1 reduces to Theorem 2 in [6].*

5.1.2 Harmonic mappings into a wedge domain

Denote by S_W the set of all univalent, harmonic, orientation-preserving mappings f of the unit disk U into the wedge domain

$$W = \{w : |\arg w| < \pi/4\},$$

with normalization $f(0) = 1$. Let $\overline{S_W}$ denote the closure of S_W in the topology of uniform convergence on compact sets. In [12], the authors described the extreme points for the closed convex hull of S_W. As an application, they obtained coefficient bounds for functions on $\overline{S_W}$. Here in the sequel, we shall adopt the notations used in [12]. Thus let P denote the Poisson kernel

$$P(z,t) := \frac{1}{2\pi}\,\mathrm{Re}\left[\frac{e^{it}+z}{e^{it}-z}\right].$$

Choose α, β and γ so that $\alpha < \beta < \gamma < 2\pi + \alpha$, and let

$$I_1 = \{e^{it} : \alpha < t < \beta\} \quad \text{and} \quad I_2 = \{e^{it} : \gamma < t < 2\pi + \alpha\}.$$

The lengths of these arcs are $|I_1| = \beta - \alpha$ and $|I_2| = 2\pi + \alpha - \gamma$. Next, consider the harmonic functions

$$U_1(z) = \frac{\pi}{|I_1|}\int_{\alpha}^{\beta} P(z,t)dt = -\frac{1}{2} + \frac{1}{|I_1|}\arg\frac{e^{i\beta}-z}{e^{i\alpha}-z}$$

and

$$U_2(z) = \frac{\pi}{|I_2|}\int_{\gamma}^{\alpha+2\pi} P(z,t)dt = -\frac{1}{2} + \frac{1}{|I_2|}\arg\frac{e^{i\alpha}-z}{e^{i\gamma}-z},$$

where the branches of the arguments are chosen so that

$$U_1(0) = U_2(0) = \frac{1}{2}.$$

Note that the boundary values of U_j are $\pi/|I_j|$ on $|I_j|$ and zero on $\partial U \backslash \overline{I_j}$.

Define

$$T_{(\alpha,\beta,\gamma)} = (1+i)U_1 + (1-i)U_2. \tag{5.6}$$

Then $T_{(\alpha,\beta,\gamma)}(0) = 1$ and

$$\hat{T}_{(\alpha,\beta,\gamma)}(e^{it}) = \lim_{r\to 1} T_{(\alpha,\beta,\gamma)}(re^{it}) = \begin{cases} \frac{(1+i)\pi}{|I_1|} & \text{if } e^{it} \in I_1; \\ \frac{(1-i)\pi}{|I_2|} & \text{if } e^{it} \in I_2; \\ 0 & \text{if } e^{it} \in \partial U \backslash [\overline{I_1} \cup \overline{I_2}]. \end{cases}$$

In [12], $T_{(\alpha,\beta,\gamma)}$ is shown to be a univalent, harmonic, orientation-preserving mapping of U onto the open triangle with vertices at the origin and the points

$$\frac{(1+i)\pi}{|I_1|} \quad \text{and} \quad \frac{(1-i)\pi}{|I_2|}.$$

The cluster sets [47, p. 1] of $T_{(\alpha,\beta,\gamma)}$ at the points $e^{i\alpha}, e^{i\beta}$ and $e^{i\gamma}$ are the respective sides of the triangle. We refer to [12] for additional details on U_1, U_2 and $T_{(\alpha,\beta,\gamma)}$. Finally, let

$$s(x) := \frac{\sin x}{x} \quad \text{for } x > 0, \text{ and } s(0) = 1.$$

The following result will be required.

Theorem 5.2. *[12, Theorem 3.8] Let*

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$$

belong to $\overline{S_W}$. *Then*

$$|a_n| \leq \sqrt{s(x_0)^2(1+\sin 2x_0)} \approx 1.3082, \quad n \geq 1,$$

where $x_0 \approx 0.5875 \in [\pi/8, \pi/4]$ *is the unique root of*

$$s'(x)(1+\sin 2x) + s(x)\cos 2x = 0.$$

Equality for any n is possible only for the function $T_{(\alpha,\beta,\gamma)}$ *with*

$$\beta = \alpha + \frac{2x_0}{n} \quad \textit{and} \quad \gamma = \alpha + 2\pi - \frac{2x_0}{n}.$$

In addition, $|b_n| \leq 1$ *for all* $n \geq 1$. *Equality in this case for any n is possible for the functions* $2\pi P(\,\cdot\,,\alpha)$.

The following lemma is also required to infer the main theorem in this section.

Lemma 5.3. *Let* $f(z) = h(z) + \overline{g(z)} = 1 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n} \in \overline{S_W}$. *Then*

$$|a_n| + |b_n| \leq \left(\sqrt{s(x_0)^2(1+\sin 2x_0)} + 1\right) \approx 2.3082, \tag{5.7}$$

$$|e^{i\mu} a_n + e^{-i\mu} b_n| \leq 2d(e^{i\mu}, \partial\mathscr{H}) = 2\cos\mu, \tag{5.8}$$

for all $-\pi/4 < \mu < \pi/4$ *and* $n \geq 1$ *where* $\mathscr{H}$ *is the right half-plane, and* x_0 *is given* *by Theorem 5.2.*

Proof. Inequality (5.7) is a direct consequence of Theorem 5.2. To show (5.8), let $W_\mu(z) = e^{i\mu}h(z) + e^{-i\mu}g(z)$. Then W_μ is analytic, $W_\mu(0) = e^{i\mu}$, and $\operatorname{Re} W_\mu(z) = \operatorname{Re}[e^{i\mu} f(z)] > 0$ for $-\pi/4 < \mu < \pi/4$.

The function

$$F(z) = \frac{1+z}{1-z}$$

maps U conformally onto the right half-plane $\mathscr{H}$. As $e^{i\mu} \in \mathscr{H}$, choose $b \in U$ so that $F(b) = e^{i\mu}$ and let

$$\varphi(z) = \frac{z+b}{1+\bar{b}z}.$$

Then $F(\varphi(0)) = e^{i\mu} = W_\mu(0)$, and hence W_μ is subordinate to $F \circ \varphi$. As $(F \circ \varphi)(U) = \mathscr{H}$ is convex, Lemma 5.1 implies (5.8). □

Theorem 5.3. *Let* $f(z) = h(z) + \overline{g(z)} = 1 + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in \overline{S_W}$. *If* $|z| \le 1/3$, *then*

$$\sum_{n=1}^{\infty} |a_n z^n| + \sum_{n=1}^{\infty} |b_n z^n| \le \frac{1}{2}\left(\sqrt{s(x_0)^2(1+\sin 2x_0)} + 1\right) \approx 1.1541, \tag{5.9}$$

and

$$\sum_{n=1}^{\infty} |e^{i\mu} a_n + e^{-i\mu} b_n| |z|^n \le d(e^{i\mu}, \partial\mathscr{H}) = \cos\mu \tag{5.10}$$

for any $-\pi/4 < \mu < \pi/4$, *where* $\mathscr{H}$ *is the right half-plane and* x_0 *is given by Theorem 5.2. The value* $1/3$ *is best possible for inequality* (5.10) *to hold.*

Proof. Since $|z| \le 1/3$, it follows that (5.7) implies (5.9), and (5.8) implies (5.10). To show $1/3$ is best, consider the harmonic function

$$\begin{aligned}T_{(\alpha,\beta,\gamma)}(z) =&(1+i)U_1(z)+(1-i)U_2(z)\\ =&-1+\frac{1+i}{|I_1|}\arg\frac{e^{i\beta}-z}{e^{i\alpha}-z}+\frac{1-i}{|I_2|}\arg\frac{e^{i\alpha}-z}{e^{i\gamma}-z}\end{aligned}$$

as given by formula (5.6). Let

$$T_{(\alpha,\beta,\gamma)}(z)=1+\sum_{n=1}^{\infty}A_nz^n+\overline{\sum_{n=1}^{\infty}B_nz^n}.$$

Comparing the coefficients gives

$$A_n=\frac{(1-i)(e^{-in\alpha}-e^{-in\beta})}{2n|I_1|}-\frac{(1+i)(e^{-in\gamma}-e^{-in\alpha})}{2n|I_2|}$$

and

$$B_n=\frac{-(1+i)(e^{-in\alpha}-e^{-in\beta})}{2n|I_1|}+\frac{(1-i)(e^{-in\gamma}-e^{-in\alpha})}{2n|I_2|}.$$

Let $x_n=n|I_1|/2$ and $y_n=n|I_2|/2$. Note that

$$\begin{aligned}\frac{(1-i)(e^{-in\alpha}-e^{-in\beta})}{2n|I_1|}&=e^{-in\alpha}\frac{(1-i)(1-e^{-in(\beta-\alpha)})}{2n|I_1|}\\ &=\frac{e^{-in\alpha}}{4x_n}(1-i)(1-e^{-2ix_n})\\ &=\frac{e^{-in\alpha}}{2x_n}(\sin^2x_n-i\sin^2x_n\\ &\quad+\cos x_n\sin x_n+i\cos x_n\sin x_n)\\ &=\frac{e^{-in\alpha}s(x_n)}{2}(1+i)(\cos x_n-i\sin x_n)\\ &=e^{-in\alpha}\frac{(1+i)}{2}e^{-ix_n}s(x_n).\end{aligned}$$

Applying the same technique to the other three terms gives

$$A_n = e^{-in\alpha}\left[\frac{(1+i)}{2}e^{-ix_n}s(x_n) + \frac{(1-i)}{2}e^{-iy_n}s(y_n)\right]$$

and

$$B_n = e^{-in\alpha}\left[\frac{(1-i)}{2}e^{-ix_n}s(x_n) + \frac{(1+i)}{2}e^{-iy_n}s(y_n)\right].$$

Therefore

$$\begin{aligned}
e^{i\mu}A_n + e^{-i\mu}B_n =& e^{i\mu}e^{-in\alpha}\left[\frac{(1+i)}{2}e^{-ix_n}s(x_n) + \frac{(1-i)}{2}e^{-iy_n}s(y_n)\right] \\
&+ e^{-i\mu}e^{-in\alpha}\left[\frac{(1-i)}{2}e^{-ix_n}s(x_n) + \frac{(1+i)}{2}e^{-iy_n}s(y_n)\right] \\
=& \frac{e^{-in\alpha}e^{-ix_n}s(x_n)}{2}\left[e^{i\mu}(1+i) + e^{-i\mu}(1-i)\right] \\
&+ \frac{e^{-in\alpha}e^{-iy_n}s(y_n)}{2}\left[e^{i\mu}(1-i) + e^{-i\mu}(1+i)\right].
\end{aligned}$$

If $\beta + \gamma = 2(\pi + \alpha)$, then $x_n = y_n$. Hence

$$\left|e^{i\mu}A_n + e^{-i\mu}B_n\right| = |e^{i\mu} + e^{-i\mu}||s(x_n)| = 2|s(x_n)|\cos\mu$$

for any $-\pi/4 < \mu < \pi/4$. Thus for $|z| > 1/3$,

$$\begin{aligned}
\sum_{n=1}^{\infty} |e^{i\mu}A_n + e^{-i\mu}B_n||z|^n > 2\cos\mu \sum_{n=1}^{\infty} |s(x_n)| \left(\frac{1}{3}\right)^n \\
\longrightarrow 2\cos\mu \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = \cos\mu
\end{aligned}$$

as $|I_1| = |I_2| \longrightarrow 0$. Hence the value $1/3$ is best possible. $\square$

5.2 Logharmonic Mappings

Denote by S_{Lh} the class consisting of univalent logharmonic maps f defined on U and of the form

$$f(z) = zh(z)\overline{g(z)}$$

with the normalization $h(0) = g(0) = 1$. This section gives emphasis to the subclass ST_{Lh}^0 consisting of functions $f \in S_{Lh}$ which maps U onto a starlike domain (with respect to the origin). Thus the linear segment joining the origin to every point $f(z)$ lies entirely in $f(U)$. Starlike logharmonic mappings is an active subject of investigation, and several recent works include those of [27, 26, 107].

5.2.1 Distortion Theorem

We first establish an integral representation for starlike logharmonic mappings.

Theorem A. *[76, Corollary 3.6] Let $p \in H(U)$ with $p(0) = 1$. Then $\operatorname{Re} p(z) > 0$ in U if and only if there is a probability measure μ on ∂U such that*

$$p(z) = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x) \quad (|z| < 1).$$

Theorem B. *[76, Theorem 3.9] Let $f \in \mathcal{S}$. Then $f \in \mathcal{S}^*$ if and only if there is a probability measure μ on ∂U so that*

$$\frac{zf'(z)}{f(z)} = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x) \quad (|z| < 1),$$

or equivalently,

$$f(z) = z \exp\left(\int_{|x|=1} -2\log(1-xz) d\mu(x) \right).$$

If $a \in H(U,U)$, then $(1+a(z))/(1-a(z))$ has positive real part for $z \in U$, and the following result follows from Theorem A.

Lemma 5.4. *If $a \in H(U,U)$ with $a(0) = 0$, then*

$$\frac{a(z)}{1-a(z)} = \int_{\partial U} \frac{xz}{1-xz} d\mu(x) \quad (|z| < 1)$$

for some probability measure μ on ∂U.

The following lemma establishes a link between starlike logharmonic functions and starlike analytic functions.

Lemma 5.5. *[4] Let $f(z) = zh(z)\overline{g(z)}$ be logharmonic in U. Then $f \in ST^0_{Lh}$ if and only if $\varphi(z) = zh(z)/g(z) \in S^*$.*

Theorem 5.4. *A logharmonic function $f(z) = zh(z)\overline{g(z)}$ belongs to ST^0_{Lh} if and only if there are two probability measures μ and ν on ∂U such that*

$$\begin{aligned} h(z) &= \exp\left(\int_{\partial U}\int_{\partial U}\left(\frac{\eta+\xi}{\eta-\xi}\log\frac{1-\xi z}{1-\eta z} - \log(1-\eta z)\right)d\mu(\eta)d\nu(\xi)\right), \\ g(z) &= \exp\left(\int_{\partial U}\int_{\partial U}\left(\frac{\eta+\xi}{\eta-\xi}\log\frac{1-\xi z}{1-\eta z} + \log(1-\eta z)\right)d\mu(\eta)d\nu(\xi)\right), \end{aligned} \tag{5.11}$$

where $\eta \neq \xi$ and $|\eta| = |\xi| = 1$; or

$$\begin{aligned} h(z) &= \exp\left(\int_{\partial U}\int_{\partial U}\left(\frac{2\eta z}{1-\eta z} - \log(1-\eta z)\right)d\mu(\eta)d\nu(\eta)\right), \\ g(z) &= \exp\left(\int_{\partial U}\int_{\partial U}\left(\frac{2\eta z}{1-\eta z} + \log(1-\eta z)\right)d\mu(\eta)d\nu(\eta)\right), \end{aligned} \tag{5.12}$$

where $|\eta| = 1$.

Proof. Let $\varphi(z) = zh(z)/g(z)$. Since the second dilatation function (Section 1.5)

$$a(z) = \frac{zg'(z)/g(z)}{1+zh'(z)/h(z)},$$

it follows that

$$\begin{aligned}\frac{z\varphi'(z)}{\varphi(z)} &= 1+\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \\ &= \frac{zg'(z)}{g(z)}\left(\frac{1-a(z)}{a(z)}\right).\end{aligned}$$

Thus

$$g(z) = \exp\left(\int_0^z \frac{a(s)}{1-a(s)} \cdot \frac{\varphi'(s)}{\varphi(s)} ds\right) \tag{5.13}$$

and

$$h(z) = \frac{\varphi(z)}{z} g(z). \tag{5.14}$$

By Lemma 5.5 and Theorem B,

$$\frac{z\varphi'(z)}{\varphi(z)} = \int_{|\eta|=1} \frac{1+\eta z}{1-\eta z} d\mu(\eta) \quad (|z|<1), \tag{5.15}$$

or equivalently,

$$\varphi(z) = z\exp\left(\int_{|\eta|=1} -2\log(1-\eta z) d\mu(\eta)\right). \tag{5.16}$$

Then (5.13), (5.15) and Lemma 5.4 imply that g can be written as

$$g(z) = \exp\left(\int_0^z \int_{\partial U}\int_{\partial U} \frac{\xi}{1-\xi s} \cdot \frac{1+\eta s}{1-\eta s} d\mu(\eta) d\nu(\xi) ds\right)$$

for some probability measures μ and ν on U.

If $\eta \neq \xi$, then

$$\begin{aligned} g(z) =& \exp\left(\int_{\partial U}\int_{\partial U}\xi\int_0^z\left(\frac{2\eta}{(\eta-\xi)(1-\eta s)}-\frac{\eta+\xi}{(\eta-\xi)(1-\xi s)}\right)dsd\mu(\eta)d\nu(\xi)\right) \\ =& \exp\left(\int_{\partial U}\int_{\partial U}\left(-\frac{2\xi}{\eta-\xi}\log(1-\eta z)+\frac{\eta+\xi}{\eta-\xi}\log(1-\xi z)\right)d\mu(\eta)d\nu(\xi)\right) \\ =& \exp\left(\int_{\partial U}\int_{\partial U}\left(\frac{\eta+\xi}{\eta-\xi}\log\frac{1-\xi z}{1-\eta z}+\log(1-\eta z)\right)d\mu(\eta)d\nu(\xi)\right). \end{aligned}$$

On the other hand, if $\eta = \xi$, then

$$\begin{aligned} g(z) =& \exp\left(\int_{\partial U}\int_{\partial U}\int_0^z\frac{\eta+\eta^2 s}{(1-\eta s)^2}dsd\mu(\eta)d\nu(\eta)\right) \\ =& \exp\left(\int_{\partial U}\int_{\partial U}\int_0^z\left(\frac{2\eta}{(1-\eta s)^2}+\frac{\eta^2 s-\eta}{(1-\eta s)^2}\right)dsd\mu(\eta)d\nu(\eta)\right) \\ =& \exp\left(\int_{\partial U}\int_{\partial U}\left(\frac{2\eta z}{1-\eta z}+\log(1-\eta z)\right)d\mu(\eta)d\nu(\eta)\right). \end{aligned}$$

The representation for h follow from g by applying (5.14) and (5.16). $\square$

Let

$$h_0(z) = \frac{1}{1-z}\exp\left(\frac{2z}{1-z}\right) = \exp\left(\sum_{n=1}^{\infty}\left(2+\frac{1}{n}\right)z^n\right),$$

and

$$g_0(z) = (1-z)\exp\left(\frac{2z}{1-z}\right) = \exp\left(\sum_{n=1}^{\infty}\left(2-\frac{1}{n}\right)z^n\right).$$

Then

$$f_0(z) = zh_0(z)\overline{g_0(z)} = \frac{z(1-\bar{z})}{1-z}\exp\left(\mathrm{Re}\left[\frac{4z}{1-z}\right]\right)$$

is the logharmonic Koebe function.

Theorem 5.5. *Let $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$. Then*

$$\frac{1}{1+|z|}\exp\left(\frac{-2|z|}{1+|z|}\right) \le |h(z)| \le \frac{1}{1-|z|}\exp\left(\frac{2|z|}{1-|z|}\right), \tag{5.17}$$

$$(1+|z|)\exp\left(\frac{-2|z|}{1+|z|}\right) \le |g(z)| \le (1-|z|)\exp\left(\frac{2|z|}{1-|z|}\right), \tag{5.18}$$

$$|z|\exp\left(\frac{-4|z|}{1+|z|}\right) \le |f(z)| \le |z|\exp\left(\frac{4|z|}{1-|z|}\right). \tag{5.19}$$

Equalities occur if and only if h, g, and f are respectively appropriate rotations of h_0, g_0 and f_0.

Proof. Since $\varphi(z) = zh(z)/g(z) \in S^*$, it follows from (5.13) that

$$g(z) = \exp\left(\int_0^z \frac{a(s)}{1-a(s)} \cdot \frac{\varphi'(s)}{\varphi(s)} ds\right),$$

and thus

$$h(z) = \frac{\varphi(z)}{z} g(z), \quad \text{and} \quad f(z) = \varphi(z)|g(z)|^2.$$

For $|z| = r$, the known estimates

$$\left|\frac{z\varphi'(z)}{\varphi(z)}\right| \le \frac{1+r}{1-r},$$

$$\left|\frac{a(z)}{z(1-a(z))}\right| \le \frac{1}{1-r},$$

and

$$|\varphi(z)| \le \frac{r}{(1-r)^2}$$

yield

$$|g(z)| \leq \exp\left(\int_0^r \frac{1}{1-s}\cdot\frac{1+s}{1-s}ds\right) = (1-r)\exp\left(\frac{2r}{1-r}\right),$$

$$|h(z)| = \left|\frac{\varphi(z)}{z}g(z)\right| \leq \frac{1}{(1-r)^2}\cdot(1-r)\exp\left(\frac{2r}{1-r}\right) = \frac{1}{1-r}\exp\left(\frac{2r}{1-r}\right),$$

and

$$|f(z)| = |\varphi(z)|\,|g(z)|^2 \leq \frac{r}{(1-r)^2}\cdot(1-r)^2\exp\left(\frac{4r}{1-r}\right) = r\exp\left(\frac{4r}{1-r}\right).$$

For the left estimates, (5.11) gives

$$\log|h(z)| = \mathrm{Re}\left[\int_{\partial U}\int_{\partial U} K(z,\xi,\eta)d\mu(\xi)d\nu(\eta)\right], \quad |\eta| = |\xi| = 1,$$

where

$$K(z,\xi,\eta) = \begin{cases} \frac{\eta+\xi}{\eta-\xi}\log\frac{1-\xi z}{1-\eta z} - \log(1-\eta z), & \text{if } \eta \neq \xi; \\ \frac{2\eta z}{1-\eta z} - \log(1-\eta z), & \text{if } \eta = \xi. \end{cases}$$

Then for $|z| = r$,

$$\begin{aligned}
\log|h(z)| =& \mathrm{Re}\left[\int_{\partial U}\int_{\partial U} K(z,\xi,\eta)d\mu(\xi)d\nu(\eta)\right] \\
\geq& \min_{\mu,\nu}\left\{\min_{|z|=r}\mathrm{Re}\left[\int_{\partial U}\int_{\partial U} K(z,\xi,\eta)d\mu(\xi)d\nu(\eta)\right]\right\} \\
=& \min\left\{\min_{|z|=r}\inf_{0<|l|\leq\pi/2}\left[-\mathrm{Im}\left[\frac{1+e^{2il}}{1-e^{2il}}\right]\arg\left(\frac{1-e^{2il}(\eta z)}{1-(\eta z)}\right)\right] - \log(1+r),\right. \\
&\left.\frac{-2r}{1+r} - \log(1+r)\right\} \\
=& \min\left\{\inf_{0<|l|\leq\pi/2}\min_{|z|=r}\left[-\mathrm{Im}\left[\frac{1+e^{2il}}{1-e^{2il}}\right]\arg\left(\frac{1-e^{2il}z}{1-z}\right)\right] - \log(1+r),\right. \\
&\left.\frac{-2r}{1+r} - \log(1+r)\right\},
\end{aligned}$$

where $e^{2il}=\overline{\eta}\xi$.

Let

$$\Phi_r(l)=\begin{cases}\min\limits_{|z|=r}\left[-\operatorname{Im}\left[\frac{1+e^{2il}}{1-e^{2il}}\right]\arg\left(\frac{1-e^{2il}z}{1-z}\right)\right]-\log(1+r), & \text{if } 0<|l|\le\pi/2;\\ \frac{-2r}{1+r}-\log(1+r), & \text{if } l=0.\end{cases}$$

Since

$$\min_{|z|=r}\left[-\operatorname{Im}\left[\frac{1+e^{2il}}{1-e^{2il}}\right]\arg\left(\frac{1-e^{2il}z}{1-z}\right)\right]=\min_{|z|=r}\operatorname{Re}\left[\frac{1+e^{2il}}{1-e^{2il}}\log\left(1+\frac{(1-e^{2il})z}{1-z}\right)\right],$$

evidently

$$\begin{aligned}&\lim_{l\to 0}\min_{|z|=r}\operatorname{Re}\left[\frac{1+e^{2il}}{1-e^{2il}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{(1-e^{2il})z}{1-z}\right)^k\right]\\ =&\min_{|z|=r}\operatorname{Re}\left[\frac{2z}{1-z}+\lim_{l\to 0}\left\{2\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}(1-e^{2il})^{k-1}\left(\frac{z}{1-z}\right)^k\right\}\right]\\ =&\min_{|z|=r}\operatorname{Re}\left[\frac{2z}{1-z}\right]=-\frac{2r}{1+r}.\end{aligned}$$

Thus $\Phi_r(l)$ is continuous in $|l|\le\pi/2$.

Moreover,

$$\begin{aligned}&\min_{|z|=r}\left[-\operatorname{Im}\left[\frac{1+e^{-2il}}{1-e^{-2il}}\right]\arg\left(\frac{1-e^{-2il}z}{1-z}\right)\right]=\min_{|z|=r}\left[-\operatorname{Im}\left[\frac{1+e^{2il}}{1-e^{2il}}\right]\arg\left(\frac{1-z}{1-e^{-2il}z}\right)\right]\\ &=\min_{|z|=r}\left[-\operatorname{Im}\left[\frac{1+e^{2il}}{1-e^{2il}}\right]\arg\left(\frac{1-e^{2il}(e^{-2il}z)}{1-(e^{-2il}z)}\right)\right]\\ &=\min_{|w|=r}\left[-\operatorname{Im}\left[\frac{1+e^{2il}}{1-e^{2il}}\right]\arg\left(\frac{1-e^{2il}w}{1-w}\right)\right]\end{aligned}$$

implies that $\Phi_r(l)$ is an even function on $|l| \le \pi/2$. Hence

$$\log|h(z)| \ge \inf_{0\le l\le \pi/2} \Phi_r(l).$$

Since

$$\max_{|z|=r} \arg\left(\frac{1-e^{2il}z}{1-z}\right) = 2\tan^{-1}\left(\frac{r\sin(l)}{1+r\cos(l)}\right),$$

this implies that

$$\log|h(z)| \ge \inf_{0\le l\le \pi/2} -2\cot(l)\tan^{-1}\left(\frac{r\sin(l)}{1+r\cos(l)}\right) - \log(1+r).$$

Evidently $\tan^{-1}(x) \le x$ for all $x \ge 0$, and so

$$\begin{aligned}\log|h(z)| &\ge \inf_{0\le l\le \pi/2}\left(\frac{-2r\cos(l)}{1+r\cos(l)} - \log(1+r)\right)\\ &\ge \frac{-2r}{1+r} - \log(1+r).\end{aligned}$$

For the lower bound of $|g(z)|$ in (5.18), a similar argument is applied to (5.12) which yields

$$\begin{aligned}\log|g(z)| &\ge \inf_{0\le l\le \pi/2}\left(\frac{-2r\cos(l)}{1+r\cos(l)} + \log(1+r)\right)\\ &\ge \frac{-2r}{1+r} + \log(1+r).\end{aligned}$$

Finally, it follows that

$$|f(z)| = |z||h(z)||g(z)| \geq \frac{r}{1+r}\exp\left(\frac{-2r}{1+r}\right)\cdot(1+r)\exp\left(\frac{-2r}{1+r}\right)$$
$$= r\exp\left(\frac{-4r}{1+r}\right),$$

which establishes (5.19). $\square$

Remark 5.3. *The upper bounds for $|h(z)|$ and $|g(z)|$ in Theorem 5.5 were also obtained by Duman [65]. Here we not only establish the sharp lower bounds, but also exhibit the extremal functions.*

Corollary 5.1. *Let $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$. Also, let $H(z) = zh(z)$ and $G(z) = zg(z)$. Then*

$$\frac{1}{2e} \leq d(0, \partial H(U)) \leq 1,$$

$$\frac{2}{e} \leq d(0, \partial G(U)) \leq 1,$$

and

$$\frac{1}{e^2} \leq d(0, \partial f(U)) \leq 1.$$

Equalities occur if and only if h, g and f are respectively suitable rotations of h_0, g_0 and f_0.

Proof. By (5.17),

$$d(0, \partial H(U)) = \liminf_{|z|\to 1} |H(z) - H(0)| = \liminf_{|z|\to 1} \frac{|H(z) - H(0)|}{|z|} \geq \frac{1}{2e}.$$

On the other hand, the minimum modulus principle shows that

$$d(0,\partial H(U)) = \liminf_{|z|\to 1} |H(z)-H(0)| = \liminf_{|z|\to 1} \frac{|H(z)-H(0)|}{|z|} \leq 1$$

since $|h(0)| = 1$. The same technique is applied to G and f to find the remaining inequalities. $\square$

5.2.2 The Bohr radius for logharmonic mappings

Consider now logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ with

$$h(z) = \exp\left(\sum_{k=1}^{\infty} a_k z^k\right) \text{ and } g(z) = \exp\left(\sum_{k=1}^{\infty} b_k z^k\right).$$

Theorem C. *[5, Theorem 3.3] Let $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$. Then*

$$|a_n| \leq 2 + \frac{1}{n} \textit{ and } |b_n| \leq 2 - \frac{1}{n}$$

for all $n \geq 1$. Equalities hold for f a rotation of the function f_0.

Our main results are the following theorems.

Theorem 5.6. *Let $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$, $H(z) = zh(z)$ and $G(z) = zg(z)$. Then*

(a)

$$|z| \exp\left(\sum_{n=1}^{\infty} |a_n||z|^n\right) \leq d(0,\partial H(U))$$

for $|z| \leq r_H \approx 0.1222$, *where* r_H *is the unique root in* $(0,1)$ *of*

$$\frac{r}{1-r}\exp\left(\frac{2r}{1-r}\right) = \frac{1}{2e},$$

(b)

$$|z|\exp\left(\sum_{n=1}^{\infty}|b_n||z|^n\right) \leq d(0,\partial G(U))$$

for $|z| \leq r_G \approx 0.3659$*, where* r_G *is the unique root in* $(0,1)$ *of*

$$r(1-r)\exp\left(\frac{2r}{1-r}\right) = \frac{2}{e}.$$

Both radii are sharp and are attained respectively by appropriate rotations of $H_0(z) = zh_0(z)$ *and* $G_0(z) = zg_0(z)$.

Proof. Note that

$$H(z) = z\exp\left(\sum_{k=1}^{\infty}a_k z^k\right) \quad \text{and} \quad G(z) = z\exp\left(\sum_{k=1}^{\infty}b_k z^k\right).$$

By Theorem C,

$$|a_n| \leq 2 + \frac{1}{n} \quad \text{and} \quad |b_n| \leq 2 - \frac{1}{n}$$

which are sharp bounds and Corollary 5.1 gives

$$d(0,\partial H(U)) \geq \frac{1}{2e} \quad \text{and} \quad d(0,\partial G(U)) \geq \frac{2}{e}.$$

Hence

$$\begin{aligned} r\exp\left(\sum_{n=1}^{\infty}|a_n|r^n\right) \leq & r\exp\left(\sum_{n=1}^{\infty}\left(2+\frac{1}{n}\right)r^n\right) \\ = & r\exp\left(\frac{2r}{1-r}-\log(1-r)\right) \\ \leq & d(0,\partial H(U)) \end{aligned}$$

if and only if

$$\frac{r}{1-r}\exp\frac{2r}{1-r} \leq \frac{1}{2e}.$$

The Bohr radius, $r_H \approx 0.1222$ is therefore the positive solution of

$$\frac{r}{1-r}\exp\frac{2r}{1-r} = \frac{1}{2e}.$$

Likewise,

$$\begin{aligned} r\exp\left(\sum_{n=1}^{\infty}|b_n|r^n\right) \leq & r\exp\left(\sum_{n=1}^{\infty}\left(2-\frac{1}{n}\right)r^n\right) \\ = & r\exp\left(\frac{2r}{1-r}+\log(1-r)\right) \\ \leq & d(0,\partial G(U)) \end{aligned}$$

if and only if

$$r(1-r)\exp\frac{2r}{1-r} \leq \frac{2}{e}.$$

Hence the Bohr radius, r_G is the positive solution of

$$r(1-r)\exp\frac{2r}{1-r} = \frac{2}{e}$$

which gives $r_G \approx 0.3659$. Finally, it is evident that both radii are attained by suitable rotations of $H_0(z)$ and $G_0(z)$, respectively. $\square$

Theorem 5.7. *Let $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$. Then, for any real t,*

$$|z| \exp\left(\sum_{n=1}^{\infty} \left|a_n + e^{it} b_n\right| |z|^n\right) \leq d(0, \partial f(U))$$

for $|z| \leq r_0 \approx 0.09078$, where r_0 is the unique root in $(0,1)$ of

$$r \exp\left(\frac{4r}{1-r}\right) = \frac{1}{e^2}.$$

The bound is sharp and is attained by a suitable rotation of the logharmonic Koebe function f_0.

Proof. By Theorem C,

$$|a_n| \leq 2 + \frac{1}{n} \text{ and } |b_n| \leq 2 - \frac{1}{n}.$$

which are sharp bounds and Corollary 5.1 gives

$$d(0, \partial f(U)) \geq \frac{1}{e^2}$$

which is also sharp. Thus

$$\begin{aligned} r \exp\left(\sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n\right) \leq & r \exp\left(4 \sum_{n=1}^{\infty} r^n\right) \\ = & r \exp\left(\frac{4r}{1-r}\right) \leq d(0, \partial f(U)) \end{aligned}$$

if and only if

$$r \exp\left(\frac{4r}{1-r}\right) \leq \frac{1}{e^2}.$$

Hence the Bohr radius, r_0 is the solution of

$$r \exp\left(\frac{4r}{1-r}\right) = \frac{1}{e^2}$$

which gives $r_0 \approx 0.09078$. Finally, it is evident that r_0 is attained by suitable rotations of the logharmonic Koebe function, f_0. $\square$

CHAPTER 6

BOHR RESEARCH AND CONCLUSION

6.1 The three famous multidimensional Bohr radii

Let $\mathbb{C}^n$ be the n-fold Cartesian product of the complex plane $\mathbb{C}$ and $U^n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_k| < 1\}$ be the unit polydisk. Then by using the standard multi-index notation, an n-variable complex power series defined on a domain G can be written as

$$\sum_{\alpha} c_\alpha z^\alpha, \quad c_\alpha \in \mathbb{C},\ z \in G, \tag{6.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ denotes an n-tuple of nonnegative integers, $|\alpha|$ denotes the sum $\alpha_1 + \cdots + \alpha_n$, $\alpha!$ denotes the product $\alpha_1!\alpha_2! \cdots \alpha_n!$, and z^α denotes the product $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. In 1989, the n-dimensional Bohr's theorem was first introduced by Dineen and Timoney [61]. They remarked that the n-dimensional Bohr radius estimation can be used to prove the connection between the existence of absolute basis and nuclearity of the underlying space.

Several years later, the n-dimensional Bohr radius $K(G)$ (also known as the *first* Bohr radius) was formulated by Boas and Khavinson [37] on a bounded complete Reinhardt domain $G \subset \mathbb{C}^n$. Denote by $K(G)$ the largest $r > 0$ such that if the series (6.1) converges in G with

$$\left| \sum_{\alpha} c_\alpha z^\alpha \right| < 1, \tag{6.2}$$

then

$$\sum_{\alpha} |c_\alpha z^\alpha| < 1,$$

holds in the scaled domain (or homothetic domain) $r \cdot G$. In particular, if G is the unit polydisk U^n, then $r \cdot U^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_k| < r\}$. They proved that for $n > 1$, $K(G)$ satisfies

$$\frac{1}{3\sqrt{n}} < K(G) < \frac{2\sqrt{\log n}}{\sqrt{n}}.$$

It is interesting to note that the Bohr radius $K(D_1^n)$ for the unit hypercone $D_1^n = \{z \in \mathbb{C}^n : |z_1| + \cdots + |z_n| < 1\}$, satisfies [15, Theorem 9]

$$\frac{1}{3e^{1/3}} < K(D_1^n) \leq 1/3,$$

which is independent of n and has the sharp upper bound $1/3$. For two bounded complete Reinhardt domains G_1 and G_2 in $\mathbb{C}^n$, the relation between the Bohr radii $K(G_1)$ and $K(G_2)$ is given by [54, Lemma 2.5]

$$\frac{1}{S(G_1, G_2)S(G_2, G_1)} K(G_2) \leq K(G_1) \leq S(G_1, G_2)S(G_2, G_1)K(G_2),$$

where

$$S(G_1, G_2) := \inf\{b > 0 : G_1 \subset bG_2\}.$$

The relation was then applied to prove the lower and upper bounds for $K(G)$ [54, Theorem 2.7 and 2.8].

Next, let $X = (\mathbb{C}^n, \|\cdot\|)$ be a Banach space. Note that the unit ball B_X in X is a bounded complete Reinhardt domain (see [54, Remark 3.9]). The application of

Banach space theory into the Bohr radii study was indeed initiated by Defant, García and Maestre. In 2003, they proved a basic link between Bohr radii $K(B_X)$ and radii of unconditionality [53, Theorem 2.2], and also expressed the bounds of $K(B_X)$ in terms of Banach-Mazur distance [53, Corollary 5.1, 5.2 and 5.3]. More detailed discussions on the relation of Bohr radii and modern Banach space theory can be found in the survey papers [51] and [52].

In 2011, Defant *et al.* showed that the Bohnenblust-Hille inequality [38] is in fact hypercontrative [57, Theorem 1], that is, for integers $m,n > 1$, the inequality

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq \left(1+\frac{1}{m-1}\right)^{m-1} \sqrt{m}\left(\sqrt{2}\right)^{m-1} \sup_{z\in U^n}\left|\sum_{|\alpha|=m} a_\alpha z^\alpha\right|$$

holds for every m-homogeneous polynomial $\sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^n$. They took advantage of this inequality and obtained new bound for $K(G)$ with $G \subset \mathbb{C}^n$ a bounded complete Reinhardt domain [57, Theorem 2]:

$$K(G) = b(n)\sqrt{\frac{\log n}{n}},$$

with $1/\sqrt{2}+o(1) \leq b(n) \leq 2$. The lower bound for $b(n)$ was then improved to $1+o(1)$ by Bayart, Pellegrino and Seoane-Sepúlveda [30], indicating $K(G)$ behaves asymptotically as $\sqrt{\log n/n}$. More precisely, they proved that

$$\lim_{n\to\infty} \frac{K(G)}{\sqrt{\frac{\log n}{n}}} = 1.$$

The *second* Bohr radius $B(G)$ was introduced by Aizenberg [15] by incorporating the supremum function. Let $G \subseteq \mathbb{C}^n$ be a bounded complete Reinhardt domain. Denote by $B(G)$ the largest number $r > 0$ such that if the series (6.1) converges in a bounded complete Reinhardt domain $G \subset \mathbb{C}^n$ and (6.2) holds in it, then

$$\sum_{\alpha} \sup_{r \cdot G} |c_\alpha z^\alpha| < 1.$$

By definition, it is clear that $B(G) \leq K(G)$. In particular, if $G = U^n$, then $B(G) = K(G)$. In the same paper, Aizenberg proved that [15, Theorem 4]

$$1 - \sqrt[n]{\frac{2}{3}} < B(G)$$

for any bounded complete Reinhardt domain G. Similarly, there was a relation between the second Bohr radii for two bounded complete Reinhardt domains G_1 and G_2 in $\mathbb{C}^n$ [54, Lemma 3.1],

$$B(G_1) \leq S(G_1, G_2) S(G_2, G_1) B(G_2),$$

which is essential in proving the lower and upper bounds for the second Bohr radius $B(G)$ [54, Corollary 3.4].

Another possible way of extending the single variable result is to consider the complex Banach space $\ell_p^n \subset \mathbb{C}^n$ with norm $\|z\|_{\ell_p^n} := (\sum_{k=1}^n |z_k|^p)^{1/p}$. Let $B_{\ell_p^n} = \{z \in \mathbb{C}^n : \|z\|_{\ell_p^n} < 1\}$ denote the unit ball of ℓ_p^n. Note that $B_{\ell_2^n}$ is the usual Euclidean unit ball D_2^n and $B_{\ell_\infty^n}$ is the unit polydisk U^n. When $n > 1$, Boas [36, Theorem 3 and Theorem 5] proved that for $1 \le p \le \infty$,

$$\begin{aligned} \frac{1}{c}\left(\frac{1}{n}\right)^{1-1/\min\{p,2\}} &\le K(B_{\ell_p^n}) < c\left(\frac{\log n}{n}\right)^{1-1/\min\{p,2\}}, \\ \frac{1}{C}\left(\frac{1}{n}\right)^{1/2+1/\max\{p,2\}} &\le B(B_{\ell_p^n}) < C\left(\frac{\log n}{n}\right)^{1/2+1/\max\{p,2\}}, \end{aligned} \tag{6.3}$$

where c and C are positive constants independent of n. Here, the unconditional basis constant and the Banach-Mazur distance serve as important tools to obtain [49, Theorem 1.1] which is an improvement for the lower estimate of $K(B_{\ell_p^n})$ in (6.3). Much later, Defant and Frerick study Bohr radii using invariants from local Banach space theory to further improve the lower estimate [50, Theorem 1.1].

The *third* n-dimensional Bohr radius, known as the arithmetic Bohr radius was introduced by Defant, Maestre and Prengel [55] due to its application in [56]. Let $G \subset \mathbb{C}^n$ be a bounded complete Reinhardt domain and $H^\infty(G)$ the class of all bounded analytic functions $f : G \to \mathbb{C}$ and $\lambda \ge 1$. Note that each $f \in H^\infty(G)$ can be written in its monomial series expansion $f(z) = \sum_\alpha c_\alpha(f) z^\alpha$. Then the λ-arithmetic Bohr radius of G with respect to $H^\infty(G)$ is denoted by

$$A(H^\infty(G),\lambda) := \sup\left\{\frac{1}{n}\sum_{k=1}^{n} r_k : \sum_\alpha |c_\alpha(f)|\, r^\alpha \le \lambda \|f\|_G,\ \forall f \in H^\infty(G)\right\},$$

where $r = (r_1, \ldots, r_n)$ denotes an n-tuple of nonnegative real numbers, $\alpha = (\alpha_1, \ldots, \alpha_n)$ denotes an n-tuple of nonnegative integers, r^α denotes the product $r_1^{\alpha_1} \cdots r_n^{\alpha_n}$, and $\|f\|_G = \sup_{z \in G} |f(z)|$. In particular, the 1-arithmetic Bohr radius is simply known as the *arithmetic Bohr radius*. In the case $1 \le \lambda \le \sqrt{2}$ and $n \ge 2$ [55, Theorem 3.1],

$$A(H^\infty(B_{\ell_1^n}),\lambda) = \frac{1}{n(3\lambda - 2\sqrt{2(\lambda^2-1)})},$$

where $B_{\ell_1^n} = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : |z_1| + \cdots + |z_n| < 1\}$. This result distinguishes the arithmetic radius from the first and second Bohr radius because the exact values of $K(G)$ and $B(G)$ are not known for any bounded complete Reinhardt domain G of dimension $n \geq 2$. A more general result is given in [55, Theorem 4.1]. Finally, the relation between first and third Bohr radius is given by [55, Theorem 4.7]

$$\max\left\{\frac{1}{3n}\frac{1}{S(B_{\ell_1^n},G)}, \frac{S(G,B_{\ell_1^n})}{n}K(G)\right\} \leq A(G,\lambda) \leq c\frac{\log n}{n}S(G,B_{\ell_2^n}),$$

where $c > 0$ is a uniform constant.

6.2 Bohr and bases in spaces of holomorphic functions

Let M be a complex manifold and $H(M)$ the space of holomorphic functions on M with basis $(\phi_n)_{n=0}^{\infty}$. A basis $(\phi_n)_{n=0}^{\infty}$ in $H(M)$ is said to have *Bohr Property (BP)* if there exist subsets $G \subset K \subset M$, where G is open and K is compact, such that

$$\sum_{n=0}^{\infty} |a_n(f)| \sup_G |\phi_n(z)| \leq |f|_K = \sup_{z\in K} |f(z)|$$

holds for any $f = \sum_{n=0}^{\infty} a_n(f)\phi_n \in H(M)$. Aizenberg, Aytuna and Djakov [21, Theorem 3.3] showed that if

(a) $\phi_0 \equiv 1$, and

(b) there is a $z_0 \in M$ such that $\phi_n(z_0) = 0$ for $n \geq 1$,

then $(\phi_n)_n^{\infty}$ has BP, where the open subset G is now a neighborhood of z_0. In fact, [21, Theorem 3.3] implies the existence of a Bohr phenomenon in $H(M)$. A generalization of [21, Theorem 3.3] was given in [20, Theorem 4] which considered manifold M with a one-parameter family of continuous seminorms in $H(M)$. In 2013, inspired by the work of Lassère and Mazzilli [89], Aytuna and Djakov [28] introduced the term Global Bohr Property. They [28, Theorem 3] proved that a basis $(\phi_n)_{n=0}^{\infty}$ has GBP if and only if one of the functions ϕ_n is a constant. A generalization of [28, Theorem 3] to Stein manifold was given in [28, Theorem 9].

Let $G \subset \mathbb{C}^n$ be a domain with $\lambda \overline{G} \subset G$ for all $0 < \lambda < 1$. Note that, if $f \in H(G)$ is a holomorphic function on G, then f has homogeneous polynomials expansion given by

$$f(z) = \sum_{k=0}^{\infty} P_k(z), \quad z \in G,$$

where P_k is a homogenous polynomial of degree k. In particular, Aizenberg [16, Lemma 1.1] proved that if $\operatorname{Re} f(z) > 0$ for all $z \in G$, then

$$|P_k(z)| \leq 2 \operatorname{Re} P_0(z), \quad z \in G, k = 1, 2, \ldots.$$

This inequality is an important tool to prove the following result [17, Theorem 2.1]: Let f be an analytic function from U into a domain $G \subset \mathbb{C}$. Further suppose the convex hull $\tilde{G}$ of G satisfies $\tilde{G} \neq \mathbb{C}$. Then

$$d(\mathcal{M}f(|z|), |f(0)|) \leq d(f(0), \partial \tilde{G})$$

for $|z| \leq 1/3$. The value $1/3$ is best provided there exists a point $p \in \mathbb{C}$ satisfying $p \in \partial \tilde{G} \cap \partial G \cap \partial D$ for some disk $D \subset G$. The result shows the extension of Bohr's theorem from the class $H(U, U)$ to the class $H(U, \tilde{G})$.

Let $U_r := \{z \in \mathbb{C} : |z| < r\}$. Also, let $H(U_r)$ be the space of holomorphic functions on U_r. Then the classical Bohr's theorem [91] can be reformulated as:

The value 3 is the smallest radius $r > 1$ such that the inequality

$$\sum_{n=0}^{\infty} |a_n z^n| < 1, \quad z \in U,$$

holds for all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(U_r)$ where $|f(z)| < 1$ on U_r. Let $K \subset \mathbb{C}$ be a continuum (that is a compact set in $\mathbb{C}$ containing at least two points such that $\overline{\mathbb{C}} \backslash K$ is simply connected). By the Riemann mapping theorem, there exists a unique function Φ_K which maps $\overline{\mathbb{C}} \backslash K$ conformally onto $\overline{\mathbb{C}} \backslash \overline{U}$ such that

(a) $\Phi_K(\infty) = \infty$, and

(b) $\Phi_K'(\infty) = \lim_{z \to \infty} \Phi_K(z)/z > 0$.

The principal part of the Laurent series of Φ_K^n at ∞ is a polynomial of degree n and is known as the n-th Faber polynomial $F_{K,n}$. It is a well known fact [113] that $(F_{K,n})_{n \geq 0}$ is a Schauder basis for all the spaces $H(\Omega_{K,r})$ where

$$\Omega_{K,r} := \{z \in \mathbb{C} \backslash K : |\Phi_K(z)| < r\} \cup K.$$

The definition of Bohr radius was given in the following [89, Theorem 3.1]:

For any continuum $K \subset \mathbb{C}$, there exists a constant $r_K > 1$ such that for all $r > r_k$ and $f = \sum_n a_n F_{K,n} \in H(\Omega_{K,r})$, if $|f(z)| < 1$ on $\Omega_{K,r}$, then

$$\sum_n |a_n| \|F_{K,n}\|_K < 1,$$

where $\|F_{K,n}\|_K = \sup_{z\in K}|F_{K,n}(z)|$. The *Bohr radius* of $(K,(\Omega_{K,r})_{r>1},(F_{K,n})_{n\geq 0})$ is then defined to be the infimum of all such $r_K > 1$. A generalization of this Bohr radius be found in [91]. In particular, if K is the domain bounded by a nondegenerate ellipse with foci ± 1, then the bounds for the Bohr radius were found by Kaptanoğlu and Sadık [84, Theorem 7 and 8]. Later on, the exact value of the radius was given by Lassère and Mazzill [90, Theorem 4.4], which is the unique solution in $[0,1]$ of

$$\sum_{n}^{\infty}\frac{4R^{2n}}{1+R^{4n}}+\sum_{m}^{\infty}\frac{4R^{2m+1}}{1+R^{4m+2}}=1.$$

6.3 Bohr and norms

Recall that for $f\in H(U,U)$ of the form (1.1), the classical Bohr's theorem can also be written in terms of its supremum norm (1.7). In 2002, Paulsen, Popescu and Singh [105, Corollary 2.8] obtained the following inequality

$$\sum_{n=0}^{\infty}|a_n|r^n \leq m(r)\|f\|_\infty, \quad 0\leq r<1,$$

where $m(r)=\inf\{M(r),(1-r^2)^{-1/2}\}$ and

$$M(r)=\sup\left\{t+(1-t^2)\frac{r}{1-r} : 0\leq t\leq 1\right\}.$$

In the same paper, the Bohr radius was shown to be $1/2$ if a_0 in (1.7) was replaced by a_0^2 [105, Corollary 2.7], and $1/\sqrt{2}$ if the $a_0=0$ [105, Corollary 2.8] (see also [22, Theorem 3]). Meanwhile, the research on finding an exact description of $m(r)$ was conducted by Bombieri [41]. He showed that

$$m(r) = \frac{3 - \sqrt{8(1-r^2)}}{r}$$

for $r \in [1/3, 1/\sqrt{2}]$, and together with Bourgain, they [42, Theorem 1.1] found that $m(r) < 1/\sqrt{1-r^2}$ for $r > 1\sqrt{2}$.

On the other hand, Bénéteau, Dahlner and Khavinson [32] showed that there is no Bohr phenomenon for the Hardy spaces H^s, $0 < s < \infty$ equipped with the usual norm $\|f\|_{H_s}$. Then for $f \in H(U)$ of the form (1.1), they conducted their study on finding $\sup\{\|f\|_{p,r} : f \in H^s, \|f\|_{H^s} \leq 1\}$, where

$$\|f\|_{p,r} = \left(\sum_{n=0}^{\infty} |a_n|^p r^n \right)^{1/p}$$

and obtained some relations between $\|f\|_{p,r}$ and $\|f\|_{H^s}$ given in [32, Theorem 2.5]. The authors also showed the effect of renorming a space on the Bohr radius [32, Theorem 3.1 and Theorem 4.2]. Djakov and Ramanujan in [63], studied the best constant r_p, $1 \leq p < \infty$ such that

$$\left(\sum_{n=0}^{\infty} |a_n|^p (r_p)^{np} \right)^{1/p} \leq \|f\|_{\infty}.$$

For $p = 1$, $r_1 = 1/3$ is exactly the classical Bohr radius and Hausdorff-Young inequality implies that $r_p = 1$ for $p \geq 2$. The best known estimate for r_p with $1 < p < 2$ was given in [63, Theorem 3].

Let X be a complex Banach space. Denote by $H^\infty(U,X)$ the Hardy space of X-

valued bounded holomorphic functions on the unit disk U. Blasco introduced and studied the Bohr radius

$$R(X) = \sup_{f \in H^\infty(U,X), \|f\|_\infty \le 1} \left\{ r \ge 0 : \sum_{n=0}^{\infty} \|x_n\| r^n \le \|f\|_\infty, \quad f(z) = \sum_{n=0}^{\infty} x_n z^n \right\}.$$

The classical Bohr radius implies $R(X) \le 1/3$ for any Banach space X but $R(\mathbb{C}_p^m) = 0$ for $1 \le p \le \infty$ whenever $m \ge 2$ and, where $\mathbb{C}_p^m$ is the space $\mathbb{C}^m$ endowed with the p-norm (see [34, Theorem 1.2]). He then defined the radius

$$R_{p,q}(X) = \sup_{f \in H^\infty(U,X), \|f\|_\infty \le 1} \left\{ r \ge 0 : \|x_0\|^p + \left(\sum_{n=1}^{\infty} \|x_n\| r^n \right)^q \le 1, \quad f(z) = \sum_{n=0}^{\infty} x_n z^n \right\}.$$

In this case, for any $m \ge 2$, Blaco obtained $R_{p,q}(\mathbb{C}_\infty^m) = 0$ for all $1 \le p, q < \infty$ [34, Proposition 2.1] and $R_{p,p}(\mathbb{C}_2^m) > 0$ for $p \ge 2$ [34, Theorem 2.2]. Further, he extended the Bohr's inequality in [105, Corollary 2.7] to the case $1 \le p \le 2$ [34, Proposition 1.4] where $R_{p,q}(\mathbb{C}) = p/(2+p)$. In his latest paper, Blasco defined the p-Bohr radius of X

$$r_p(X) = \sup_{f \in H^\infty(U,X), \|f\|_\infty \le 1} \left\{ r \ge 0 : \sum_{n=0}^{\infty} \|x_n\|^p r^{np} \le \|f\|_\infty^p, \quad f(z) = \sum_{n=0}^{\infty} x_n z^n \right\}$$

and has the relation $R_{p,p}(X) \le r_p(X) \le 2^{1-1/p} R_{p,p}(X)$. Further $r_p(X) > 0$ if and only if X is a p-uniformly $\mathbb{C}$-convex complex Banach space (see [35, Theorem 1.10]).

By replacing scalar valued holomorphic functions by Banach space, Defant, Maestre and Schwarting defined a new Bohr radius in [58]. Let $v : X \to Y$ be a bounded operator between complex Banach spaces X and Y, $n \in \mathbb{N}$ and $\lambda \ge \|v\|$. The *λ-Bohr radius of v* $K_n(v, \lambda)$ is the supremum of all $r \ge 0$ such that for all holomorphic functions

$f(z) = \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha z^\alpha$ on $U^n \subset \mathbb{C}^n$,

$$\sup_{z\in rU^n} \sum_{\alpha\in\mathbb{N}_0^n} \|v(c_\alpha)z^\alpha\|_Y \leq \lambda \sup_{z\in U^n} \left\| \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha z^\alpha \right\|_X .$$

If v is the identity on X, then write $K_n(X,\lambda)$. Further if $X = \mathbb{C}$ and $\lambda = 1$, then write K_n. Note that K_n is exactly the first Bohr radius as defined earlier by Boas and Khavinson. They proved that [58, Theorem 4.1] if X is a finite dimensional Banach space and $\lambda > 1$, then there exist constants $C, B > 0$ such that for each n

$$C\frac{\lambda-1}{2\lambda-1}\sqrt{\frac{\log n}{n}} \leq K_n(X,\lambda) \leq B\lambda^2\sqrt{\frac{\log n}{n}};$$

here B is a universal constant and C is a constant that depends only on X.

6.4 More extensions of Bohr's theorem

There are still many possible directions of extending Bohr's theorem. For example, in [83], Kaptanoğlu studied the Bohr phenomenon for elliptic equations by considering the case of harmonic functions for the Laplace-Beltrami operator. The Bohr radii for classes of harmonic, separately harmonic and pluriharmonic functions were evaluated in [19]. Extension of Bohr's theorem to uniform algebra can also be found in [103]. The *Dirichlet-Bohr radius* was defined in [43] which relates the Bohr radius to the Dirichlet series while earlier work in this direction can be found in [29]. The interested reader may also refer to a recent survey paper on Bohr's inequality [13].

A paper by Dixon [62] gave a great impetus to the development of Bohr's inequality in operator theory. In the paper, he used the classical Bohr's theorem to construct a non-unital Banach algebra which is not an operator algebra but satisfies the non-unital von Neumann's inequality. The construction was later expanded by Paulsen, Popescu and Singh [105, Theorem 2.10]. To be precise, if $(A, \|\cdot\|)$ is a Banach algebra and letting $|||a|||_r = r^{-1}\|a\|$, for $0 < r \leq 1/3$, then $(A, |||\cdot|||_r)$ is a Banach algebra that satisfies the non-unital von Neumann inequality. Popescu [108] wrote a good paper on operator theoretic multivariable Bohr study. In the paper, the operator-valued coefficients version of Bohr's theorem [108, Theorem 2.6] extended some results of [104, Theorem 2.1] which considered the single variable case.

A sharp Bohr's theorem can also be found in the study of skew field of quaternions. Let $\mathbb{H}$ denote the skew field of quaternions. Then each $q \in \mathbb{H}$ can be written in the form

$$q = q_0 + q_1 i + q_2 j + q_3 k, \quad q_0, q_1, q_2, q_3 \in \mathbb{R},$$

where $\{1, i, j, k\}$ is a basis of $\mathbb{H}$ satisfying

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, \\ &ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik. \end{aligned}$$

Indeed, every $q \in \mathbb{H}$ can be written as $q = x + yI$, for $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$ where $\mathbb{S}$ is given by

$$\mathbb{S} = \left\{ q \in \mathbb{H} : q^2 = -1 \right\}.$$

For each $I \in \mathbb{S}$, denote by L_I the plane $\mathbb{R} + I\mathbb{R}$ (isomorphic to $\mathbb{C}$) and by Ω_I the intersection $\Omega \cap L_I$ whenever Ω is a domain in $\mathbb{H}$. Then a function $f : \Omega \to \mathbb{H}$ is called *slice regular* if, for all $I \in \mathbb{S}$, its restriction f_I to Ω_I has continuous partial derivatives and satisfies

$$\left(\frac{\partial}{\partial x}+I\frac{\partial}{\partial y}\right)f_I(x+yI)=0$$

for all $x+yI\in\Omega_I$. The following result [59, Theorem 4.1] was proved by Rocchetta, Gentil and Sarfatt:

Let $\mathbb{B}=\{q\in\mathbb{H}:|q|<1\}$ be the unit ball of $\mathbb{H}$. Also, let $f(q)=\sum_{n\geq 0}q^n a_n$ be a slice regular function on $\mathbb{B}$, continuous on the closure $\mathbb{B}$ such that $|f(q)|<1$ for all $|q|\leq 1$. Then

$$\sum_{n\geq 0}|q^n a_n|<1$$

for all $|q|\leq 1/3$. Moreover $1/3$ is the largest value for which the statement is true. For related works in this area, one can refer to [74, 75].

The connection between Hadamard real part theorem and Bohr's theorem can be seen in [87], in which Kresin and Maz'ya introduced the Bohr-type real part estimates and proved the following theorem by applying ℓ_p norm on the remainder of power series expansion [87, Corollary 1]:

Let $f(z)=\sum_{n=0}^{\infty}a_n z^n\in H(U)$ such that

$$\sup_{|\zeta|<1}\operatorname{Re}\left[e^{-i\arg f(0)}f(\zeta)\right]<\infty,$$

where $\arg f(0)$ is replaced by zero if $f(0)$=0. Then for any $q\in(0,\infty]$, integer $m\geq 1$, and $|z|\leq r_{m,q}$, the inequality

$$\left(\sum_{n=m}^{\infty}|a_n z^n|^q\right)^{1/q}\leq\sup_{|\zeta|<1}\operatorname{Re}\left[e^{-i\arg f(0)}f(\zeta)\right]-|f(0)|$$

holds, where $r_{m,q} \in (0,1)$ is the root of the equation $2^q r^{mq} + r^q - 1 = 0$ if $0 < q < \infty$, and $r_{m,\infty} := 2^{-1/m}$. The radius $r_{m,q}$ is best possible. With $(q,m) = (1,1)$, the result becomes the sharp inequality obtained by Sidon [112] and it contains the classical Bohr's inequality. For more discussion on Bohr type real part estimates, refer to [88, Chapter 6].

On the other hand, Guadarrama [73] considered the *polynomial Bohr radius* defined by

$$R_n = \sup_{p \in \mathcal{CP}_n} \left\{ r \in (0,1) : \sum_{k=0}^{n} |a_k| r^k \le \|p\|_\infty, \quad p(z) = \sum_{k=0}^{n} a_k z^k \right\}$$

where $\mathcal{CP}_n$ consists of all the complex polynomials of degree at most n. She showed that

$$C_1 \frac{1}{3^{n/2}} \le R_n - 1/3 \le C_2 \frac{\log n}{n}$$

for some positive constants C_1 and C_2. Few years later, Fournier [70] computed and obtained an explicit formula for R_n by applying the notion of bounded-preserving operators. The following result concerning the asymptotic behaviour of R_n was proved only recently in [45]:

$$\lim_{n \to \infty} n^2 \left(R_n - \frac{1}{3} \right) = \frac{\pi^2}{3}.$$

Consider polynomials in the setting of weighted Hardy spaces. Fix weights $\beta_n > 0$ with $\beta_0 = 1$ and define the weighted Hardy space

$$H^2(\beta) =: \left\{ \sum_{n=0}^{\infty} a_n z^n \in H(U) : \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}$$

with inner product $\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2$. Note that the functions $e_n = \beta_n^{-1} z^n$ will form an orthonormal basis of $H^2(\beta)$. If $H^2(\beta)$ has Bohr phenomenon, then for all n large enough (see [9, Theorem 5.2])

$$\frac{1}{\sqrt{2}} < R_n < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}(2^{2n+2} - 2n - 2)}.$$

A discussion on the multivariable weighted Hardy spaces can be found in Section 3 of [105].

Finally, let X, Y be complex Banach spaces and let B_Y be the unit ball in Y. For domains $G \subset X$, $D \subset Y$, let $H(G, D)$ be the set of holomorphic mappings from G into D. For $f \in H(G, D)$, let $D^k f(x)$ denote the k-th Fréchet derivative of f at $x \in G$. A complex Banach space X is called a JB^*-triple if there exists a triple product $\{\cdot, \cdot, \cdot\} : X^3 \to X$ which is conjugate linear in the middle variable but linear and symmetric in the other variables and satisfies

(i) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\}$+{x,y,{a,b,z}},

(ii) the map $a \Box a : X \to X$ defined by $a \Box a(x) = \{a, a, x\}$ is Hermitian with non-negative spectrum,

(iii) $\|\{a, a, a\}\| = \|a\|^3$

for $a, b, x, y, z \in X$. Hamada and Honda proved the following result [77, Theorem 3.2] which is a refinement of [78, Theorem 3.1]:

Let X be a complex Banach space and Y be a JB^*-triple. Let G be a bounded balanced domain in X, that is, $zG \subseteq G$ for all $z \in \overline{U}$, and let B_Y be the unit ball in Y. Let

$f : G \to B_Y$ be a holomorphic mapping. If $a = f(0)$, then

$$\sum_{n=0}^{\infty} \frac{\|D\phi_a(a)[D^k f(0)(z^k)]\|}{k!\|D\phi_a(a)\|} < 1$$

for $z \in (1/3)G$, where $\phi_a \in \mathrm{Aut}(B_Y)$ such that $\phi_a(a) = 0$. Moreover, the constant $1/3$ is best possible. Note that [77, Remark 3.3] if B_Y is one of the four classical domains in the sense of Hua [82], then B_Y is the unit ball of a J^*-algebra. Since J^*-algebra is a JB^*-triple, it follows that [77, Theorem 3.2] is a generalization of the result obtained by Liu and Wang [93]. Another generalization was given by Roos [109] who extended the result in [93] to bounded circled symmetric domains.

6.5 Conclusion

The purpose of this thesis is to further generalize the Bohr's theorem in distance form which might serve as a platform to solve the n-dimensional Bohr problem, $n \geq 2$. Till this day, the exact n-dimensional Bohr radius remains unknown even though its asymptotic value is known to be $\sqrt{\log n/n}$.

Bohr radius for the subordination class $R(\alpha, \gamma, h)$ is obtained. If h is a convex function, then the radius r is the smallest positive root of the equation

$$\sum_{n=1}^{\infty} \frac{1}{(1+\mu n)(1+\nu n)} r^n = \frac{1}{2}.$$

Let $\alpha = \gamma = 0$ (implying $\mu = \nu = 0$) and h be the half-plane function $z/(1-z)$. Then the Bohr radius for this case is $1/3$. Now, if h is a starlike function with respect to

$h(0)$, then the radius r is the smallest positive root of the equation

$$\sum_{n=1}^{\infty} \frac{1}{(1+\mu n)(1+\nu n)} r^n = \frac{1}{4}.$$

Putting $\alpha = \gamma = 0$ and taking h to be the Koebe function $z/(1-z)^2$, it follows that the Bohr radius is $3-2\sqrt{2}$. Both Bohr radii $1/3$ and $3-2\sqrt{2}$ are in fact known Bohr radii.

Earlier Bohr's theorems on the convex sets and the subordination classes of a univalent function inspires to investigate the Bohr's theorem for the class of analytic functions from U to concave-wedge domains W_α. For each domain W_α, there corresponds a Bohr radius $(2^{\frac{1}{\alpha}}-1)/(2^{\frac{1}{\alpha}}+1)$. The Bohr radii $1/3$ and $3-2\sqrt{2}$ corresponds to cases W_1 and W_2 respectively. In this thesis, the result requires the constant term to be positive but later it is shown that the theorem holds true for any constant in [24].

As the counterpart of the result by Abu-Muhanna and Ali on the class of analytic functions from U to the exterior of the closed unit disk, Bohr's theorem for the class of analytic functions from U to the punctured unit disk U_0 is obtained with Bohr radius $1/3$. Although a condition is imposed on the constant term, the constraint can be improved with better coefficient estimates or even be discarded once the Krzyż conjecture is proved. Indeed, the constraint is improved by replacing the Euclidean distance with the spherical chordal distance.

The classical Bohr's theorem makes its appearance in the spherical geometry with respect to the spherical chordal distance and the Bohr radius again has the value $1/3$. On the other hand, the Bohr's theorem for the class of analytic self-maps of U^h exists in the Poincaré disk model with respect to the hyperbolic distance. Furthermore, the

Bohr result obtained by Aizenberg is shown to be true in this model. The Bohr radii for both cases are shown to be $\tanh(1/2)/3$ implying the invariance of Bohr radius in the hyperbolic geometry.

The thesis also presents the Bohr's theorem for the class of complex-valued harmonic functions from U into a bounded domain in $\mathbb{C}$. The class has the Bohr radius $1/3$ and the result is a generalization of earlier Bohr's theorem for the class of complex-valued harmonic self-map of U. The same Bohr radius $1/3$ is obtained for the class of complex-valued harmonic functions from U into a fixed convex wedge. Finally, the Bohr-type inequality with sharp radius is established for a certain class of univalent logharmonic functions from U onto a starlike domain with respect to origin.

The Bohr radius $\tanh(1/2)/3$ in hyperbolic geometry is due to U^h which is the disk centered at the origin of radius $\tanh(1/2)$. Thus with a slight modification, the theorem remains true for the class of analytic self-maps of U and the Bohr radius value is again $1/3$. Hence this implies the Bohr radius $1/3$ for this class of functions is unaffected by the choice of geometry. Future research can be embarked to show that the invariance of Bohr radius $1/3$ in any geometry with constant Gaussian curvature.

It is also interesting to introduce the n-dimensional distance form Bohr's theorems which might have an exact Bohr radius. From there, it is possible to search for classes of analytic function with several variables which have the Bohr phenomenon and show the invariance of the Bohr radius among those classes. Lastly, distance form Bohr's theorems should be related to the existing n-dimensional Bohr results as well as other fields of researches to further enrich the field of study.

REFERENCES

[1] Z. Abdulhadi. Close-to-starlike logharmonic mappings. *Internat. J. Math. Math. Sci.*, 19(3):563–574, 1996.

[2] Z. Abdulhadi and R. M. Ali. Univalent logharmonic mappings in the plane. *Abstr. Appl. Anal.*, 2012. (Article ID 721943, 32 pp.).

[3] Z. Abdulhadi and D. Bshouty. Univalent functions in $h \cdot \overline{h}(d)$. *Trans. Amer. Math. Soc.*, 305(2):841–849, 1988.

[4] Z. Abdulhadi and W. Hengartner. Spirallike logharmonic mappings. *Complex Variables Theory Appl.*, 9(2-3):121–130, 1987.

[5] Z. Abdulhadi and W. Hengartner. Univalent harmonic mappings on the left half-plane with periodic dilatations. In H. M. Srivastava and S.Owa, editors, *Univalent functions, fractional calculus, and their applications (Kōriyama, 1988)*, pages 13–28, Ellis Horwood Ser. Math. Appl. Horwood, Chichester, 1989.

[6] Y. Abu-Muhanna. Bohr's phenomenon in subordination and bounded harmonic classes. *Complex Var. Elliptic Equ.*, 15(11):1–8, 2010.

[7] Y. Abu-Muhanna and R. M. Ali. Bohr's phenomenon for analytic functions into the exterior of a compact convex body. *J. Math. Anal. Appl.*, 379(2):512–517, 2011.

[8] Y. Abu-Muhanna and R. M. Ali. Bohr phenomenon for analytic functions and the hyperbolic metric. *Math. Nachr.*, 286(11-12):1059–1065, 2013.

[9] Y. Abu-Muhanna and G. Gunatillake. Bohr phenomenon in weighted Hardy-Hilbert spaces. *Acta Sci. Math. (Szeged)*, 78(3-4):517–528, 2012.

[10] Y. Abu-Muhanna and D. J. Hallenbeck. A class of analytic functions with integral representations. *Complex Var. Theory Appl.*, 19(4):271–278, 1992.

[11] Y. Abu-Muhanna and A. Lyzzaik. The boundary behaviour of harmonic univalent maps. *Pacific J. Math.*, 141(1):1–20, 1990.

[12] Y. Abu-Muhanna and G. Schober. Harmonic mappings onto convex domains. *Canad. J. Math.*, 39(6):1489–1530, 1987.

[13] Y. Abu-Muhanna, R. M. Ali, and S. Ponnusamy. On the Bohr inequality. In N. K. Govil, R. Mohapatra, M. A. Qazi, and G. Schmeisser, editors, *Progress*

in Approximation Theory and Applicable Complex Analysis: In Memory of Q.I. Rahman, pages 269–300. Springer International Publishing, Cham, 2017.

[14] L. V. Ahlfors. *Complex analysis*. McGraw-Hill Book Co., New York, third edition, 1978. An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.

[15] L. Aizenberg. Multidimensional analogues of Bohr's theorem on power series. *Proc. Amer. Math. Soc.*, 128(4):1147–1155, 2000.

[16] L. Aizenberg. Generalization of Carathéodory's inequality and the Bohr radius for multidimensional power series. In V. Ya. Eiderman and M. V. Samokhin, editors, *Selected topics in complex analysis*, volume 158 of Oper. Theory Adv. Appl., pages 87–94. Birkhäuser, Basel, 2005.

[17] L. Aizenberg. Generalization of results about the Bohr radius for power series. *Stud. Math.*, 180(2):161–168, 2007.

[18] L. Aizenberg. Remarks on the Bohr and Rogosinski phenomena for power series. *Anal. Math. Phys.*, 2(1):69–78, 2012.

[19] L. Aizenberg and N. Tarkhanov. A Bohr phenomenon for elliptic equations. *Proc. Amer. Math. Soc. (3)*, 82(2):385–401, 2001.

[20] L. Aizenberg, A. Aytuna, and P. Djakov. An abstract approach to Bohr's phenomenon. *Proc. Amer. Math. Soc.*, 128(9):2611–2619, 2000.

[21] L. Aizenberg, A. Aytuna, and P. Djakov. Generalization of a theorem of Bohr for bases in spaces of holomorphic functions of several complex variables. *J. Math. Anal. Appl.*, 258(2):429–447, 2001.

[22] L. A. Aĭzenberg, I. B. Grossman, and Yu. F. Korobeĭnik. Some remarks on the Bohr radius for power series. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (10):3–10, 2002.

[23] R. M. Ali, S. K. Lee, K. G. Subramanian, and A. Swaminathan. A third-order differential equation and starlikeness of a double integral operator. *Abstract and Applied Analysis*, 2011. (Article ID 901235, 10 pp.).

[24] R. M. Ali, R. W. Barnard, and A. Yu. Solynin. A note on the Bohr's phenomenon for power series. *J. Math. Anal. Appl.*, 449(1):154–167, 2017.

[25] J. W. Anderson. *Hyperbolic geometry*. Springer Undergraduate Mathematics Series. Springer-Verlag London, Ltd., London, second edition, 2005.

[26] M. Aydoğan. Some results on Janowski starlike log-harmonic mappings of complex order *b*. *Gen. Math.*, 17(4):171–183, 2009.

[27] M. Aydoğan and Y. Polatoğlu. A certain class of starlike log-harmonic mappings. *J. Comput. Appl. Math.*, 270:506–509, 2014.

[28] A. Aytuna and P. Djakov. Bohr property of bases in the space of entire functions

and its generalizations. *Bulletin of the London Mathematical Society*, 45(2):411–420, 2013.

[29] R. Balasubramanian, B. Calado, and H. Queffélec. The Bohr inequality for ordinary Dirichlet series. *Stud. Math.*, 175(3):285–304, 2006.

[30] F. Bayart, D. Pellegrino, and J. B. Seoane-Sepúlveda. The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$. *Adv. Math.*, 264:726–746, 2014.

[31] A. F. Beardon and D. Minda. The hyperbolic metric and geometric function theory. In S. Ponnusamy, T. Sugawa, and M. Vuorinen, editors, *Quasiconformal mappings and their applications*, pages 9–56. Narosa, New Delhi, 2007.

[32] C. Bénéteau, A. Dahlner, and D. Khavinson. Remarks on the Bohr phenomenon. *Comput. Methods Funct. Theory*, 4(1):1–19, 2004.

[33] L. Bieberbach. Über die koeffizienten derjenigen potenzreihen, welche eine schlichte abbildung des einheitskreises vermitteln. *Semesterberichte Preuss. Akad. Wiss.*, 38:940–955, 1916.

[34] O. Blasco. The Bohr radius of a Banach space. In G. Curbera, G. Mockenhaupt, and W. J. Ricker, editors, *Vector measures, integration and related topics*, volume 201 of Oper. Theory Adv. Appl., pages 59–64. Birkhäuser, Basel, 2010.

[35] O. Blasco. The p-Bohr radius of a Banach space. *Collect. Math.*, page to appear, 2016. doi:10.1007/s13348-016-0181-3.

[36] H. P. Boas. Majorant series. *Korean Math. Soc.*, 37(2):321–337, 2000.

[37] H. P. Boas and D. Khavinson. Bohr's power series theorem in several variables. *Proc. Amer. Math. Soc.*, 125(10):2975–2979, 1997.

[38] H. F. Bohnenblust and E. Hille. On the absolute convergence of Dirichlet series. *Ann. of Math.*, 32(2):600–622, 1931.

[39] H. Bohr. Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorieder Dirichletschen Reihen $\sum \frac{a_n}{n^s}$. *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl.*, 4:441–488, 1913.

[40] H. Bohr. A theorem concerning power series. *Proc. Lond. Math. Soc.*, s2-13(1): 1–5, 1914.

[41] E. Bombieri. Sopra un teorema di H. Bohr e G. Ricci sulle funzioni maggioranti delle serie di potenze. *Boll. Un. Mat. Ital. (3)*, 17:276–282, 1962.

[42] E. Bombieri and J. Bourgain. A remark on Bohr's inequality. *Int. Math. Res. Not.*, (80):4307–4330, 2004.

[43] D. Carando, A. Defant, D. García, M. Maestre, and P. Sevilla-Peris. The Dirichlet–Bohr radius. *Acta Arith.*, 171(1):23–37, 2015.

[44] X. Chen and T. Qian. Non-stretch mappings for a sharp estimate of the Beurling-Ahlfors operator. *J. Math. Anal. Appl.*, 412(2):805–815, 2014.

[45] C. Chu. Asymptotic Bohr radius for the polynomials in one complex variable. In J. Mashreghi, E. Fricain, and W. Ross, editors, *Invariant subspaces of the shift operator*, volume 638 of Contemp. Math., pages 39–43. Amer. Math. Soc., Providence, RI, 2015.

[46] J. Clunie and T. Sheil-Small. Harmonic univalent functions. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 9:3–25, 1984.

[47] E. F. Collingwood and A. J. Lohwater. *The Theory of Cluster Sets*, volume 56 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1966.

[48] L. de Branges. A proof of the Bieberbach conjecture. *Acta Math.*, 154(1-2): 137–152, 1985.

[49] A. Defant and L. Frerick. A logarithmic lower bound for multi-dimensional bohr radii. *Israel J. Math.*, 152(1):17–28, 2006.

[50] A. Defant and L. Frerick. The Bohr radius of the unit ball of ℓ_p^n. *J. reine angew. Math.*, 660:131–147, 2011.

[51] A. Defant and C. Prengel. Harald Bohr meets Stefan Banach. In J. M. F. Castillo and W. B. Johnson, editors, *Methods in Banach space theory*, volume 337 of London Math. Soc. Lecture Note Ser., pages 317–339. Cambridge Univ. Press, Cambridge, 2006.

[52] A. Defant and U. Schwarting. Bohr's radii and strips – a microscopic and a macroscopic view. *Note Mat.*, 31(1):87–101, 2011.

[53] A. Defant, D. García, and M. Maestre. Bohr's power series theorem and local Banach space theory. *J. reine angew. Math.*, 557:173–197, 2003.

[54] A. Defant, D. García, and M. Maestre. Estimates for the first and second Bohr radii of Reinhardt domains. *J. Approx. Theory*, 128(1):53–68, 2004.

[55] A. Defant, M. Maestre, and C. Prengel. The arithmetic Bohr radius. *Q. J. Math.*, 59(2):189–205, 2008.

[56] A. Defant, M. Maestre, and C. Prengel. Domains of convergence for monomial expansions of holomorphic functions in infinitely many variables. *J. Reine Angew. Math.*, 634:13–49, 2009.

[57] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes, and K. Seip. The Bohnenblust-Hille inequality for homogenous polynomials is hypercontractive. *Ann. of Math. (2)*, 174(1):512–517, 2011.

[58] A. Defant, M. Maestre, and U. Schwarting. Bohr radii of vector valued holomorphic functions. *Adv. Math.*, 231(5):2837–2857, 2012.

[59] C. Della-Rocchetta, G. Gentili, and G. Sarfatti. The Bohr Theorem for slice regular functions. *Math. Nachr.*, 285(17-18):2093–2105, 2012.

[60] S. Dineen. *The Schwarz lemma*. The Clarendon Press, Oxford University Press, New York, 1989. Oxford Science Publications.

[61] S. Dineen and R. M. Timoney. Absolute bases, tensor products and a theorem of Bohr. *Studia Math.*, 94(3):227–234, 1989.

[62] P. G. Dixon. Banach algebras satisfying the non-unital von Neumann inequality. *Bull. Lond. Math. Soc.*, 27(4):359–362, 1995.

[63] P. B. Djakov and M. S. Ramanujan. A remark on Bohr's theorem and its generalizations. *J. Anal.*, 8:65–77, 2000.

[64] H. Dörrie. *100 great problems of elementary mathematics: Their history and solution*. Dover Publications, Inc., New York, 1965. Translated by David Antin.

[65] E. Y. Duman. Some distortion theorems for starlike log-harmonic functions. *RIMS Kokyuroku*, 1772:1–7, 2011.

[66] P. Duren. *Harmonic mappings in the plane*, volume 156 of Cambridge Tracts in Mathematics. Cambridge Univ. Press, Cambridge, 2004.

[67] P. Duren and G. Schober. A variational method for harmonic mappings onto convex regions. *Complex Variables Theory Appl.*, 9(2-3):153–168, 1987.

[68] P. Duren and G. Schober. Linear extremal problems for harmonic mappings of the disk. *Proc. Amer. Math. Soc.*, 106(4):967–973, 1989.

[69] P. L. Duren. *Univalent functions*, volume 259 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, 1983.

[70] R. Fournier. Asymptotics of the Bohr radius for polynomials of fixed degree. *J. Math. Anal. Appl.*, 338(2):1100–1107, 2008.

[71] R. Fournier and S. Ruscheweyh. On the Bohr radius for simply connected plane domains. In J. Mashreghi, T. Ransford, and K. Seip, editors, *Hilbert spaces of analytic functions*, volume 51 of CRM Proc. Lecture Notes, pages 165–171. Amer. Math. Soc., Providence, RI, 2010.

[72] S. Gong. *The Bieberbach conjecture*, volume 12 of AMS/IP Studies in Advanced Mathematics. Amer. Math. Soc., Providence, RI, 1999. Translated from the 1989 Chinese original and revised by the author.

[73] Z. Guadarrama. Bohr's radius for polynomials in one complex variable. *Comput. Methods Funct. Theory*, 5(1):143–151, 2005.

[74] K. Gürlebeck and J. Morais. On the development of Bohr's phenomenon in the context of quaternionic analysis and related problems. In W. Tutschke and H. S. Le,

editors, *Algebraic structures in partial differential equations related to complex and Clifford analysis*, pages 9–24. Ho Chi Minh City Univ. Educ. Press, Ho Chi Minh City, 2010.

[75] K. Gürlebeck and J. P. Morais. Bohr type theorems for monogenic power series. *Comput. Methods Funct. Theory*, 9(2):633–651, 2009.

[76] D. J. Hallenbeck and T. H. MacGregor. *Linear problems and convexity techniques in geometric function theory*, volume 22 of Monographs and studies in mathematics. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[77] H. Hamada and T. Honda. Some generalizations of Bohr's theorem. *Math. Methods Appl. Sci.*, 35(17):2031–2035, 2012.

[78] H. Hamada, T. Honda, and G. Kohr. Bohr's theorem for holomorphic mappings with values in homogeneous balls. *Israel J. Math.*, 173(1):177–187, 2009.

[79] G. H. Hardy and M. Riesz. *The general theory of Dirichlet series*. Cambridge Univ. Press, Cambridge, 1915.

[80] W. Hengartner and G. Schober. On the boundary behavior of orientation-preserving harmonic mappings. *Complex Variables Theory Appl.*, 5(2-4):197–208, 1986.

[81] W. Hengartner and G. Schober. Harmonic mappings with given dilatation. *J. London Math. Soc.*, s2-33(3):473–483, 1986.

[82] L. K. Hua. *Harmonic analysis of functions of several complex variables in the classical domains*. Amer. Math. Soc., Providence, RI, 1963. Translated from the Russian by Leo Ebner and Adam Korányi.

[83] H. T. Kaptanoğlu. Bohr phenomena for Laplace-Beltrami operators. *Indag. Math. (N.S.)*, 17(3):407–423, 2006.

[84] H. T. Kaptanoğlu and N. Sadık. Bohr radii of elliptic region. *Russ. J. Math. Phys.*, 12(3):363–368, 2005.

[85] K.-T. Kim and H. Lee. *Schwarz's lemma from a differential geometric viewpoint*, volume 2 of IISc Lecture Notes Series. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

[86] W. Koepf and D. Schmersau. Bounded nonvanishing functions and Bateman functions. *Complex Var.*, 25(3):237–259, 1994.

[87] G. Kresin and V. Maz'ya. Sharp Bohr type real part estimates. *Comput. Methods Funct. Theory*, 7(1):151–165, 2006.

[88] G. Kresin and V. Maz'ya. *Sharp real-part theorems. A unified approach*, volume 1903 of Lecture Notes in Mathematics. Springer, Berlin, 2007. Translated from the Russian and edited by T. Shaposhnikova.

[89] P. Lassère and E. Mazzilli. Bohr's phenomenon on a regular condenser in the complex plane. *Comput. Methods Funct. Theory*, 12(1):31–43, 2012.

[90] P. Lassère and E. Mazzilli. The Bohr radius for an elliptic condenser. *Indag. Math. (N.S.)*, 24(1):83–102, 2013.

[91] P. Lassère and E. Mazzilli. Estimates for the Bohr radius of a Faber-Green condenser in the complex plane. *Constr. Approx.*, 45(3):409–426, 2017.

[92] H. Lewy. On the non-vanishing of the Jacobian in certain one-to-one mappings. *Bull. Amer. Math. Soc.*, 42(10):689–692, 1936.

[93] T. Liu and J. Wang. An absolute estimate of the homogeneous expansions of holomorphic mappings. *Pacific J. Math.*, 231(1):155–166, 2007.

[94] Zh. Mao, S. Ponnusamy, and X. Wang. Schwarzian derivative and Landau's theorem for logharmonic mappings. *Complex Var. Elliptic Equ.*, 58(8):1093–1107, 2013.

[95] S. S. Miller and P. T. Mocanu. *Differential subordinations*, volume 225 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 2000.

[96] S. S. Miller, P. T. Mocanu, and M. O. Reade. Bazilevic functions and generalized convexity. *Rev. Roumaine Math. Pures Appl.*, 19:213–224, 1974.

[97] D. Minda. Applications of hyperbolic convexity to Euclidean and spherical convexity. *J. Analyse Math.*, 49:90–105, 1987.

[98] Z. Nehari. The elliptic modular function and a class of analytic functions first considered by Hurwitz. *Amer. J. Math.*, 69:70–86, 1947.

[99] J. C. C. Nitsche. *Lectures on minimal surfaces. Vol. 1. Introduction, fundamentals, geometry and basic boundary value problems*. Cambridge Univ. Press, Cambridge, 1989. Translated from the German by J. M. Feinberg.

[100] R. Osserman. *A survey of minimal surfaces*. Dover Publications, Inc., New York, 2nd edition, 1986.

[101] H. E. Özkan and Y. Polatoğlu. Bounded log-harmonic functions with positive real part. *J. Math. Anal. Appl.*, 399(1):418–421, 2013.

[102] V. I. Paulsen and D. Singh. A simple proof of Bohr's inequality. preprint.

[103] V. I. Paulsen and D. Singh. Bohr's inequality for uniform algebras. *Proc. Amer. Math. Soc.*, 132(12):3577–3579, 2004.

[104] V. I. Paulsen and D. Singh. Extensions of Bohr's inequality. *Bull. Lond. Math. Soc.*, 38(6):991–999, 2006.

[105] V. I. Paulsen, G. Popescu, and D. Singh. On Bohr's inequality. *Proc. Lond.*

Math. Soc.(3), 85(2):493–512, 2002.

[106] G. Pick. über eine Eigenschaft der konformen Abbildung kreisförmiger Bereiche. *Math. Ann.*, 77(1):1–6, 1915.

[107] Y. Polatoḡlu, E. Y. Duman, and H. E. Özkan. Growth theorems for perturbated starlike log-harmonic mappings of complex order. *Gen. Math.*, 17(4):185–193, 2009.

[108] G. Popescu. Multivariable Bohr inequalities. *Trans. Amer. Math. Soc.*, 359(11): 5283–5317, 2007.

[109] G. Roos. Harald Bohr's theorem for bounded symmetric domains. *Tambov University Reports. Series: Natural and Technical Sciences*, 16(6-2):1729–1737, 2011.

[110] S. Ruscheweyh. New criteria for univalent functions. *Proc. Amer. Math. Soc.*, 49:109–115, 1975.

[111] S. Ruscheweyh and T. Sheil-Small. Hadamard products of schlicht functions and the Polya-Schoenberg conjecture. *Comment. Math. Helv.*, 48:119–135, 1973.

[112] S. Sidon. Über einen Satz von Herrn Bohr. *Math. Z.*, 26(1):731–732, 1927.

[113] P. K. Suetin. *Series of Faber Polynomials*, volume 1 of Analytical Methods and Special Functions. Gordon and Breach Science Publishers, Amsterdam, 1998. Translated from the 1984 Russian original by E. V. Pankratiev.

[114] R. Szász. The sharp version of a criterion for starlikeness related to the operator of Alexander. *Ann. Polon. Math.*, 94(1):1–14, 2008.

[115] M. Tomić. Sur un théorème de H. Bohr. *Math. Scand.*, 11:103–106, 1962.

[116] D. G. Yang and J. L. Liu. On a class of analytic functions with missing coefficients. *Appl. Math. Comput.*, 215(9):3473–3481, 2010.

LIST OF PUBLICATIONS

[1] Y. Abu-Muhanna, R. M. Ali, Z. C. Ng, and S. F. M. Hasni. Bohr radius for subordinating families of analytic functions and bounded harmonic mappings. *J. Math. Anal. Appl.*, 420(1):124–136, 2014. doi:10.1016/j.jmaa.2014.05.076.

[2] Y. Abu-Muhanna, R. M. Ali, and Z. C. Ng. Bohr radius for the punctured disk. *Math. Nachr.*, page in press, 2017. doi:10.1002/mana.201600094.

[3] R. M. Ali and Z. C. Ng. The bohr inequality in the hyperbolic plane. *Complex Var. Elliptic Equ.*, page in press, 2017. doi:10.1080/17476933.2017.1385070.

[4] R. M. Ali, Z. Abdulhadi, and Z. C. Ng. The Bohr radius for starlike logharmonic mappings. *Complex Var. Elliptic Equ.*, 61(1):1–14, 2016. doi:10.1080/17476933.2015.1051477.

[5] Z. C. Ng and R. M. Ali. A Bohr phenomenon on the punctured unit disk. *AIP Conf. Proc.*, 1750:050002, 7 pp., 2016. doi:10.1063/1.4954590.